Chapter 3

Nonlinear Sausage instability

The objective, in this chapter, is to investigate the nonlinear development of the sausage instability on fast EMHD time scales by numerically solving two dimensional nonlinear EMHD equations with appropriate initial and boundary conditions.

3.1 Introduction

In the last chapter we studied azimuthally symmetric two dimensional shear driven modes using linearized EMHD [1] equations. In the linearization process of the EMHD equations, the terms containing the dependent variables with powers higher than one were regarded small and were neglected, and hence dependent variables occur in the equations to no higher than the first power. The process of linearization simplified the analysis by enabling us to treat each Fourier component separately with the understanding that any non-sinusoidal perturbation can be handled by adding up the appropriate distribution of Fourier components. This works as long as the perturbation amplitude is small enough so that the linear equations are valid. But in case of an instability, the perturbation amplitudes remain small for very short time and grow, however small initially, to large values with time. Once perturbation amplitudes attain sufficiently large values, linear equations can not be used to describe them. Apart from this, there are certain intrinsically nonlinear features like evolution of perturbed energy, formation of nonlinear structures and/or turbulence, change in average quantities etc. which can not be covered in the linear theory even
in small amplitude limit because linear theory does not permit interaction among various modes. Therefore, even though the linear analysis is useful in understanding the basic physics of the excitation of the unstable mode, one needs to solve nonlinear equations also for the complete description of the mode.

Earlier nonlinear studies of Bulanov et al. [54,55] on sausage instability in EMHD are on a primitive level and various features of the nonlinear stage of the instability remain unclear. The physics of the formation of the structures that develop in the nonlinear stage of the instability is not clear. The saturation mechanism of the instability has been explained very qualitatively. The physics of nonlinear modifications of the mean profiles have not been addressed at all. Apart from these, the form of initial shear flow profile is also not clear in their simulations. In this chapter, we address all these questions and provide clear answers to them.

A complete study of the sausage mode requires investigation of the two dimensional nonlinear EMHD equations along with the proper initial and boundary conditions. The nonlinear equations are more difficult to be treated analytically as compared to linear equations. For example, in the linear case, analysis could be simplified by taking Fourier transform of the equations but no such simplification is possible in nonlinear case. Another way to attempt a solution of nonlinear equations is to solve them using numerical techniques, which is the objective of this chapter.

To attempt a numerical solution of nonlinear EMHD equations, we cast the EMHD equation for the evolution of the generalized vorticity ($\nabla^2 B - B$, where $B$ is the magnetic field) in a form of two dimensional continuity equation which can then be solved using a generalized continuity equation solver package, LCPFCT (Laboratory for Computational Physics, Flux-Corrected Transport) [56] coupled with the time-step splitting method [61]. LCPFCT is a name given to a collection of Fortran subroutines which implements "Flux-Corrected Transport" algorithm [62, 63] to solve one dimensional generalized continuity equation in various geometries. The output of the LCPFCT gives the source term $f(x,z)$ for the Helmholtz equation, $\nabla^2 B - B = f(x,z)$, in magnetic field. This Helmholtz equation is solved for the magnetic field $B(x,z)$, using a two dimensional Helmholtz solver [66] from FISHPACK library of Fortran subroutines. The output of the Helmholtz solver is then coupled to second EMHD equation, namely Ampere’s law, to obtain the electron
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fluid velocity.

This chapter is organized as follows. In the next section we cast the two dimensional nonlinear EMHD equations in a form in which they are integrated numerically. Initial and boundary conditions for the simulations are presented in section (3.3). In section (3.4) we briefly mention the numerical methods used and outline the algorithm of the numerical scheme. In section (3.5) we discuss various criteria for choosing the simulation parameters. In section (3.6), results of the nonlinear evolution of sausage instability are presented. We attempt to understand the nonlinear evolution of the instability using quasi-linear theory in section (3.7) and finally the whole chapter is summarized in section (3.8).

3.2 Nonlinear EMHD Equations

Dimensionless EMHD equations (A.11) and (A.12) in collisionless limit can be written as,

$$\frac{\partial}{\partial t} \left( \nabla^2 \vec{B} - \vec{B} \right) = \vec{\nabla} \times [\vec{v} \times (\nabla^2 \vec{B} - \vec{B})] \tag{3.1}$$

$$\vec{v} = -\vec{\nabla} \times \vec{B} \tag{3.2}$$

First of the above equations is the evolution equation of generalized vorticity ($\nabla \times \{\vec{v} - \vec{A}\} = \nabla^2 \vec{B} - \vec{B}$) obtained by taking curl of electron momentum equation and using Faraday's law. Second equation is Ampere's law in which displacement current has been neglected under the assumption $\omega \ll \omega_{pe}^2/\omega_{ce}$. Normalization in the above equations is same as in chapter (2) i.e. length by electron skin depth $d_e = c/\omega_{pe}$, magnetic field by some arbitrary $B_{00}$ and time by the inverse of electron cyclotron frequency $\omega_{ce}^{-1}$ corresponding to the magnetic field $B_{00}$.

