Chapter 3

DIVERSIFICATION IN ORDER LEVEL LOT SIZE INVENTORY MODEL
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3.1 INTRODUCTION:

In this, chapter an order level lot-size inventory model is developed. The results are derived by making the same assumptions those defined in chapter 2 by allowing shortages. The shortage cost per unit remains same for both the suppliers. Expressions for total cost is developed as function of the order levels and order quantities from each supplier when both the orders are placed simultaneously and when orders are placed one by one to both the suppliers.

The chapter concludes with a discussion of a numerical example for each situations and a comparison of the single source versus diversification. Next section deals with construction of mathematical model for stated situations explaining symbols used explicitly.

3.2 MATHEMATICAL MODEL:

In this section, we assume that annual demand is $R$ units and we may order from two independent sources in which case $K$ is the total ordering cost when both sources are used. $Q_i > 0, i=1,2$ units are ordered from source $i$. It is also assumed that shortages are allowed $S_i, i=1,2$ are the order levels.

The unit price charged by the supplier $i$ is $C_i$ per unit and inventory carrying charge fraction is $I$ per annum for both the sources.
SITUATION I: When units are ordered simultaneously to both the sources.

When $Q_{il}$, $(i = 1, 2)$ units are ordered from each supplier, the total amount ordered is $Q_{1l} + Q_{2l}$. Since $S_{il}$, $(i = 1, 2)$ are the order levels, total initial inventory at the beginning of each cycle is $S_{1l} + S_{2l}$. Then the total cost, $TC_l$, incurred per time unit consists of following costs:

a) Purchase Cost

$$PC = \frac{R(C_1Q_{1l} + C_2Q_{2l})}{Q_{1l} + Q_{2l}}$$

b) Ordering Cost

$$OC = \frac{KR}{Q_{1l} + Q_{2l}}$$

c) Inventory Holding Cost

$$IHC = I \left( \frac{C_1^2S_{1l}^2 + C_2^2S_{2l}^2}{2(Q_{1l} + Q_{2l})} + \frac{(C_1 + C_2)S_{1l}S_{2l}}{Q_{1l} + Q_{2l}} \right)$$

d) Shortage Cost

$$SC = \frac{\pi}{2(Q_{1l} + Q_{2l})}(Q_{1l} + Q_{2l} - S_{1l} - S_{2l})^2$$

Using (a) - (b), $TC_l$, total cost of an inventory system per time unit is a function of order levels $S_{1l}$ and $S_{2l}$ and procurement quantities $Q_{1l}$ and $Q_{2l}$. Thus

$$TC_l(Q_{1l}, Q_{2l}, S_{1l}, S_{2l}) = PC + OC + IHC + SC \quad (3.1)$$

To minimize $TC_l$ subject to $S_{1l} > 0$, $S_{2l} > 0$, $Q_{1l} > 0$ and $Q_{2l} > 0$ (since diversification is in effect, i.e. both sources are being used) would give us the optimal values...
of order levels and order quantities which are to be ordered from each sources.

To minimize \( TC_i \), we perform partial differentiations and obtain the following:

\[
\frac{\partial TC_i}{\partial S_{1i}} = 0
\]

\[
S_{1i} = \frac{\pi (Q_{1i} + Q_{2i}) - \alpha S_{2i}}{\beta}
\]

where

\[ \beta = C_1 I + \tau \text{ and } \]

\[ \alpha = C_2 I + \beta \]

\[
\frac{\partial TC_i}{\partial S_{2i}} = 0
\]

\[
S_{2i} = \frac{Y}{X} (Q_{1i} + Q_{2i})
\]

where

\[
X = 2\alpha \beta (C_1 + C_2) I + C_2 I \pi (\beta - \alpha) - I (C_1 \alpha^2 + C_2 \beta^2)
\]

\[
Y = \pi I (C_1 (\beta - \alpha) + C_2 (2\beta - \pi))
\]

\[
Z = \frac{C_1 I}{2\beta} (\pi - \alpha \frac{Y}{X})^2 + \frac{C_2 I Y^2}{2X^2} + \frac{(C_1 + C_2) I Y}{\beta X} (\pi - \alpha \frac{Y}{X}) + \frac{\pi}{2} \left( 1 - \frac{\pi - \alpha \frac{Y}{X}}{\beta} - \frac{Y}{X} \right)^2
\]

