Chapter 5

Kink Instability: Nonlinear Studies

The objective, in this chapter, is to investigate the nonlinear development of the kink instability on fast EMHD time scales by numerically solving three dimensional nonlinear EMHD equations with appropriate initial and boundary conditions.

5.1 Introduction

In the last chapter, we investigated the linear physics of the three dimensional kink instability. It was learnt that when general three dimensional perturbations are considered, the sheared current profile is driven unstable even by those perturbations which have scale lengths shorter than the typical shear scale (local limit). These studies clearly demonstrate that in the presence of three dimensional perturbations there is a significant widening of the wave-number domain for which modes are unstable. Also kink mode is unstable even in the absence of the curvature of flow velocity profile and is driven by the transverse gradient in the flow velocity. These features of the three dimensional kink instability are in contrast with those of two dimensional sausage instability in which a particular mode is unstable only under very restricted conditions [37,40]. It basically gets driven only if the perturbation scales are typically longer than the shear scale (i.e. in the nonlocal limit). The mode is driven by electron inertia effects and requires curvature (second spatial
derivative) of the electron velocity flow profile. Because of these restrictions on the unstable perturbations, the sausage instability is saturated rather easily due to the approach of the equilibrium velocity profile to a form which violates one or more of the aforementioned conditions for the instability [38]. The coherent structures, as observed in the nonlinear state of the sausage instability, are due to the two dimensionality of the problem which allows two non dissipative square invariants $\int \int B_y^2 + (\nabla B_y)^2 dx dz$ and $\int \int (\nabla B_y)^2 + (\nabla^2 B_y)^2 dx dz$, where $B_y$ is the $y$-component of magnetic field [38]. The existence of these two invariants conspires to have a dominant transfer of power in the magnetic fields towards long scales. Sausage instability is thus easy to saturate and one is left with a shallow and/or a linearly varying sheared current profile in the final state. This leads to an incomplete stopping of the flow of electrons in the current channel.

Based on the relaxation of instability criteria in three dimensions it can be argued that the nonlinear development of the instability in the three dimensional case would lead to a highly turbulent state, and may cause significant anomalous damping of the electron flow.

Earlier nonlinear studies by Drake et al. [8,9] on kink instability in EMHD [1], are on a primitive level and incomplete. They have reported the widening of the shear layer due to the instability. However, various features of interest in the nonlinear stage of the instability like saturation mechanism, modification of mean flow profiles, turbulence characterization etc. remain to be explored.

In this chapter we present a detailed 3-dimensional EMHD fluid simulation of current shear driven modes in electron current channels. The simulations indeed show development of turbulent electromagnetic fluctuations and a comparatively much larger reduction in the directed flow of electrons as compared to the two dimensional case.

This chapter is organized as follows. In the next section we cast the three dimensional nonlinear EMHD equations in the form in which they are integrated numerically. Initial and boundary conditions for the simulations will be presented in section(5.3). In section(5.4) we give an overview of the numerical methods used. In section(5.5), results of the nonlinear simulation will be presented. Section5.6 contains the summary of the whole chapter.
5.2 Three Dimensional Nonlinear EMHD Equations

We are interested in the study of nonlinear evolution of kink instability on EMHD time scales in three dimensional slab geometry. In three dimensional slab geometry, all the components of the magnetic field ($B_x, B_y$ and $B_z$) and of velocity ($v_x, v_y$ and $v_z$) are required. This is in contrast to the case of the sausage instability studied in chapters (2) and (3) in two dimensional slab geometry in which taking only $y$-component of magnetic field ($B_y$) as finite was sufficient, for the reason that the other components get decoupled from $B_y$.

