Chapter 3

A LOT-SIZE MODEL WITH PARTIAL BACKLOGGING WHEN THE AMOUNT RECEIVED IS UNCERTAIN FOR DETERIORATING ITEMS
3.1 Introduction

Economic lot-size models for deteriorating items have been studied extensively in recent years. In these models it is usually assumed that the quantity received is same as the quantity ordered. In practice, this is not always true. The quantity received may not be same as the quantity ordered due to various reasons such as rejection during inspection, damage or breakage during transportation, high production yields, etc.

Gor and Shah (1994) developed a lot-size model for deteriorating items by allowing shortages. Silver (1976) has extended the simple Wilson's lot-size model to include the case where the quantity received is not necessarily equal to the quantity ordered. Two separate cases were considered to account for the variability in the quantity received. Kalro and Gohil (1982) extended Silver's model by allowing shortages.

In this chapter, a lot-size model with mixture of backorders and lost sales for continuously deteriorating items is developed for a similar situation. The quantity received is assumed to be a random variable following a probability distribution with the standard deviation \( \sigma \) given by \( \sigma^2 = \sigma_0^2 + \sigma_1^2 Q^2 \) where \( Q \) is the quantity ordered, which covers both the cases considered by Silver (1976) and Kalro and Gohil (1982), viz.,

- when the standard deviation of the quantity received is independent of the quantity ordered (for \( \sigma_1 = 0 \)) and

- when the standard deviation of the quantity received is proportional to the quantity ordered (for \( \sigma_0 = 0 \)).
3.2 Mathematical Model

3.2.1 Assumptions

- The rate of demand is constant.
- Shortages are partially backordered and remaining part of shortages comes under lost-sales.
- The quantity received is a random variable.
- The item under consideration is continuously deteriorating.

3.2.2 Notations

- $Y$ = The random variable representing the quantity received
- $E(Y|Q) = bQ$ = The expected value of $Y$, given $Q$ units are ordered, where $b$ is the bias factor
- $\sigma^2$ = Variance of $Y$, given $Q$ units are ordered
- $T(Y|Q)$ = The cycle length when $Y$ units are received, given $Q$ units are ordered

3.2.3 Decision Variables

- $Q$ = The ordered quantity
- $s$ = The total demand during the stockout period in a cycle
3.2.4 Determining The Objective Function

If a quantity \( Q \) is ordered each time, the expected quantity received will be \( E(Y|Q) = bQ \), where \( b \) is the bias factor. If the expected quantity received is less than or equal to the ordered quantity due to various reasons as mentioned above, then, \( 0 \leq b \leq 1 \).

If the expected quantity received is greater than the ordered quantity, then \( b > 1 \).

The variance of the quantity received is given by the following equation where \( \sigma_0^2 \) and \( \sigma_1^2 \) are non-negative constants.

\[
\sigma^2 = \sigma_0^2 + \sigma_1^2 Q^2
\]

For a given \( Y \), the inventory level at the beginning of the scheduling period will be \( (Y - \alpha s) \). If this quantity can satisfy the demand up to the time \( t_1 \), \( (t_1 \leq T(Y|Q)) \), we have:

\[
T(Y|Q) = t_1 + \frac{s}{R}
\]

where

\[
t_1 = \frac{1}{\theta} \log \left( 1 + \frac{\theta(Y - \alpha s)}{R} \right)
\]

The total cost per cycle is given by the following equation.

\[
C(Y) = A + CY + C_1 I_1(Y) + C_{21} I_{21}(Y) + C_{22} I_{22}(Y) + CD(Y)
\]

where

\[
I_1(Y) = \frac{1}{\theta^2} \left[ (Y - \alpha s)\theta - R \log \left( 1 + \frac{\theta(Y - \alpha s)}{R} \right) \right]
\]
\[ I_{21}(Y) = \frac{\sigma_0^2}{2R} \]

\[ I_{22}(Y) = (1 - \alpha)s \text{ and} \]

\[ D(Y) = Y - \alpha s - R_l \]

In equation (3.2), the first term on the R.H.S is the ordering cost, the second term is the inventory purchase cost, the third term is the inventory carrying cost, the fourth term is the cost due to backordered shortages, the fifth term is the cost due to lost-sales and the sixth term is the cost due to deterioration. It is not easy to work with present form of equations (3.1) and (3.2). Therefore, assuming \( \theta \) to be very small, we take the linear approximations of the terms involving \( \theta \) neglecting higher powers of \( \theta \).