\[
\frac{\partial TC_i}{\partial Q_{1i}} = 0, \text{ gives }
\]

\[
Q_{1i} = -Q_{2i} + \sqrt{Q_{2i}^2 - (ZQ_{2i}^2 + RQ_{2i}(C_1 - C_2) - KR)/Z}
\]

\[
\frac{\partial TC_i}{\partial Q_{2i}} = 0, \text{ gives }
\]

\[
\frac{R(C_2 - C_1)}{(Q_{1i} + Q_{2i})^2} \left( Q_{1i} - Q_{2i} \frac{\partial Q_{1i}}{\partial Q_{2i}} \right) + \left( 1 + \frac{\partial Q_{1i}}{\partial Q_{2i}} \right) \left( Z - \frac{KR}{(Q_{1i} + Q_{2i})^2} \right) = 0
\]
where
\[
\frac{\partial Q_{1i}}{\partial Q_{2i}} = \frac{2Q_{2i} - (2ZQ_{2i} + R(C_1 - C_2))/Z}{2\sqrt{Q_{2i}^2 - (ZQ_{2i} + RQ_{2i}(C_1 - C_2) - KR)/Z}} - 1
\]

Eq (3.5) can be solved for \( Q_{2i} \) by suitable numerical methods and hence eq. (3.4) will
give \( Q_{1i} \), eq. (3.3) will give \( S_{2i} \) and eq. (3.2) will give optimum value of \( S_{1i} \) and \( TC_i \)
can be obtained from eq. (1).

**Proposition (P-1):**

Order level \( S_{1i} \) is a decreasing function of order level \( S_{2i} \).

**Proposition (P-2):**

Order level \( S_{2i} \) is an increasing function of \( Q_{1i} \) and \( Q_{2i} \) both if
\[
\frac{C_i^2I^2 + C_i\alpha^2 + C_2\beta^2}{2\alpha(C_1 + C_2)} + 1 < \frac{C_i I}{\pi}
\]

To see whether diversification is worthwhile one need to compare \( TC_i^* \) with \( TC_i^* \),
the minimum cost incurred if source \( i=1,2 \) is used exclusively. Letting \( K_i \) to be order
cost if source \( i \) (i = 1,2) is used, we have Optimal order levels

\[
S_i^* = \sqrt{\frac{2RK_i\pi}{C_i I(C_i I + \pi)}} \quad (3.6)
\]

Optimal order quantity

\[
Q_i^* = \sqrt{\frac{(C_i I + \pi)}{C_i I\pi}2RK_i} \quad (3.7)
\]

The corresponding minimum cost is

\[
TC_i^* = \sqrt{2RK_i}\frac{C_i I\pi}{(C_i I + \pi)} \quad (3.8)
\]
A comparison of eqs. (3.1) and (3.8) reveals that the optimal solution to the diversification problem will depend on which of $TC_1^*$, $TC_i^*$, or $TC_2^*$ is the smallest, given the parameters $R$, $K$, $K_i$, $C_i$, and $I$. We will discuss numerical example where the comparison and interdependence of parameters is studied.

**SITUATION II:** When order is placed one after the other to two sources.

Arguing, as above, in this situation, the total cost $TC_{II}$ of an inventory system per time unit is given by

$$
TC_{II}(Q_{1II}, Q_{2II}, S_{1II}, S_{2II}) = \frac{KR}{Q_{1II} + Q_{2II}} + \frac{C_1 S_1^2}{2(Q_{1II} + Q_{2II})} + \frac{R(C_1 Q_{1II} + C_2 Q_{2II})}{Q_{1II} + Q_{2II}} + \frac{\pi}{2} \left( \frac{(Q_{1II} - S_{1II})^2 + (Q_{2II} - S_{2II})^2}{Q_{1II} + Q_{2II}} \right)
$$

To minimize $TC_{II}$, we perform partial differentiation and obtain the following.