We now write the dimensionless EMHD equations (A.11) and (A.12) neglecting collisions.

$$\frac{\partial}{\partial t} (\nabla^2 \vec{B} - \vec{B}) = \vec{\nabla} \times [\vec{v} \times (\nabla^2 \vec{B} - \vec{B})] \quad (5.1)$$

$$\vec{v} = -\vec{\nabla} \times \vec{B} \quad (5.2)$$

First of the above equations is the evolution equation of generalized vorticity ($\nabla \times \{\vec{v} - \vec{A}\} = \nabla^2 \vec{B} - \vec{B}$) obtained by taking curl of electron momentum equation and using Faraday’s law. Second equation is Ampere’s law in which displacement current has been neglected under the assumption $\omega \ll \omega_{pe}/\omega_{ce}$. Normalization in the above equations is same as in the last chapter i.e. length by electron skin depth $d_e = c/\omega_{pe}$, magnetic field by some arbitrary $B_{00}$ and time by the inverse of electron cyclotron frequency $\omega_{ce}^{-1}$ corresponding to the magnetic field $B_{00}$.

Now using the vector identity for $\nabla \times (\vec{A} \times \vec{B})$ [72], the evolution equation (5.1) of the generalized vorticity can be written as,

$$\frac{\partial}{\partial t} (\nabla^2 \vec{B} - \vec{B}) + (\vec{v} \cdot \nabla)(\nabla^2 \vec{B} - \vec{B}) = [(\nabla^2 \vec{B} - \vec{B}).\nabla]\vec{v} \quad (5.3)$$

For each component of the equation (5.3), the second term on the LHS and the only term on the RHS, by virtue of the relations $\nabla.\vec{v} = 0$ and $\nabla.\vec{B} = 0$, can be written as $\nabla.\{\vec{v}(\nabla^2 B_j - B_j)\}$ and $\nabla.\{[(\nabla^2 \vec{B} - \vec{B})v_j]\}$ respectively, where $j$ stands for $x$, $y$ or $z$. Expanding these divergence operators in their cartesian geometry form in each
component of the equation (5.3), we get following set of equations.

\[
\begin{align*}
\frac{\partial}{\partial t} (\nabla^2 B_x - B_x) + \frac{\partial}{\partial y} \{v_y (\nabla^2 B_x - B_x)\} + \frac{\partial}{\partial z} \{v_z (\nabla^2 B_x - B_x)\} \\
= \frac{\partial}{\partial y} \{v_y (\nabla^2 B_y - B_y)\} + \frac{\partial}{\partial z} \{v_z (\nabla^2 B_y - B_y)\} \\
(5.4)
\end{align*}
\]

\[
\begin{align*}
\frac{\partial}{\partial t} (\nabla^2 B_y - B_y) + \frac{\partial}{\partial x} \{v_x (\nabla^2 B_y - B_y)\} + \frac{\partial}{\partial z} \{v_z (\nabla^2 B_y - B_y)\} \\
= \frac{\partial}{\partial x} \{v_x (\nabla^2 B_y - B_y)\} + \frac{\partial}{\partial z} \{v_z (\nabla^2 B_y - B_y)\} \\
(5.5)
\end{align*}
\]

\[
\begin{align*}
\frac{\partial}{\partial t} (\nabla^2 B_z - B_z) + \frac{\partial}{\partial y} \{v_y (\nabla^2 B_z - B_z)\} + \frac{\partial}{\partial x} \{v_x (\nabla^2 B_z - B_z)\} \\
= \frac{\partial}{\partial x} \{v_x (\nabla^2 B_z - B_z)\} + \frac{\partial}{\partial y} \{v_y (\nabla^2 B_z - B_z)\} \\
(5.6)
\end{align*}
\]

In obtaining equations (5.4), (5.5) and (5.6) the term,

\[
\frac{\partial}{\partial x_j} [v_j (\nabla^2 B_j - B_j)]
\]

appearing on the LHS and RHS of the \( j \)th component of the equation (5.3) has been canceled. For this reason, we see only two partial space-derivatives, on LHS and RHS of the equations (5.4-5.6). Each of the equations (5.4), (5.5) and (5.6) is in the form of two dimensional continuity equation with the source terms but this pair of two dimensions is different in each of these equations \((y - z \text{ or } x - y \text{ or } x - z)\), thereby making the over all problem three dimensional due to the coupling between these equations.