For our simulations, we shall use two dimensional slab geometry in which direction along y-coordinate is taken to be the direction of symmetry i.e. there are variations only along x and z directions. This means,

$$\frac{\partial}{\partial y} = 0 \Rightarrow \vec{\nabla} \equiv \hat{x} \frac{\partial}{\partial x} + \hat{z} \frac{\partial}{\partial z}. \tag{3.3}$$

This geometry can be considered as the slab representation of the cylindrical geometry (see Fig. (2.1) in chapter(2)) of the electron current channels observed in 3D
PIC simulation of Sentoku et al. [30,31], if the shear scale length of the equilibrium flow and the damping scale length of the modes are smaller than the channel radius. In the slab representation of the cylindrical geometry, the radial direction of current shear is taken as \( \hat{x} \), the azimuthal direction as \( \hat{y} \) and the current flow direction as \( \hat{z} \). Therefore, in the terminology of cylindrical geometry, sausage like perturbations are azimuthally symmetric perturbations.

Now since, as mentioned in chapter (2), only \( y \)-component of the magnetic field \( B_y \) needs to be finite for the study of sausage mode, equation (3.1) using the vector identity,

\[
\vec{\nabla} \times (\vec{A} \times \vec{B}) = \vec{A}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{A}) + (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B}
\]

can be written as,

\[
\frac{\partial}{\partial t}(\nabla^2 B_y - B_y) + (\vec{v} \cdot \vec{\nabla})(\nabla^2 B_y - B_y) = 0
\]

In writing the above equation we have used the fact that \( \vec{\nabla} \cdot \vec{B} = 0 \) and \( \vec{\nabla} \cdot \vec{v} = 0 \). Again using \( \vec{\nabla} \cdot \vec{v} = 0 \), the second term on the left hand side of the above equation can be written as \( \vec{\nabla}.[(\nabla^2 B_y - B_y)\vec{v}] \). Therefore evolution equation of generalized vorticity in EMHD can be written in the form of a two dimensional continuity equation in \( x - z \) plane as follows.

\[
\frac{\partial}{\partial t}(\nabla^2 B_y - B_y) + \frac{\partial}{\partial x}\{v_x(\nabla^2 B_y - B_y)\} + \frac{\partial}{\partial z}\{v_z(\nabla^2 B_y - B_y)\} = 0 \tag{3.4}
\]

Equations (3.2) and (3.4) are the equations which we shall solve numerically to study the nonlinear evolution of the sausage mode by employing continuity equation solver LCPFCT [56] which gives values of \( \nabla^2 B_y - B_y \) at each space point. To find the values of \( B_y \) at each space point we employ Helmholtz solver [66]. Then we calculate electron velocity using Ampere’s law (3.2).

### 3.3 Initial and Boundary Conditions

We are interested in studying the nonlinear evolution of the two dimensional shear driven sausage instability on EMHD time scales. For this purpose we choose a sheared electron flow configuration in equilibrium, add some initial small perturbations to it to hasten the instability and then allow the equations to evolve under proper initial and boundary conditions.
3.3.1 Initial Conditions

Initial conditions for our simulation are obtained by adding initial perturbations of small amplitudes to the equilibrium variables. In slab geometry, we take equilibrium electron velocity \( \bar{v}_0 \) to be sheared along \( x \) and directed along \( z \). The corresponding equilibrium magnetic field \( \bar{B}_0 \) will be along \( y \) and can be obtained by integrating equation (3.2) for a given equilibrium electron velocity. Therefore equilibrium variables depend only on \( x \) coordinate.

\[
\bar{v}_0 = v_0(x) \hat{z} \\
\bar{B}_0 = B_0(x) \hat{y}
\]

The scalar variables in the equations (3.4) and (3.2) are \( B_y, v_x \) and \( v_z \) (since only \( y \)-component of the magnetic field needs to be finite for the study of sausage mode and correspondingly only two components of electron velocity \( v_x \) and \( v_z \) are finite). In equilibrium only \( B_y \) and \( v_z \) are finite and vary along \( x \) only. But in perturbations all the three variables will be finite and vary along both the \( x \) and \( z \) directions. Therefore, the initial conditions for the simulation obtained by adding small initial perturbations to the equilibrium variables are following.

\[
\bar{v}(x, z, t = 0) = v_0(x) \hat{z} + v_{z1}(x, z, t = 0) \hat{x} + v_{x1}(x, z, t = 0) \hat{z} \\
\bar{B}(x, z, t = 0) = B_0(x) \hat{y} + B_{y1}(x, z, t = 0) \hat{y}
\]

where suffix '1' denotes the perturbed variables.