Then the expected value of \( T \), given \( Q \) is

\[
E(T(Y)|Q) = \frac{1}{2R^2} \theta (b^2 + \sigma_1^2)Q^2 - \theta \alpha^2 s^2 + 2\theta b\alpha Qs
\]

\[
+ 2RbQ + 2R(1 - \alpha)s + \theta \sigma_0^2 \]  \hspace{1cm} (3.3)

The expected cost during the cycle for a given \( Q \) is given by

\[
E(C(Y)|Q) = \frac{C_1 \theta \alpha^3 s^3}{3R^2} - \frac{C_1 \theta (b^3 + 3b\sigma_1^2)Q^3}{3R^2}
\]

\[
+ \frac{C_1 \theta \alpha (b^2 + \sigma_1^2)Q^2 s}{R^2} - \frac{C_1 \alpha^2 \theta bQs^2}{R^2}
\]

\[
+ \alpha [(C_1 + C\theta)\alpha + C_{21}] s^2
\]

\[ \frac{2R}{3} \]

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Therefore, the expected cost per unit time is given by

\[
K(Q, s) = \frac{E(T|Q)}{E(T|Q)}
\]  \hspace{1cm} (3.5)

### 3.2.5 Determining The Optimal Solution

We wish to find absolute minimum of \( K(Q, s) \) from equation (3.5) in the region \( 0 < Q < \infty \) and \( s \geq 0 \). The optimum values of \( Q \) and \( s \) i.e., \( Q_0 \) and \( s_0 \) and hence minimum expected cost per unit time i.e., \( K_0 \) are given by the following equations. \( Q_0 \) and \( s_0 \) are the solution of the following system of equations

\[
\frac{\partial K(Q, s)}{\partial Q} = 0
\]

\[
\frac{\partial K(Q, s)}{\partial s} = 0
\]

satisfying the following condition.

\[
\frac{\partial^2 K(Q, s)}{\partial s^2} - \frac{\partial^2 K(Q, s)}{\partial Q \partial s} > 0
\]  \hspace{1cm} (3.6)
From \( \frac{\partial k(Q,s)}{\partial q} = 0 \), we get the following equation.

\[
A_1 Q^3 + B_1 Q^2 s + D_1 Q s^2 + E_1 s^3 + F_1 Q^2 \\
+ G_1 Q s + H_1 s^2 + J_1 Q + L_1 s + M_1 = 0
\]

where

\[A_1 = -40b^2(b^2 + 3\sigma_1^2)C_1,\]

\[B_1 = 60b(b^2(2\alpha - 1) + \sigma_1^2(4\alpha - 3)),\]

\[D_1 = 3\alpha(\sigma_1^2 + b^2)[(1 - \alpha)C_1 + C_{21}],\]

\[E_1 = \alpha(2(2\alpha - 3)C_1 - 3C_{21}),\]

\[F_1 = 3Rb(\sigma_1^2 + b^2)(C_1 + 2\theta C),\]

\[G_1 = 6R(1 - \alpha)(\sigma_1^2 + b^2)(C_1 + \theta(C_1 + C_{21})),\]

\[H_1 = -3R\theta(2 - \alpha)C_1 + 2\theta(1 - \alpha)C_{21} + C_{21} + 2\theta C),\]

\[J_1 = 6R\theta(\sigma_1^2 + b^2)A,\]

\[L_1 = 6b(b^2(1 - \alpha)(C - C_{22}) - \theta(\sigma_2^2C_1 + R\alpha A)],\text{ and}\]