Separating equation (5.2) into components,

\[
\begin{align*}
v_x &= \frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y} \\
v_y &= \frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z} \\
v_z &= \frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} \\
(5.7) \quad (5.8) \quad (5.9)
\end{align*}
\]

The three dimensional system of coupled equations (5.4-5.9) in the six variables \((B_x, B_y, B_z, v_x, v_y, v_z)\) is the one which is solved numerically to study the nonlinear evolution of the kink instability in EMHD. This system of equation is different from that for the two dimensional sausage instability [38] not only in the dimensionality and number of variables but also in the form of the continuity equations (5.4-5.6) which have source terms in this case.
5.3 Initial and Boundary Conditions

To study the time evolution of the equations (5.4-5.9), initial and boundary conditions of the variables to be evolved are required. In this section we specify the initial and boundary conditions used in our simulations.

5.3.1 Initial Conditions

Our interest in this chapter is in the study of nonlinear evolution of the three dimensional small amplitude perturbations growing on a sheared flow configuration due to the presence of the kink instability. The stability of sheared flow configurations against three dimensional perturbations was examined in the last chapter for two choices of equilibrium flow profiles, namely, step-function profile and tanh-profile. Here, for the purpose of numerical simulations, we choose tanh-profile for the equilibrium flow, as step-function profile is not suitable for numerical studies. Therefore,

$$\bar{u}_0(x) = V_0 \tanh(x/e) \hat{z}, \quad (5.10)$$

where $V_0$ is the magnitude of the velocity far away from the shear region which is located around $x = 0$ and $2e$ is the width of the shear region. This profile has been illustrated in Fig.(2.6) in chapter (2). The profile represents the two oppositely directed streams of electron fluid reversing the direction in a distance of $2e$.

The corresponding magnetic field,

$$\bar{B}_0(x) = -V_0 e \log(\cosh(x/e)) \hat{y}, \quad (5.11)$$

is obtained by integrating the relation $v_0(x) = -dB_0(x)/dx$, which is the Ampere’s law in equilibrium. It can be seen from the expression of the magnetic field and velocity that $v_0 \rightarrow V_0$ and $B_0 \rightarrow -\infty$ as $x \rightarrow \pm \infty$. The justification for choosing these profiles in spite of their unphysical behaviour as $x \rightarrow \pm \infty$ can be given by saying that we are interested in the evolution of the perturbations only in the region which is in the vicinity of the shear layer around $x = 0$.

Now we add small initial perturbations to the equilibrium specified above. This
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gives the initial conditions for our simulations.

\[ \bar{v}(x, y, z, t = 0) = v_0(x) \hat{x} + v_{z1}(x, y, z, t = 0) \hat{z}, \]
\[ \bar{B}(x, y, z, t = 0) = B_0(x) \hat{y} + B_{z1}(x, y, z, t = 0) \hat{z}, \]

Here subscripts '0' and '1' represent the equilibrium and perturbed quantities respectively. We have chosen following form of \(B_{z1}(x, y, z, t = 0)\), \(B_{y1}(x, y, z, t = 0)\).

\[ B_{z1}(x, y, z, t = 0) = B_{y1}(x, y, z, t = 0) = B_1 e^{-x^2/\sigma^2} \sin(k_y y) \sin(k_z z), \]

Form of the \(B_{z1}\) can be obtained by using the relation \(\nabla \cdot \bar{B} = 0\).

\[ B_{z1}(x, y, z, t = 0) = - \int \left( \frac{\partial B_{z1}}{\partial x} + \frac{\partial B_{y1}}{\partial y} \right) dz \]
\[ = - \frac{B_1 e^{-x^2/\sigma^2} \cos(k_z z)}{k_z} \left[ 2 \frac{x \sin(k_y y)}{\sigma^2} - k_y \cos(k_y y) \right] \]

Having known \(B_{z1}\), \(B_{y1}\) and \(B_{z1}\), the forms of \(v_{z1}\), \(v_{y1}\) and \(v_{z1}\) can be obtained using equations (5.7-5.9). These forms of initial perturbations are consistent with the boundary conditions mentioned below.