We choose equilibrium velocity profile of the electron fluid to be \( \tanh \) in \( x \).

\[
\bar{v}_0 = V_0 \tanh(x/\epsilon) \hat{z}.
\]

This profile (illustrated in Fig. (2.6) in chapter (2)) represents a sheared flow configuration of the electron fluid in which electron fluid velocity changes smoothly from the value \(-V_0\) (in the region \( x < 0 \)) to the value \(+V_0\) (in the region \( x > 0 \)) within a small region, typically of the width of \( 2\epsilon \), around \( x = 0 \). Therefore, \( 2\epsilon \) is the width of the shear layer around \( x = 0 \) and \( V_0 \) is the magnitude of the velocity far away from the shear layer. Corresponding magnetic field obtained from the Ampere's law is,

\[
\bar{B}_0 = -V_0 \epsilon \log[\cosh(x/\epsilon)] \hat{y}
\]
The expressions of the velocity $v_0(x)$ and the magnetic field $B_0(x)$ show that, as $x \to \pm \infty$, velocity becomes constant and magnetic field blows up. The choice of these profiles in spite of their unphysical behaviour can be justified because we are interested only in those perturbations which are confined to the shear layer and are zero for large $|x|$. Such a study gives insight into the basic destabilization physics of a shear layer.

The form of the initial perturbations is following.

\[
B_{y1}(x, z, t = 0) = B_1 \exp(-x^2/\sigma^2) \sin(k_z z)
\]

\[
v_{x1}(x, z, t = 0) = B_1 k_z \exp(-x^2/\sigma^2) \cos(k_z z)
\]

\[
v_{z1}(x, z, t = 0) = \frac{2x}{\sigma^2} B_1 \exp(-x^2/\sigma^2) \sin(k_z z)
\]

Here $B_1$ is the amplitude of the initial perturbations, $k_z$ is the wave number in the $z$ direction and $\sigma$ is typical scale length of the variation of initial perturbations in the $x$ direction. The forms of the variables $B_{y1}(x, z, t = 0)$, $v_{x1}(x, z, t = 0)$ and $v_{z1}(x, z, t = 0)$ are consistent with the Ampere's law (3.2). The initial value of $\nabla^2 B_y - B_y$ required for the solution of the equation (3.4) will be,

\[
(\nabla^2 B_y - B_y)_{t=0} = [\nabla^2 B_0(x) - B_0(x)] + [\nabla^2 B_{y1}(x, z, t = 0) - B_{y1}(x, z, t = 0)]
\]

\[
= -\frac{V_0}{\epsilon \cosh^2(x/\epsilon)} + V_0 \epsilon \log[\cosh(x/\epsilon)]
\]

\[-B_1 \exp(-x^2/\sigma^2) \sin(k_z z)[1 + k_z^2 + \frac{2}{\sigma^2} \{1 - \frac{2x^2}{\sigma^2}\}]
\]

These initial perturbations decay away from $x = 0$ and tend to zero for large values of $|x|$. In simulations, we choose the value of $\sigma$ to be of the order of the shear width $\epsilon$ so that the initial perturbations are confined to the shear layer. Also, the value of $B_1$ is taken to be small enough to keep the initial evolution of the instability in the regime of linear theory.

### 3.3.2 Boundary Conditions

We are interested in the study of the perturbations which are confined to the shear layer and vanish far away from the layer. Therefore, boundary conditions in the $x$ direction have been so chosen that the values of the perturbed variables at the
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x-boundaries remain zero for all time and for all the values of z. The initial perturbations chosen above also satisfy these boundary conditions. Thus, at the two x-boundaries (say $x = x_{b1}$ and $x = x_{b2}$) the values of the variables remain equal to their equilibrium values there.

$$v_{x1}(x = x_{b1}, z, t) = 0 \Rightarrow v_x(x = x_{b1}, z, t) = 0$$

$$v_{z1}(x = x_{b1}, z, t) = 0 \Rightarrow v_z(x = x_{b1}, z, t) = V_0 \tanh(x_{b1}/\epsilon)$$

$$B_{y1}(x = x_{b1}, z, t) = 0 \Rightarrow B_y(x = x_{b1}, z, t) = -V_0 \epsilon \log[cosh(x_{b1}/\epsilon)]$$

To mimic the infinite extent of the electron flow along z, we choose periodic boundary conditions in z direction.

3.4 Numerical Method

Equation (3.4) is a continuity equation in a two dimensional plane. This equation is solved by employing the continuity equation solver LCPFCT (Laboratory for Computational Physics, Flux Corrected Transport) [56] with two stage Runge-Kutta time integration [61]. LCPFCT is a package of Fortran subroutines which implements the Flux Corrected Transport (FCT) algorithm [62-65] to solve a generalized continuity equation. LCPFCT solves one dimensional continuity equation but our equation is two dimensional. Therefore, to solve two dimensional continuity equation using one dimensional continuity equation solver, we adopt time-step splitting method [61] which combines several one dimensional calculations to create a multidimensional calculation. The time step for the integration is chosen adaptively using Courant condition [61]. The output of the LCPFCT gives the value of $\nabla^2 B_y - B_y$. We use a two dimensional FISHPACK Helmholtz solver (HWSCRT) [66] to obtain the values of magnetic field from the output of the LCPFCT. Then components of the velocity can be calculated using equations (3.2). In this way we calculate all the components of the velocity and magnetic field.
3.5 Simulation Parameters