$\frac{\partial TC_{II}}{\partial S_{1II}} = 0$ implies

$$S_{1II} = \frac{Q_{1II} \pi}{C_1 I + \pi}$$

(3.10)

$\frac{\partial TC_{II}}{\partial S_{2II}} = 0$ implies

$$S_{2II} = \frac{Q_{2II} \pi}{C_2 I + \pi}$$

(3.11)

$\frac{\partial TC_{II}}{\partial Q_{1II}} = 0$ implies

$$Q_{1II} = - Q_{2II} + \sqrt{Q_{2II}^2 - \frac{2}{\alpha_1} (\gamma_1 Q_{2II} - \beta_1 Q_{2II}^2 - KR)}$$

(3.12)
where

\[ \alpha_1 = \frac{C_1 I \pi (C_1 I + \pi)}{(C_1 I + \pi)^2} \]

\[ \beta_1 = \frac{C_2 I}{2} \left( \frac{\pi}{C_1 I + \pi} \right)^2 + \left( \frac{C_2 I}{C_2 I + \pi} \right)^2 \pi \]

\[ \gamma_1 = R(C_1 - C_2) \]

\[ \frac{\partial T_{11}}{\partial Q_{211}} = 0 \] implies

\[
- \frac{K R(1 + \frac{\partial Q_{111}}{\partial Q_{211}})}{(Q_{111} + Q_{211})^2} + \frac{C_1 I \left( \frac{\pi}{C_1 I + \pi} \right)^2 Q_{111} \frac{\partial Q_{111}}{\partial Q_{211}}}{Q_{111} + Q_{211}} + \frac{C_2 I \left( \frac{\pi}{C_2 I + \pi} \right)^2 Q_{211}}{Q_{111} + Q_{211}}
\]

\[
- \frac{\left( C_1 I \left( \frac{\pi}{C_1 I + \pi} \right)^2 + C_2 I \left( \frac{\pi}{C_2 I + \pi} \right)^2 \right) \left( \frac{\partial Q_{111}}{\partial Q_{211}} + 1 \right)}{2(Q_{111} + Q_{211})^2}
\]

\[
+ \frac{\pi}{Q_{111} + Q_{211}} \left( \frac{C_1 I Q_{111}}{C_1 I + \pi} \right) \frac{\partial Q_{111}}{\partial Q_{211}} + \left( \frac{C_2 I Q_{211}}{C_2 I + \pi} \right) \frac{\partial Q_{211}}{\partial Q_{211}}
\]

\[
- \left( \frac{C_1 I Q_{111}}{C_1 I + \pi} \right)^2 + \left( \frac{C_2 I Q_{211}}{C_2 I + \pi} \right)^2 \frac{\pi \left( 1 + \frac{\partial Q_{111}}{\partial Q_{211}} \right)}{2(Q_{111} + Q_{211})^2}
\]

\[
+ \frac{R(C_1 \frac{\partial Q_{111}}{\partial Q_{211}} + C_2)}{Q_{111} + Q_{211}} - \frac{R(C_1 Q_{211} + C_2 Q_{211}) \left( 1 + \frac{\partial Q_{111}}{\partial Q_{211}} \right)}{(Q_{111} + Q_{211})^2} - 1
\]

\[ 3.13 \]

where

\[ \frac{\partial Q_{111}}{\partial Q_{211}} = \frac{Q_{211} - \frac{1}{\alpha_1} (\gamma_1 - 2 \beta_1 Q_{211})}{\sqrt{Q_{211}^2 - \frac{2}{\alpha_1} (\gamma_1 Q_{211} - \beta_1 Q_{211}^2 - K R)}} - 1 \]

Eq. (3.13) can be solved for \( Q_{211} \) be suitable numerical method and \( Q_{111} \) can be obtained from eq. (3.12), order level \( S_{211} \) from eq. (3.11) and order level \( S_{111} \) from
eq. (3.10) and $TC_{II}$ can be obtained from eq. (3.9). In situation II, we have following proportion.

Proposition (P-3):
Order levels $S_{1II}$ and $S_{2II}$ are increasing functions of procurement quantities $Q_{1II}$ and $Q_{2II}$ respectively.

Proposition (P-4):
The procurement quantity $Q_{III}$ is an increasing function of $Q_{2III}$.

To see whether diversification is worthwhile, one need to compare $TC^*_1$ given in eq. (3.9) and $TC^*_i$ ($i = 1, 2$) given in eq. (3.8).

The above comparison is also discussed by a numerical example.

NUMERICAL EXAMPLE:

In this section, we establish effect of various parameters on optimum order levels, optimum order quantities and total costs of both the situations.