5.3.2 Boundary Conditions

Boundary conditions in \(x\)-direction are determined by the fact that we are interested in the region in the vicinity of shear layer. Therefore the boundary conditions in \(x\) are such that the perturbations vanish at \(x\)-boundaries, which are away from the shear region. In \(y\) and \(z\) directions we choose periodic boundary conditions. The form of the initial perturbations \(B_{z1}\) and \(B_{y1}\) is \(B_1 \exp(-x^2/\sigma^2) \sin(\pi y/l_y) \sin(\pi z/l_z)\). Such a perturbation vanishes at \(x\)-boundaries and is periodic in \(y\) and \(z\) directions with periods equal to \(2l_y\) and \(2l_z\) respectively, where \(2l_y\) and \(2l_z\) are simulation box size in \(y\) and \(z\) directions.

5.4 Numerical Method

Equations(5.4-5.6) are continuity equations in a three dimensional plane. These equations are solved by employing the continuity equation solver LCPFCT(Laboratory
for Computational Physics, Flux Corrected Transport) [56] with two stage Runge-Kutta time integration [61]. LCPFCT is a package of Fortran subroutines which implements the Flux Corrected Transport (FCT) algorithm [62-65] to solve a generalized continuity equation. LCPFCT solves one dimensional continuity equation but our equations are three dimensional. Therefore, to solve three dimensional continuity equations using one dimensional continuity equation solver, we adopt time-step splitting method [61] which combines several one dimensional calculations to create a multidimensional calculation. The time step for the integration is chosen adaptively using Courant condition [61]. The output of the LCPFCT gives the value of $\nabla^2 B_j - \Delta_j$, where $j$ stands for $x$, $y$ or $z$. We use a three dimensional FISHPACK Helmholtz solver (HW3CRT) [73] to obtain the values of various components of magnetic field from the output of the LCPFCT. Then components of the velocity can be calculated using equations (5.7-5.9). In this way we calculate all the components of the velocity and magnetic field.

5.5 Nonlinear results

In this section we present the results of the nonlinear fluid simulation of the kink instability in EMHD. The simulation box size for the results presented in this section is $2l_x \times 2l_y \times 2l_z = 5 \times 5 \times 5$ with equal grid spacing $dx = dy = dz = .0625$ in all of the three $x$, $y$ and $z$ directions. Also value of the shear width $\epsilon$ has been chosen to be equal to .3 with $V_0 = 5$.

Fig(5.1) shows the evolution of the average perturbed energy $E_1 = (1/8l_x l_y l_z) \int \int \int (B_1^2 + V_1^2) dx dy dz$, which is the sum of the average magnetic and kinetic energies, in the log-scale. It can be seen from the Fig.(5.1) that in the beginning (during the linear phase) the energy in perturbations grows exponentially (exponential growth represented by the linear part of the solid curve in Fig.(5.1)) with time due to the presence of unstable modes in the system. As the amplitude of the perturbations increases with time, the nonlinear interaction among various modes dominates and leads to the saturation of the instability. The growth rate of the instability, which is half of the slope of the linear part of the solid curve in Fig.(5.1), is dominated by that of the fastest growing mode present in the system. For the choice of parameters $\epsilon = .3$
and $V_0 = 5$ the value of the maximum growth rate $\gamma_{max}$, as obtained from linear theory [39] is 2.68. The slope of the broken line, plotted by the side of the energy evolution curve in Fig.(5.1), is equal to $2\gamma_{max} = 2 \times 2.68$ which matches well with slope of the solid curve. Thus the growth rate of the instability as observed in the simulation matches with that predicted by the linear theory. This validates our simulations. In Fig.(5.2) we show the development of instability leading to a turbulent state in nonlinear regime, by plotting coloured contours in $x - y$ plane at $z = 0$, of the $z$-component of the velocity $v_z$. At $t = 0$, blue color represents the electron velocity in negative $z$ direction while red color represents the electron velocity in positive $z$ direction. Transition from negative velocity to positive velocity takes place in a small shear width represented by the small region around $x = 0$ in which colour changes from blue to red. As can be seen from the Fig.(5.2) that the instability first develops in the shear layer around $x = 0$ leading to generation of short scale structures and hence making shear layer turbulent. Then this turbulence spreads