\[M_1 = -3Rb(\sigma_2^2(2\theta C + C_1) + 2RA].\]
From $\frac{\partial k(Q,s)}{\partial s} = 0$, we have

$$A_2 Q^3 + B_2 Q s^2 + D_2 Q s^2 + E_2 s^3 + F_2 Q^2 + G_2 Q s + H_2 s^2 + J_2 Q + L_2 s + M_2 = 0 \quad (3.8)$$

where

$$A_2 = 2\theta b[3\sigma_1^2 + (2\alpha + 1)b^2]C_1,$$

$$B_2 = -3\theta \alpha [4\alpha b^2 C_1 + (\sigma_1^2 + b^2)C_{21}],$$

$$D_2 = 3\alpha^2 \theta b[C_{21} + 2(2\alpha - 1)C_1],$$

$$E_2 = 4\theta \alpha^3(1 - \alpha)C_1,$$

$$F_2 = -3R[(1 - \alpha)(\sigma_1^2 + b^2)(\theta C_{22} + C_1 + \theta C) - 2\alpha^2(C_1 + 2\theta C)],$$

$$G_2 = 6R\alpha b[C_{21} + \alpha(C_1 + 2\theta C)],$$

$$H_2 = 3R\alpha(1 - \alpha)[\alpha(C_1 + \theta C + \theta C_{22}) + C_{21}],$$

$$J_2 = 6b[\theta(\sigma_0^2 C_1 - R\alpha A) + H^2(1 - \alpha)(C_{22} - C')],$$

$$L_2 = 3\theta \alpha(2R\alpha A - \sigma_0^2 C_{21}),$$

and

$$M_2 = -3R(1 - \alpha)[2RA + \sigma_0^2(\theta C_{22} + C_1 + \theta C)].$$

The above equations are solved for the optimal values of $Q$ and $s$, i.e., $Q_0$ and $s_0$.  

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and the expected minimum cost per unit time $K_0$ is obtained. Now we consider the two special cases regarding the standard deviation of $Y$ considered by Silver (1976) and Kalro and Gohil (1982).

3.2.6 Special Cases

The two cases viz.,

- the standard deviation of the quantity received is independent of the quantity ordered,

- the standard deviation of the quantity received is proportional to the quantity ordered,

are the special cases of the mathematical model developed. Taking $\sigma_1 = 0$ and $\sigma_0 = 0$ in the mathematical model, we get the above mentioned two cases respectively. The expected cost per unit time from equation (3.5) in both the cases respectively will be given by the following equations.

$$K_i(Q, s) = \left( A_i Q^3 + B_i Q^2 s + D_i Q s^2 + E_i s^3 + I_i Q^2 \\
+ G_i Q s + H_i s^2 + L_i Q + J_i s + L_i \right) / \\
(M_i Q^2 + N_i Q s + O_i s^2 \\
+ P_i Q + R_i s + T_i) \; , \; i = 1, 2$$ (3.9)

where

$A_1 = -2C_1 \theta b^3$,

$B_1 = 6C_1 \theta ab^2$, 

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\[ D_1 = -6C_1 \alpha^2 \theta b, \]

\[ E_1 = 2C_1 \theta \alpha^3, \]

\[ F_1 = 3R(C_1 + C\theta)b^2, \]

\[ G_1 = -6R(C_1 + \theta C)\alpha b, \]

\[ H_1 = 3R\alpha[\alpha(C_1 + C\theta) + C_{21}], \]

\[ I_1 = (6R^2C\theta - 6C_1 \theta b\sigma_0^2), \]

\[ J_1 = [6R^2C_{22}(1 - \alpha) + 6C_1 \theta \alpha \sigma_0^2], \]

\[ L_1 = 6R^2A + 3R(C_1 + C\theta)\sigma_0^2, \]

\[ M_1 = -3\theta b^2, \]

\[ N_1 = 6\theta b\alpha, \]

\[ O_1 = -3\theta \alpha^2, \]

\[ P_1 = 6Rb, \]