We now choose the values of the various parameters of the simulation based on certain criteria. The simulation parameters to be chosen are,

1. equilibrium parameters \( i.e. \, \epsilon \) and \( V_0 \),

2. parameters related to initial perturbations \( i.e. \, B_z, \sigma \) and \( k_z \),

3. simulation box size \( 2l_x \times 2l_z \) (from \( -l_x \) to \( +l_x \) in \( x \) direction and from \( -l_z \) to \( +l_z \) in \( z \) direction), and

4. grid resolutions \( dx \) and \( dz \) (in \( x \) and \( z \) directions respectively).

The desired value of the shear width \( \epsilon \) is of the order of few tenth of the electron skin depth \( (d_e = c/\omega_{pe}) \), where \( \omega_{pe} \) is electron plasma frequency. In our normalization this corresponds to \( \epsilon \sim 0.1 - 1.0 \). The normalized value of \( V_0 \) is chosen to be 5.0. This corresponds to the value \( 5d_e\omega_{ce} \) in unnormalized units, where \( \omega_{ce} \) is electron gyration frequency.

\[
\epsilon = 0.1 - 1.0 \\
V_0 = 5.0
\]

The boundary conditions in the \( x \)-direction require the perturbations to vanish at the \( x \)-boundaries. Since the value of \( \epsilon \) will be sufficiently smaller than \( l_x \), we choose \( \sigma \), the typical scale length of the initial perturbations in the \( x \) direction, to be of the order of \( \epsilon \) so that the initial perturbations, being proportional to \( \exp(-x^2/\sigma^2) \), decays away from the central shear region around \( x = 0 \) and approaches zero at the \( x \)-boundaries at \( x = \pm l_x \).

\[
\sigma \sim \epsilon
\]

The boundary conditions in the \( z \)-direction are periodic. Therefore, possible values of \( k_z \), the wave number of the initial perturbations in the \( z \) direction, are integer multiple of the minimum wave number \( k_z^{min} = 2\pi/2l_z \) (corresponding to the maximum wavelength \( \lambda_z^{max} = 2l_z \) that the system can accommodate with the
constraint of the periodicity) present in the system. We choose to give initial power to the maximum wavelength \( \lambda = \lambda_2^{\text{max}} \) i.e. \( k_z = k_z^{\text{min}} = \pi/l_z \).

\[
k_z = \frac{\pi}{l_z}
\]

The value of the \( B_1 \) is chosen so small that the initial perturbation amplitudes remain in the linear regime. The typical magnitude of the equilibrium magnetic field can be estimated as \( V_0 \epsilon \sim 0.5 - 5.0 \) (for \( V_0 = 5 \) and \( \epsilon = 0.1 - 1.0 \)) which is of the order of unity. We choose \( B_1 = 1.0 \times 10^{-3} \), which is three order of magnitude smaller than the equilibrium value. So we have,

\[
B_1 = 1.0 \times 10^{-3}
\]

To decide on the values of the simulation box size and grid resolution we need to consider various criterion ensuring that all the important scales are resolved and included in the system. To begin with, we have in \( x \) direction the equilibrium scale length \( \epsilon \) which should be sufficiently smaller than \( l_x \) (at least 4 to 5 times smaller) to represent well the \( \text{tanh} \)-profile of equilibrium velocity.

\[
l_x \geq 5\epsilon
\]

Another restriction that comes because of \( \epsilon \) is on the grid resolution \( dx \) in the \( x \) direction. The value of \( dx \) should be so small that it can resolve the total shear width \( 2\epsilon \) of the \( \text{tanh} \)-profile of equilibrium electron velocity. We consider \( dx \) to be small enough if it is at least five times smaller than \( 2\epsilon \). Therefore.

\[
dx \leq \frac{2\epsilon}{5} \quad (3.5)
\]