**Figure 5.1**: Plot of the net perturbed energy $E_i = v_i^2 + B_i^2$ in log scale with time. The slope of the broken line shown in the figure corresponds to the maximum linear growth rate for this system.
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Figure 5.2: Contours of the z component of the electron velocity in the plane $z = 0$ at various times. This figure shows development and spreading of the turbulence.

Figure 5.3: Colored contours of the z component of the velocity showing existence of the short scale structures in different two dimensional planes.
in the stable regions and finally the whole \( x - y \) plane becomes turbulent. Fig(5.3) shows the existence of short scale structures in various 2-D planes. A few thick lines in the figure are stream lines of electron flow and show the three dimensional character of the instability.

Power spectrum of the various variables also show the transfer of the power to short scale structures. Fig(5.4) shows, at different times, the power spectrum densities(PSD) \( P_{k_x}, P_{k_y} \) and \( P_{k_z} \) on \( y \)-axis in log-scale for the \( y \) component \( B_{y1} \) of
the perturbed magnetic field with $k_x$, $k_y$ and $k_z$ respectively, where,

$$P_{k_x} = \frac{\int \int P(k_x, k_y, k_z, t)dk_ydk_z}{\int \int dk_xdk_ydk_z}$$

$$P_{k_y} = \frac{\int \int P(k_x, k_y, k_z, t)dk_zdk_x}{\int \int dk_xdk_ydk_z}$$

$$P_{k_z} = \frac{\int \int P(k_x, k_y, k_z, t)dk_xdk_y}{\int \int dk_xdk_ydk_z}$$

with,

$$P(k_x, k_y, k_z, t) = \frac{B_{yk}(k_x, k_y, k_z, t)B^*_{yk}(k_x, k_y, k_z, t)}{N_xN_yN_z},$$

where $N_x$, $N_y$ and $N_z$ are number of grid points in $x$, $y$ and $z$ direction respectively.

**Figure 5.5**: Power spectrum densities (a)$P_{k_x}$ (b)$P_{k_y}$ and (c)$P_{k_z}$ averaged over time in the stationary state, with $k_x$, $k_y$ and $k_z$ respectively for the variable $B_{y1}$. See text for the definition of $P_{k_x}$, $P_{k_y}$ and $P_{k_z}$.

At $t = 0$, spectra correspond to the power given to the perturbations initially to hasten the instability. The form of the initial perturbed $B_{y1}$ is Gaussian in $x$, that is why we see a broad profile of the $P_{k_x}$ at $t = 0$ while $P_{k_y}$ and $P_{k_z}$ peaks at
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$k_y = \pi/l_y = 1.256$ and $k_z = \pi/l_z = 1.256$ respectively because initial power in $y$ and $z$ direction was given only in these wave numbers. As time passes, spectral power transfer takes place towards higher $k_x$, $k_y$ and $k_z$ as seen from the broadening of the spectra with time in Fig.(5.4) and finally the spectra become stationary. In the stationary state we have averaged this spectra in time and plotted it in the Fig.(5.5). The figure shows the power law dependence of the spectrum.