We consider the following parametric values:

Order cost, $K = $ 25.00 per order
Demand rate, $R = 1000$ units per year
Unit cost, $C_1 = $ 20.00 per unit charged by supplier 1
Unit cost, $C_2 = $ 30.00 per unit charged by supplier 2
Inventory carrying charge fraction, $I = 10$ % per annum
Shortage cost, $\pi = $ 34.00 per unit

We assume that the second source charge more, $C_2 > C_1$
Using this data, the optimal order levels, order quantities and total cost are as given in TABLE : 1.

**TABLE : 1**

<table>
<thead>
<tr>
<th></th>
<th>Situation - I</th>
<th>Situation - II</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q_1 )</td>
<td>851/order</td>
<td>699/order</td>
</tr>
<tr>
<td>( Q_2 )</td>
<td>160/order</td>
<td>45/order</td>
</tr>
<tr>
<td>( S_1^* )</td>
<td>350</td>
<td>632</td>
</tr>
<tr>
<td>( S_2^* )</td>
<td>526</td>
<td>41</td>
</tr>
<tr>
<td>( TC^* )</td>
<td>$23332.94</td>
<td>$21264.91</td>
</tr>
</tbody>
</table>

Now suppose that the orders can be placed at one source. If the order cost at individual suppliers, are, e.g. \( K_1 = \$13.00 \) per order and \( K_2 = \$12.00 \) per order, with the other data as above, we obtain

\[
\begin{align*}
Q_1^* &= 117.32 \text{ per order} \quad S_1^* = 110.81 \text{ units} \quad TC_1^* = \$24651.65 \\
Q_2^* &= 93.31 \text{ per order} \quad S_2^* = 85.74 \text{ units} \quad TC_2^* = \$24687.26
\end{align*}
\]

Comparing these \( TC_i^* \) \((i = 1, 2)\) values with \( TC_I^* \) and \( TC_{II}^* \) obtained when two suppliers are used \(( K = 25.00)\). We see that it would be preferable to use both the suppliers and situation II.

It is interesting to note when the values of \( K_1 \) and \( K_2 \) are interchanged, e.g. \( K_1 = \$12.00 \) per order and \( K_2 = \$13.00 \) per order, the optimal solution is to use both the suppliers and situation II, since, in this case we get
$Q_1^* = 112.72$ per order $S_1^* = 106.46$ units $TC_1^* = $ 24841.21

$Q_2^* = 97.16$ per order $S_2^* = 89.24$ units $TC_2^* = $ 24896.01

We conducted sensitivity analysis of changes in inventory carrying charge fraction and have observed the changes in the optimal decisions. These results are presented in TABLE : 2 (for situation - I).
TABLE : 2

Effect of changes in $I$ on decision variables (Situation I)

<table>
<thead>
<tr>
<th>$I$</th>
<th>$Q_1^*$</th>
<th>$Q_2^*$</th>
<th>$S_1^*$</th>
<th>$S_2^*$</th>
<th>$TC_i^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>815</td>
<td>160</td>
<td>350</td>
<td>526</td>
<td>23332.94</td>
</tr>
<tr>
<td>0.11</td>
<td>810</td>
<td>180</td>
<td>352</td>
<td>529</td>
<td>23686.01</td>
</tr>
<tr>
<td>0.12</td>
<td>788</td>
<td>190</td>
<td>345</td>
<td>517</td>
<td>23934.45</td>
</tr>
<tr>
<td>0.13</td>
<td>782</td>
<td>210</td>
<td>346</td>
<td>520</td>
<td>24281.54</td>
</tr>
<tr>
<td>0.14</td>
<td>763</td>
<td>220</td>
<td>340</td>
<td>510</td>
<td>24524.48</td>
</tr>
<tr>
<td>0.15</td>
<td>756</td>
<td>240</td>
<td>341</td>
<td>512</td>
<td>24865.78</td>
</tr>
</tbody>
</table>

It can be observed that, as inventory carrying charge fraction per annum increases, optimum quantities to be purchased from supplier 1 decreases whereas units to be procured from supplier 2 and total cost of an inventory system increases significantly.

The change in demand rate or shortage cost has negligible changes an optimal decisions for situation I.