\[ R_1 = 6R(1 - \alpha)S, \]
\[ T_1 = -30\sigma_0^2, \]
\[ A_2 = -2C_1\theta b(b^2 + 3\sigma_1^2), \]
\[ B_2 = 6C_1\theta\alpha(b^2 + \sigma_1^2), \]
\[ D_2 = -6C_1\alpha^2\theta b, \]
\[ E_2 = 2C_1\theta\alpha^3, \]
\[ F_2 = 3R(C_1 + C0)(b^2 + \sigma_1^2), \]
\[ G_2 = -6RC_1\alpha b, \]
\[ H_2 = 3R[(C_1 + C0)\alpha^2 + \alpha C_{21}], \]
\[ I_2 = 6R^2C_{22}(1 - \alpha), \]
\[ J_2 = 6CbR^2, \]
\[ L_2 = 6R^2\lambda, \]
\[ M_2 = -30(b^2 + \sigma_1^2), \]
\[ N_2 = 6b\alpha, \]
\[ O_2 = -3\theta \alpha^2, \]

\[ P_2 = 6Rb, \]

\[ R_2 = 6R(1 - \alpha), \] and

\[ T_2 = 0. \]

If we assume that there is no deterioration of units in the inventory, then the above mentioned models, whose expected costs per unit time are given by equations (3.9), reduce to that of Case 1 and Case 2 in Kalro and Gohil (1982). Taking \( \theta = 0 \), \( C_{21} \) as shortage cost per unit backordered per unit time, \( C_{22} \) as lost profit due to unit lost-sales and no fixed penalty cost per unit short in equations (3.9), we get Case 1 and Case 2 for both, the Complete Backordering situation (\( \alpha = 1 \)) and the Partial Backordering situation (\( \alpha < 1 \)) considered by Kalro and Gohil (1982).

The results obtained in the mathematical model can be reduced to that of the classical lot-size model with complete backordering Naddor (1966), by putting \( \alpha = 1 \) and \( \theta = 0 \). Further, if we consider \( \sigma_1 = \sigma_0 = 0 \), i.e., the amount received is equal to the amount ordered and that there is no deterioration of quantity in inventory, then we get the standard EOQ model with partial backordering.

### 3.3 A Hypothetical Problem

Consider an inventory model with the following parameters.

\[ R = 1000 \text{ units per year} \]
\( C = \$ 50 \) per unit

\( C_1 = \$ 9 \) per unit per year

\( C_{21} = \$ 10 \) per unit per year

\( C_{22} = \$ 15 \) per unit

\( A = \$ 250 \) per order

The optimal order quantity \( Q_0 \) and the optimal number of units of total shortages \( s_0 \) which will give the minimum expected total cost \( K_0 \) (excluding the purchase cost), are computed for different values of \( \sigma_0^2, \sigma_1^2, b, \theta \) and \( \alpha \). All these values are tabulated below.

**Table 3.1 : Effect of variations in \( b \)**

\( (\alpha = 0.95; \ \theta = 0.01; \ \sigma_0 = 30; \ \sigma_1 = 0.2) \)

<table>
<thead>
<tr>
<th>( b )</th>
<th>( s )</th>
<th>( Q )</th>
<th>( K )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>211</td>
<td>448</td>
<td>2832</td>
</tr>
<tr>
<td>0.9</td>
<td>222</td>
<td>273</td>
<td>2625</td>
</tr>
</tbody>
</table>

Due to increase in the bias factor \( b \), there is

- increase in the total shortages \( s_0 \) allowed during the scheduling period, (fig 3.1)

- decrease in both the order quantity \( Q_0 \) (fig 3.2) and

- the expected cost per unit time \( K_0 \) (fig 3.3).