We now try to understand the cascading of the power in various directions. For this purpose we define rms wave numbers in the following way.

$$\langle k_x \rangle = \sqrt{\frac{\sum k_x \sum k_y \sum k_z k_x^2 |Q(k_x, k_y, k_z, t)|^2}{\sum k_x \sum k_y \sum k_z |Q(k_x, k_y, k_z, t)|^2}}$$

$$\langle k_y \rangle = \sqrt{\frac{\sum k_x \sum k_y \sum k_z k_y^2 |Q(k_x, k_y, k_z, t)|^2}{\sum k_x \sum k_y \sum k_z |Q(k_x, k_y, k_z, t)|^2}}$$

$$\langle k_z \rangle = \sqrt{\frac{\sum k_x \sum k_y \sum k_z k_z^2 |Q(k_x, k_y, k_z, t)|^2}{\sum k_x \sum k_y \sum k_z |Q(k_x, k_y, k_z, t)|^2}}$$

Here $Q$ stands for any of the following variables $v_x, v_y, v_z, B_x, B_y$ and $B_z$. In Fig.(5.6), we show the change in these rms wave numbers with time for the variable $B_{y1}$. The figure shows that $\langle k_y \rangle$ and $\langle k_z \rangle$ remain at their initial values for some time in the beginning and then increase while $\langle k_x \rangle$ first decreases to a value similar to those of $\langle k_y \rangle$ and $\langle k_z \rangle$, and then increases. It means that initially the power in the $y$ and $z$ directions remains in the same longest (permitted by periodicity in $y$ and $z$ directions) scale in which it was given initially and then transfers to shorter scales while power in $x$ direction, which was initially in shorter scales also, first transfers to the longer scales and then to shorter scales. It can also be seen from this figure that at later times $\langle k_x \rangle$ and $\langle k_y \rangle$ attain similar values but $\langle k_z \rangle$ attain a smaller value implying that the power transfers more efficiently in $x$ and $y$ direction as compared to $z$ direction. To quantify this anisotropy in power transfer, we define a number R...
Figure 5.6: Plot of rms wave numbers \( \langle k_x \rangle \), \( \langle k_y \rangle \) and \( \langle k_z \rangle \) for the variable \( B_y \) with time as defined in the text as follows.

\[
R_{xy}(t) = \frac{\sum_{k_x} \sum_{k_y} \sum_{k_z} k_x^2 |Q(k_x, k_y, k_z, t)|^2}{\sum_{k_x} \sum_{k_y} \sum_{k_z} k_y^2 |Q(k_x, k_y, k_z, t)|^2}
\]
\[
R_{xz}(t) = \frac{\sum_{k_x} \sum_{k_y} \sum_{k_z} k_z^2 |Q(k_x, k_y, k_z, t)|^2}{\sum_{k_x} \sum_{k_y} \sum_{k_z} k_x^2 |Q(k_x, k_y, k_z, t)|^2}
\]
\[
R_{yz}(t) = \frac{\sum_{k_x} \sum_{k_y} \sum_{k_z} k_y^2 |Q(k_x, k_y, k_z, t)|^2}{\sum_{k_x} \sum_{k_y} \sum_{k_z} k_z^2 |Q(k_x, k_y, k_z, t)|^2}
\]

If the spectrum is isotropic in any two directions, corresponding ratio \( R \) will be close to unity. Any deviation from unity is an indication of anisotropy in the spectrum. Fig(5.7) shows the anisotropic factor \( R \) with time for the variable \( B_{z1} \). As indicated by the Fig.(5.6), the spectra is isotropic in \( x - y \) but is anisotropic in \( y - z \) and \( x - z \). The anisotropy can be explained by arguing that flow in the \( z \) direction permits only slow cascade of power towards shorter scales.
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Anisotropy factor $R$ for $B_y$

![Plot of anisotropy factors $R_{xy}$, $R_{yz}$, and $R_{xz}$ for the variable $B_y$ with time.](image)