The next concern is about including in the simulation as many growing modes as possible. We learnt from the linear analysis in the chapter(2) that the growth rate of the instability remains finite only for \( k_z < k_z^c = 1/\epsilon \) and peaks around \( k_z \sim k_z^m = 0.5/\epsilon \). Therefore in our simulation, we should include the modes with wave numbers \( k_z \) such that \( 0 < k_z < 1/\epsilon \). The minimum value of \( k_z \) present in the system is determined by the length \( l_z \) of the simulation box in \( z \) direction and is given by \( k_z^{\text{min}} = \pi/l_z \) while the maximum value of the wave number \( k_z \) that can be represented in the system is determined by the Nyquist wave length [61] spanning
two grid points and is given by $k_{z}^{\text{max}} = 2\pi/2dz = \pi/dz$. Therefore to include as many wave numbers as possible in the range $0 < k_{z} < 1/\epsilon$, we need to increase the value of $l_{z}$ (to reduce the value of $k_{z}^{\text{min}}$) and at the same time decrease the value of $dz$ (to increase the value of $k_{z}^{\text{max}}$). The upper limit $1/\epsilon$ of the interval $0 < k_{z} < 1/\epsilon$ is rather easy to achieve because $k_{z}^{\text{max}} = \pi/dz$ is either greater than $1/\epsilon$ or similar to it for $\epsilon = 0.1 - 1.0$ and $dz < 1$. On the other hand as we try to approach the lower limit, $k_{z}^{\text{min}} \to 0$, $l_{z} \to \infty$. This results in an enormous increase in the number of grid points for which we are computationally limited. Because of this limitation we choose to include at least, as many fastest growing modes as possible. The fastest growing modes have their wave numbers lying around $k_{z}^{m} = .5/\epsilon$. The minimum value of the wavenumber $k_{z}^{\text{min}}$ present in the system should be sufficiently smaller than $k_{z}^{m}$ while the maximum value $k_{z}^{\text{max}}$ should be sufficiently larger than $k_{z}^{m}$ to cover as many fastest growing modes as possible in the range $(k_{z}^{\text{min}}, k_{z}^{\text{max}})$. Therefore,

$$k_{z}^{\text{min}} < \frac{.5}{\epsilon} < k_{z}^{\text{max}}$$

The condition $k_{z}^{\text{min}} < .5/\epsilon$ gives following restriction on the values of $l_{z}$.

$$l_{z} > 2\pi\epsilon$$

From the condition $k_{z}^{\text{max}} > .5/\epsilon$ we get,

$$dz < 2\pi\epsilon$$

We shall demonstrate most of our results for two values of $\epsilon$ viz. $\epsilon = 0.3$ and $\epsilon = 0.5$. For $\epsilon = 0.3$, the simulation box size is chosen to be $2l_{x} \times 2l_{z} = 5.0 \times 5.0$ and for $\epsilon = 0.5$ it is $2l_{x} \times 2l_{z} = 10.0 \times 10.0$. The value of each of the $dx$ and $dz$ is chosen to be 0.0625 for both the values of $\epsilon$. Such a choice satisfies the above mentioned conditions on $l_{x}$, $l_{z}$ and $dx$, $dz$.

In the next section we present the results of the simulation. The simulation parameters for these results are the same as mentioned above unless stated otherwise.
Figure 3.1: Quiver plot of the velocity at different times for $\epsilon = 0.3$.

Figure 3.2: Contours of the magnetic field $B_y$ at different times for $\epsilon = 0.3$.
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Figure 3.3: The plot of \( \log(E_1)(E_1 \text{ being the average density of perturbed energy as defined in the text}) \) with time for (a) \( \epsilon = 0.3 \), (b) \( \epsilon = 0.5 \). The dotted reference line has been drawn for comparison and indicates the maximum growth rate of the instability as predicted by the linear theory.

3.6 Results

3.6.1 Code Validation: Comparison with Linear Results

The simulations show the development of an instability leading to the formation of convective cell patterns in the flow velocity as shown in the Fig. (3.1). The constant contours for the magnetic field can be seen from Fig. (3.2), which show an evolution from an initial straight line configuration to a sausage like structure. To understand the growth characteristics of this particular mode we plot \( \ln E_1(t) \) vs. time in Fig. (3.3) for (a) \( \epsilon = 0.3 \) and (b) \( \epsilon = 0.5 \). It can be seen from this figure that in the beginning (during the linear phase) the energy in perturbations grows exponentially (exponential growth represented by the linear part of the solid curve in Fig.(3.3)) with time due to the presence of unstable modes in the system. As the amplitude of the perturbations increases with time, the nonlinear interaction among various modes dominates and leads to the saturation of the instability. The growth rate of the instability, which is half of the slope of the linear part of the solid curve
in Fig.(3.3), is dominated by that of the fastest growing mode present in the system. For $\epsilon = 0.3$ and $V_0 = 5$, the value of the maximum growth rate $\gamma_{\text{max}}$, as obtained from linear theory is 2.72 while for $\epsilon = 0.5$ and $V_0 = 5$ it is 1.35. The slopes of the broken lines, plotted by the side of the energy evolution curves in subplots (a) and (b) of Fig.(3.3), are equal to $2\gamma_{\text{max}} = 2 \times 2.68$ and $2\gamma_{\text{max}} = 2 \times 1.35$. These slopes match well with the slopes of the corresponding solid curves. Thus the growth rate of the instability as observed in the simulation matches with that predicted by the linear theory. This validates our simulations.

After the initial growth phase the perturbed field energy is seen to approach a saturated state. This happens as a result of nonlinear modification of the shear in the $z$ independent equilibrium velocity profile. We shall discuss this topic in the next section.