**Table 3.2 : Effect of \( \alpha \)**

\( (b = 0.75; \ \theta = 0.01; \ \sigma_0 = 30; \ \sigma_1 = 0.2) \)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( s )</th>
<th>( Q )</th>
<th>( K )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>149</td>
<td>398</td>
<td>1680</td>
</tr>
<tr>
<td>0.95</td>
<td>219</td>
<td>321</td>
<td>2667</td>
</tr>
</tbody>
</table>

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Increase in the value of backordering fraction $\alpha$ shows

- increase in $Q_0$ (fig 3.5)

- decrease in both $s_0$ (fig 3.4) and $K_0$ (fig 3.6).

Table 3.3: Effect of $\theta$

$(b = 0.75; \alpha = 0.95; \sigma_0 = 30; \sigma_1 = 0.2)$

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$s_0$</th>
<th>$Q_0$</th>
<th>$K_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>220</td>
<td>327</td>
<td>2628</td>
</tr>
<tr>
<td>0.05</td>
<td>218</td>
<td>307</td>
<td>2783</td>
</tr>
</tbody>
</table>

Increase in the deterioration fraction $\theta$ is causing

- decrease in both $s_0$ and $Q_0$ (fig 3.7 and fig 3.8) and

- increase in $K_0$ (fig 3.9).

Table 3.4: Effect of $\sigma_0$

$(b = 0.75; \alpha = 0.95; \theta = 0.01; \sigma_1 = 0.2)$

<table>
<thead>
<tr>
<th>$\sigma_0$</th>
<th>$s_0$</th>
<th>$Q_0$</th>
<th>$K_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>218</td>
<td>317</td>
<td>2662</td>
</tr>
<tr>
<td>50</td>
<td>222</td>
<td>329</td>
<td>2667</td>
</tr>
</tbody>
</table>

- There is increase in $s_0$, $Q_0$ and $K_0$ with increase in $\sigma_0$ (fig 3.10, fig 3.11 and fig 3.12).

Table 3.5: Effect of $\sigma_1$

$(b = 0.75; \alpha = 0.95; \theta = 0.01; \sigma_0 = 30)$

<table>
<thead>
<tr>
<th>$\sigma_1$</th>
<th>$s_0$</th>
<th>$Q_0$</th>
<th>$K_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>228</td>
<td>343</td>
<td>2526</td>
</tr>
<tr>
<td>0.5</td>
<td>193</td>
<td>250</td>
<td>3300</td>
</tr>
</tbody>
</table>
Increase in the value of $\sigma_1$ is resulting in

- decrease of both $s_0$ and $Q_0$ (fig 3.13 and fig 3.14) and

- increase in $K_0$ (fig 3.15).

### 3.4 Concluding Remarks

In this chapter a lot-size inventory model is developed for deteriorating items under the assumption that quantity received is not same as the quantity ordered. Shortages are allowed and are partially backordered in this model. The quantity received ($Y$) is taken as a random variable following some distribution function ($f(Y|Q)$) for a given ordered quantity ($Q$). The cases, of standard deviation of the above probability distribution being independent of the quantity ordered and being proportional to the quantity ordered, are the special cases of the present model. The results obtained in this model reduce to that of Kalro and Gohil (1982) for nondeteriorating items, i.e., by substituting $\theta = 0$. 
fig 3.1: Effect of \( b \) on \( s_0 \)
fig 3.2: Effect of $b$ on $Q_0$
fig 3.3: Effect of $b$ on $K_0$
fig 3.4: Effect of $\alpha$ on $s_0$
**Fig 3.5:** Effect of $\alpha$ on $Q_0$
fig 3.6: Effect of $\alpha$ on $K_0$
fig 3.7: Effect of $\theta$ on $s_0$
fig 3.8: Effect of $\theta$ on $Q_0$
fig 3.9: Effect of $\theta$ on $K_0$
fig 3.10: Effect of $\sigma_0$ on $s_0$
Fig 3.11: Effect of $\sigma_0$ on $Q_0$
Figure 3.12: Effect of $\sigma_0$ on $K_0$
fig 3.13: Effect of $\sigma_1$ on $s_0$
fig 3.14: Effect of $\sigma_1$ on $Q_0$
fig 3.15: Effect of $\sigma_1$ on $K_0$