**Figure 5.7:** Plot of anisotropy factors $R_{xy}$, $R_{yz}$, and $R_{xz}$ for the variable $B_y$ with time.

It is interesting to contrast 3d simulations results presented here with our earlier 2d ($\partial/\partial y = 0$) simulations results [38] presented in chapter (3). In 2d simulations the power transfers to longer scales leading to a nonlinear coherent state. In Fig. (5.8) we show $P_{k_x}$ and $P_{k_y}$ averaged over time in the stationary nonlinear state of both of the two and three dimensional simulations, with $k_x$ and $k_z$. The initial power levels in 2d simulations were taken to be same as those in 3d simulations. It is clear from the figure that the longer wavelengths have more power in 2d case as compared to the 3d case while opposite is true for shorter wavelengths. This feature is visible also from Fig. (5.9) in which we have plotted rms wave numbers $\langle k_x \rangle$ and $\langle k_z \rangle$ for 2d case along with those for 3d case. The value of the rms wave numbers is small for 2d case as compared to the 3d case implying dominant transfer of power towards longer scales in 2d. This figure also shows that in 2d $\langle k_x \rangle$ reduces from its initial value to a value smaller than one while $\langle k_z \rangle$ does not change from its initial value.
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Figure 5.8: Plot of $P_{k_x}$ and $P_{k_z}$ averaged over time in stationary state in log scale for the two and three dimensional simulations.

Figure 5.9: Plot of rms wave numbers $\langle k_x \rangle$ and $\langle k_z \rangle$ for the variable $B_{y1}$ with time for two and three dimensional simulations.
Figure 5.10: Mean flow profile of electron velocity at various times for (a) three dimensional simulations (b) two dimensional simulations.

In Fig.(5.2), we saw that development of the instability in the shear layer leads to turbulence which spreads in the whole $x-y$ plane. It can also be seen from this figure that the magnitude of the $z$-component of velocity $v_z$ also reduces simultaneously in the unstable region. This indicates that the profile of the mean flow, which was initially $V_0 \tanh(x/\epsilon)$, is getting flattened in the shear region with time. To illustrate this point we plot profiles of mean flow $<v_z>$ as a function of $x$ at different times, in the subplot(a) of Fig(5.10), where,

$$<v_z> = \frac{1}{4|l_y|l_z} \int_{-l_z}^{l_z} \int_{-l_y}^{l_y} v_z(x, y, z, t) dy dz$$

To concentrate on the shear region, the mean flow profiles in this figure have been plotted for the range of $x$ from $-1$ to $+1$. At $t = 0$, the mean flow profile corresponds to $V_0 \tanh(x/\epsilon)$, shear in which leads to instability. We observe that as time passes,
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Figure 5.11: Growth rate of the two and three dimensional instability with time for the mean profiles that the system has at different times as a result of the nonlinear modifications in two and three dimensional simulations.

The profile flattens and finally saturates at a less sheared state. This happens because the system is unstable due to the presence of shear in the equilibrium flow profile and tries to attain an equilibrium state which is less sheared. So in an attempt to obtain such an equilibrium state, growing modes back react on the profile to flatten it. However, the final saturated state is still sheared and could be amenable to instability. To check if the modified mean profiles are unstable, we have solved linearized eigen value equations [39] for this system numerically for the modified profiles to obtain the growth rates of the instability growing on these profiles. In Fig.(5.11) we plot the growth rate (dotted line) maximized over all the \( k_x \), \( k_y \) and \( k_z \) present in the system, with time. At \( t = 0 \), the growth rate corresponds to the initial equilibrium profile and has been shown to match with the one obtained from simulation in Fig.(5.1). We see that the growth rate reduces with time to a value much smaller than its initial value but remains finite and after \( t \approx 6 \) becomes independent of time. It is clear from this figure that the saturated profile is still...
unstable. Therefore the saturation does not occur for the reason that system is no longer unstable but because the direct nonlinear cascade provides coupling of the unstable modes to damped short scale structures. In numerical simulations, the damping of short scale modes is due to numerical viscosity associated with finite grid size.