In next part, we observe changes in demand rate (TABLE : 4) inventory carrying charge fraction (TABLE : 3) and shortage cost (TABLE : 5) on optimal decisions for situation II.
TABLE : 3

Effect of changes in I on decision variables (Situation II)

<table>
<thead>
<tr>
<th>I</th>
<th>$Q_1^*$</th>
<th>$Q_2^*$</th>
<th>$S_1^*$</th>
<th>$S_2^*$</th>
<th>$TC_i^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>669</td>
<td>45</td>
<td>632</td>
<td>41</td>
<td>21264.91</td>
</tr>
<tr>
<td>0.02</td>
<td>579</td>
<td>36</td>
<td>541</td>
<td>32</td>
<td>21192.36</td>
</tr>
<tr>
<td>0.12</td>
<td>495</td>
<td>28</td>
<td>462</td>
<td>25</td>
<td>21110.75</td>
</tr>
<tr>
<td>0.13</td>
<td>437</td>
<td>23</td>
<td>406</td>
<td>20</td>
<td>21057.71</td>
</tr>
<tr>
<td>0.14</td>
<td>381</td>
<td>18</td>
<td>352</td>
<td>16</td>
<td>20985.89</td>
</tr>
<tr>
<td>0.15</td>
<td>342</td>
<td>15</td>
<td>314</td>
<td>13</td>
<td>20943.05</td>
</tr>
</tbody>
</table>

It can be observed that increase in inventory carrying charge fraction, results decreases in optimal order quantities, order levels and total cost per time unit.

The comparison of TABLE : 2 and TABLE : 3 indicates that situation II is better to be in practices as it lowers the total cost of an inventory system per time unit.
TABLE : 4

Effect of changes in R on decision variables (Situation II)

<table>
<thead>
<tr>
<th>R</th>
<th>$Q_1^*$</th>
<th>$Q_2^*$</th>
<th>$S_1^*$</th>
<th>$S_2^*$</th>
<th>$TC_i^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>669</td>
<td>45</td>
<td>632</td>
<td>41</td>
<td>21264.91</td>
</tr>
<tr>
<td>1200</td>
<td>836</td>
<td>60</td>
<td>789</td>
<td>55</td>
<td>25579.28</td>
</tr>
<tr>
<td>1400</td>
<td>1003</td>
<td>75</td>
<td>947</td>
<td>68</td>
<td>29895.06</td>
</tr>
<tr>
<td>1600</td>
<td>1163</td>
<td>89</td>
<td>1099</td>
<td>81</td>
<td>34198.36</td>
</tr>
<tr>
<td>1800</td>
<td>1324</td>
<td>103</td>
<td>1250</td>
<td>94</td>
<td>38501.20</td>
</tr>
<tr>
<td>2000</td>
<td>1884</td>
<td>117</td>
<td>1401</td>
<td>107</td>
<td>42803.74</td>
</tr>
</tbody>
</table>

It can be easily seen that increases in demand rate, results significant increase in all optimal parameters, viz, order quantities, order levels and total cost per time unit.

TABLE : 5

Effect of changes in $\pi$ on decision variables (Situation II)

<table>
<thead>
<tr>
<th>$\pi$</th>
<th>$Q_1^*$</th>
<th>$Q_2^*$</th>
<th>$S_1^*$</th>
<th>$S_2^*$</th>
<th>$TC_i^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>34</td>
<td>669</td>
<td>45</td>
<td>632</td>
<td>41</td>
<td>21264.91</td>
</tr>
<tr>
<td>35</td>
<td>653</td>
<td>43</td>
<td>618</td>
<td>39</td>
<td>21236.95</td>
</tr>
<tr>
<td>36</td>
<td>632</td>
<td>40</td>
<td>599</td>
<td>36</td>
<td>21198.83</td>
</tr>
<tr>
<td>37</td>
<td>610</td>
<td>37</td>
<td>579</td>
<td>34</td>
<td>21159.07</td>
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<tr>
<td>38</td>
<td>588</td>
<td>34</td>
<td>558</td>
<td>31</td>
<td>21117.47</td>
</tr>
<tr>
<td>39</td>
<td>564</td>
<td>31</td>
<td>537</td>
<td>28</td>
<td>21073.81</td>
</tr>
</tbody>
</table>

Increase in shortage cost also leads to decrease in optimal procurement quantities, optimal order levels and total cost of an inventory system per time unit.
3.3 CONCLUSIONS:

In this chapter, we have sited a complete analysis of the diversification situation in the order-level lot-size inventory model under two different situations, namely, when both the orders are placed simultaneously and when second order is placed the first order is depleted. The analytic solutions are obtained for the optimal order quantities from two sources, optimal order levels and the minimum average cost in terms of all parameter values.