### 3.6.2 Nonlinear regime

As the amplitude of unstable modes acquires appreciable level, nonlinear terms in the evolution equation start playing a role. The nonlinearity permits interaction amongst a variety of excitation scales, which ultimately leads to the formation of nonlinear structures and/or turbulence. This cannot happen in the linear domain, where each and every scale operates independently. Here we illustrate by numerical simulation the emergence of long scale structural patterns in the nonlinear domain. We also show that the equilibrium $z$ independent profile approaches a stable configuration ultimately.

In Fig.(3.4) we plot the constant magnetic field contours at various times for $\epsilon = 0.25$. Other parameters for this figure are $V_0 = 5$, $l_x = 5$ and $l_z = 10$. In this figure the multiple structures present initially correspond to the fastest growing linear mode. We observe that these multiple structures merge and form a single island structure of the size of simulation box length later in time. This approach towards the longest scale permitted by the system in the nonlinear phase is found to be an intrinsic feature of the magnetic field $B_y$ evolution governed by equations (3.2) and (3.4) and is not specific to some set of parameters alone. This happens essentially due to the inverse cascade of power towards long scales, a phenomena which is well known in the context of 2-dimensional hydrodynamics and also in certain
Figure 3.4: The plot showing the coalescence of several magnetic islands with time at $t = 0, t = 5, t = 7, t = 10, t = 25, t = 40$ in subplots (a), (b), (c), (d), (e) and (f) respectively. For this case the parameters are $\epsilon = 0.25, V_0 = 5, l_x = 5$ and $l_z = 10$.

magnetized plasma models. Basically, the nonlinear terms responsible for coupling amidst various scales in the vorticity evolution equation for 2-d hydrodynamics and the polarization drift nonlinearity of magnetized plasmas have similar form, viz. $\hat{y} \times \nabla \psi \cdot \nabla \nabla^2 \psi$ (here $\hat{y}$ is the direction of symmetry, and $\psi$ represents the velocity and electrostatic potential for 2-d hydro-fluid and plasma problem respectively), which supports two mean square invariants ($\int \int (\nabla \psi)^2 dx dz$ and $\int \int (\nabla^2 \psi)^2 dx dz$) in the non-dissipative limit. Such a conservation puts some restriction on the process of power transfer, which yields a predominant transfer of power towards long scales [67, 68]. The restriction on the power transfer due to the presence of two integral square
invariants mentioned above can be understood in the following way.

The conservation of the two square invariants in Fourier space can be written as,

\[
\delta \int \int E_k dk_x dk_z = \int \int \delta E_k dk_x dk_z = 0 \quad (3.6)
\]
\[
\delta \int \int k^2 E_k dk_x dk_z = \int \int k^2 \delta E_k dk_x dk_z = 0, \quad (3.7)
\]

where \(k\) is the wave number, \(E_k = k^2 \psi_k^2\) represents the energy in the wave number \(k\), \(\delta E_k\) is the change in \(E_k\) due to the nonlinear interactions and \(\psi_k\) is the Fourier transformed amplitude of the real space variable \(\psi\) corresponding to the wavenumber \(k\). The nonlinear processes transfer the energy from the range of wave numbers, where the energy was concentrated initially, to lower and higher wave numbers. The value of \(\delta E_k\) will be negative for the range of wave numbers where most of the energy was concentrated initially and positive for lower and higher wave numbers. It is clear from the relations (3.6) and (3.7) that these two relations cannot be satisfied simultaneously if the positive values of \(\delta E_k\) are large for higher values of \(k\) (which is the case of direct cascade) as compared to the lower values of \(k\). The reason for this is the disproportionate change in the positive and negative values of \(k^2 \delta E_k\) (given condition (3.6) holds) leading to the violation of condition (3.7). However, in case of inverse cascade that is when positive values of \(\delta E_k\) are small for higher wave numbers as compared to the lower wave numbers, the two conditions can be satisfied simultaneously. Therefore, presence of the two square invariants does not allow direct cascade of power.

Now we shall show that our equations also support two integral square invariants similar to those discussed above. The evolution Eq.(3.4) can be rewritten as

\[
\frac{\partial}{\partial t} (\nabla^2 B - B) + \dot{\mathbf{y}} \times \nabla B \cdot \nabla \nabla^2 B = 0
\]

Note, that the nonlinear term here is identical to the form we discussed above. Multiplying Eq.(3.8) by \(B\) and integrating over space leads to

\[
\frac{\partial}{\partial t} \int \int \{ B^2 + (\nabla B)^2 \} \, dx \, dz = 0 \quad (3.9)
\]

implying that \(\int \int \{ B^2 + (\nabla B)^2 \} \, dx \, dz\) is an invariant. Similarly when Eq.(3.8) is multiplied by \(\nabla^2 B\) it can be shown that the second invariant supported by our equation is \(\int \int \{ (\nabla B)^2 + (\nabla^2 B)^2 \} \, dx \, dz\). In Fourier space, these two invariants can
be written as $\int \int E_k dk_x dk_z$ and $\int \int k^2 E_k dk_x dk_z$, where $E_k = B_k^2 + k^2 B_k^2$. Therefore, dominant transfer of power in the magnetic fields takes place towards long scales as our simulations show.