We again contrast 3d simulation results presented here with the 2d simulation [38] results. In subplot(b) of Fig.(5.10), we show flattening of equilibrium velocity profile with time in 2d simulations. By comparing subplots (a) and (b) we see that flattening of the velocity profile is less in 2d simulations as compared to 3d simulations. Also flattening of the profile saturates earlier in 2d case, leaving a profile which is more sheared as compared to the 3d case. Again, to check the stability of the modified mean profiles in 2d we plot in Fig.(5.11) the maximum growth rate (solid line) of the instability present in the system with time. It can be seen from the figure that unlike 3d case the growth rate reduces to zero quite early, implying that the system is no longer unstable to two dimensional perturbations. This happens because, in 2d, there exists only one unstable mode which is driven by the curvature in the equilibrium velocity profile and the instability exists only for the wave numbers for which \( k_z \epsilon < 1 \) [37,40]. Therefore in that case, saturation of the instability occurs rather easily, either by getting rid of the curvature in the velocity profile or by violating the condition \( k_z \epsilon < 1 \) even for smallest possible \( k_z \) permitted by the periodicity in the z-direction, whichever occurs first [38]. On the other hand in 3d case, in addition to the 2d unstable mode(\( k_y = 0 \)), there exist a number of other unstable modes also which are driven by the gradient in the equilibrium velocity profile and are unstable even for \( k_z \epsilon > 1 \) [39]. Therefore, for the saturation of instability to occur in 3d simulations, the mean flow profile should not only be free from curvature but also be gradient free. Therefore saturation of the instability in 3d does not take place as early as in 2d case. For these reasons flattening of the mean profile is more effective in 3 dimensions.

The flattening of the profile can be analyzed quantitatively by plotting the average current,

\[ J(t) = \int_0^1 (v_z(x,t))dx, \]  

(5.12)

flowing in one direction, with time. Fig(5.12) shows such a plot in which we have
Figure 5.12: Plot of forward current $J$ with time. Solid line is for two dimensional simulation while broken line is the result of three dimensional simulation.

Plotted $J$ both for 2d and 3d simulations. Initial period during which current does not change appreciably can be recognized as linear phase of the instability. After the linear phase, current starts dropping. It is clear from the Fig.(5.12) that, although the rate of the current reduction in the two cases is almost same initially, the total current reduction is more in three dimensional simulation because the saturation of the instability occurs later in three dimensional case than in 2 dimensional case and current reduction continues at almost the same rate till saturation. It can also be seen from the Fig.(5.12) that current starts dropping earlier in 3d case. This happens because of the presence of the large number of unstable modes in the 3d simulations.
5.6 Summary

In this chapter we presented a detailed 3-dimensional EMHD fluid simulation of kink instability. The simulations show the development of the instability in the current shear layer. In the linear regime of the instability, the characteristic growth rates of the instability as observed in the simulation match with those predicted by the linear theory [39]. In the nonlinear regime the interaction among a large number of modes gives rise to the generation of electromagnetic turbulence. The power spectra of this turbulence show the cascading of power towards shorter scale and exhibit a power-law behavior. The various diagnostics used to characterize the turbulence show that the cascade of power towards shorter scales in the flow direction is less as compared to the other two directions, implying the anisotropy in the turbulence. We also investigate the effect of the turbulence on the mean flow profile. The investigations reveal that the presence of the turbulence in 3 dimensions leads to a faster and more effective reduction in the directed flow of electrons, as compared to the 2 dimensional case in which there was no turbulence.

The reduction in the magnitude of $J$ via the generation of electromagnetic turbulence is directly related to the loss of directed kinetic energy of electrons and has implications for the stopping of energetic electrons in fast ignition. We shall discuss the relevance of our work to fast ignition in the next chapter.