We now focus on understanding the issue of the saturation at later times of the linear instability, which is evident from the plots of Fig. 3.3. For this purpose we investigate the modification of the $z$ independent electron flow velocity (the shear in whose initial profile was responsible for the instability) defined by the following expression

$$<v_z> \tilde{z} = \frac{1}{2l_z} \int_{-l_z}^{l_z} \bar{v}_z(x,z) dz \quad (3.10)$$

In Fig. 3.5 we plot the evolution of $<v_z>$ profile (as a function of $x$) with time. At $t = 0$, the mean flow profile corresponds to $V_0 \tanh(x/\epsilon)$, shear in which leads to instability. We observe that as time passes, the profile flattens and finally saturates at a less sheared state. This happens because the system is unstable due to the

![Figure 3.5](image-url)
presence of shear in the equilibrium flow profile and tries to attain an equilibrium state which is less sheared. So in an attempt to obtain such an equilibrium state, growing modes back react on the profile to flatten it. However, the final saturated state is still sheared and could be amenable to instability. To check if the modified mean profiles are unstable, we have solved linearized eigen value equations for the sausage mode (as described in chapter-2) numerically for the modified profiles to obtain the growth rates of the instability growing on these profiles. In Fig.(3.6) we plot the growth rate maximized over all the $k_x$ and $k_z$ present in the system, with time. At $t = 0$, the growth rate corresponds to the initial equilibrium profile and has been shown to match with the one obtained from simulation in Fig.(3.3). We see that the growth rate reduces with time and becomes zero ultimately, approximately at the same time at which the saturation occurs in Fig. (3.3). So, it is clear from this figure that the saturated profiles are no longer unstable. The saturation of the
instability can now be easily interpreted on the basis of this flattening. Basically the flattening of the profile results in an effective value of the shear scale length \( \epsilon_{\text{eff}} \) which is larger than \( \epsilon \). It thus becomes increasingly difficult to satisfy the condition of \( k_z \epsilon_{\text{eff}} < 1 \) even for the longest mode \( k_z = k_z^{\text{min}} = \pi/l_z \) in the box, for instability. This halts the linear growth of the modes. To confirm that this indeed is the mechanism for the saturation of the instability we carried out simulation runs for other set of parameters for which the value of the effective shear scale length \( \epsilon_t \) to stabilize the longest mode of the system is much larger than the system length in the \( x \)-direction i.e. \( \epsilon_t = 1/k_z^{\text{min}} >> l_x \). For this case (\( \epsilon = 0.1, V_0 = 5, l_x = 0.5 \) and \( l_z = 10 \)) as shown in Fig. 3.7, \( \langle v_z \rangle \) keeps evolving and acquires a straight line profile connecting the two values at boundary points. Thus our observations show that so long as \( \epsilon_t = 1/k_z^{\text{min}} \) is larger than \( l_z \) the final \( \langle v_z \rangle \) profile is a straight line. The absence of second derivative for such cases then halts the instability. However, if the box size \( 2l_x \) is sufficiently large then the profile acquires a shape such that finally the effective shear scale length \( \epsilon_{\text{eff}} \approx \epsilon_t \). Thus the saturation of the instability in our model occurs either (for large \( l_x \)) because the modified velocity shear is too weak.

\[ \varepsilon = 0.1, \ V_0 = 5 \]
\[ l_x = 0.5, \ l_z = 10 \]

**Figure 3.7:** Approach of initial equilibrium profile to straight line configuration. Parameters for this run are \( \varepsilon = 0.1, \ V_0 = 5, \ l_x = 0.5 \) and \( l_z = 10 \).
even to make the longest mode unstable (i.e. \( k_{z}^{\text{min}} \epsilon_{\text{eff}} \gtrsim 1 \)) or (for small \( L_x \)) because the curvature due to velocity shear totally disappears (\( v''_z = 0 \)). The flattening of the profile can be analyzed quantitatively by plotting the average current,

\[
J(t) = \int_{0}^{L_x} \langle v_z(x, t) \rangle \, dx, \tag{3.11}
\]

flowing in one direction, with time. Fig(3.8) shows such a plot in which we have plotted \( J \) for \( \epsilon = 0.3 \) and 0.5. Initial period during which current does not change appreciably can be recognized as linear phase of the instability. After the linear phase, current starts dropping. It is noted that \( J \) is reduced but does not go to zero. Thus, although there is collective stopping of the electron flow, it does not lead to complete stopping. We ascribe the incompleteness of the stopping to two dimensionality of our simulations. In two dimensions, the tendency for the power is to accumulate at long scales (limited by simulation box sizes) leading to artificial coherence effects and lack of turbulence as the longest scale hits the box. We speculate that the initial development of the velocity profile in 2d simulations thus gives us a reasonable indication of the overall collective stopping power due to the devel-
opment of EMHD turbulence; this can however best be pinned down only in actual three dimensional EMHD simulations where turbulence would have an opportunity to fully develop.

3.7 Quasilinear Analysis

The simulation results show a flattening of the initial shear profile of the $z$ directed flow. A repetition of the simulation by allowing a self consistent evolution of the boundary points also yields similar adjustment of the shear structure. The fact that the overall magnitude of the velocity (e.g. the values at the edges) remains largely unaltered suggests that the evolution of $\vec{V}$ occurring via nonlinear scale interactions can be mocked up by an effective viscous or even higher order derivative dissipation. The viscous and/or higher derivative effective dissipation of the flow can be understood by the following simplified quasilinear analysis.

We represent the sheared $z$ independent electron flow velocity and the corresponding magnetic field by

\[ v_0 \hat{z} = v_q \exp(iq_x x) + c.c \]
\[ B_0 \hat{y} = B_q \exp(iq_x x) + c.c \]  

We next study the evolution of this $z$ independent flow as a result of nonlinear interactions with other scales represented by

\[ \dot{B}(x, z, t) = \sum_{k_x, k_y} B_k(t) \exp(i k_x x + i k_z z) \]  

(3.13)

Here $\dot{B}$ is that part of the $B$ field which depends on $x$ and $z$ both. Now, taking the $q^{th}$ Fourier component of Eq.(3.4) we obtain

\[ (1 + q_x^2) \frac{d}{dt} B_q = q_z \sum k_z k_1^2 < B_k B_{k_1} > \]  

(3.14)

Here $\vec{k_1} = q_z \vec{x} - \vec{k}$. The right hand side signifies the generation of the $q^{th}$ mode as a result of nonlinear coupling between two Fourier modes. The angular average are meant for the ensemble averaging. Our next task is to express the correlator
\( < B_k B_{k1} > \), in terms of the power spectrum e.g. \( \sim | B_k |^2 \). For this we make use of a quasilinear approximation and write

\[
< B_k B_{k1} > = < B_k \delta B_{k1} > + < \delta B_k B_{k1} >
\]  

(3.15)

where \( \delta B_k \) is that part of the \( k^{th} \) Fourier mode of \( B \) field which gets generated by the interaction of \( q_x \) with \( k_1 \). The interaction of remaining other modes which also generate \( k^{th} \) Fourier mode is essentially responsible for the eddy decorrelation effect, leading to a finite spectral correlation time say \( \tau_k \) [69,70]. For simplicity we will assume that \( \tau_k = \tau \) independent of \( k \). Thus the equation for \( \delta B_k \) can be written as

\[
(1 + k^2) \delta B_k = \tau q_x k_{1z} (k_1^2 - q_x^2) B_q B_{-k_1}
\]  

(3.16)

Since there are no variations parallel to the magnetic field (i.e. along \( y \)) the real frequency in the evolution of \( \delta B_k \) is zero. Using Eqs(3.14), (3.15) and (3.16) we can write

\[
(1 + q_x^2) \frac{dB_q}{dt} = q_x^2 \tau \sum k^2 T_k \{ \frac{k_z (k^2 - q_x^2)}{(1 + k^2)} | B_k |^2 + \frac{k_{1z} (k_1^2 - q_x^2)}{(1 + k_1^2)} | B_{k_1} |^2 \} B_q
\]  

(3.17)

For simplicity we assume that typically the spectral strengths are such that we have \( | B_k |^2 \approx | B_{k1} |^2 \). Since \( k_{1z} = - k_z \) it is clear from the above expression that the right hand side will have terms which have powers of \( q_x^2 \) and higher. This establishes that the \( z \) independent flow being \( V_q = -i q_x B_q \) suffers a viscous and/or a higher derivative damping. The effective viscosity coefficient can be obtained from Eq.(3.17) by collecting the coefficient of terms containing powers of \( q_x^4 \). This gives

\[
| \mu_{eff} | = \tau \sum \frac{k_z^2}{(1 + k^2)^2} (1 + 2k^2)(k^2 + 4k_z^2) | B_k |^2
\]  

(3.18)

### 3.8 Summary

We have presented here a detailed fluid simulation study to understand the evolution of fast modes associated with shear in the electron flow velocity in a simplified two dimensional slab geometry. We observe development of sausage like structures with characteristic growth rates in agreement with the predictions of the linear theory. The coherent structures observed in the nonlinear state of the instability are formed...
due to the presence of two non-dissipative mean square invariants supported by two dimensional EMHD equations. The instability saturates in the nonlinear state either by getting rid of the curvature in the velocity profile or by violating the condition \( k_z\varepsilon_{eff} < 1 \) even for smallest possible \( k_z \) permitted by the periodicity in the z-direction, whichever occurs first. The flattening of the electron flow profile caused by these shear driven unstable excitations can have crucial implications to several frontier areas of plasma research, as it provides for a collective stopping mechanism of electron flow.