Chapter 2
Mathematical Preliminaries

This chapter is devoted to a discussion on mathematical preliminaries encompassing several areas in analysis. For the convenience of the reader we include some standard results which one may have the occasion to use in the thesis. We begin with a section on analysis in normed linear spaces. By and large, this section will feature notions of measurability of functions taking values in a Banach space and properties of the Bochner integral. Some important results involving complex valued analytic functions have also been included.

2.1 Analysis in Normed Linear Spaces
2.1.1 Analytic Functions

Theorem 2.1.1.1 (Vitali's convergence theorem). Let \( f_n(z) \) be a sequence of functions, each regular in a region \( D \); let

\[
|f_n(z)| \leq M
\]
for every $n$, and $z$ in $D$, and let $f_n(z)$ tend to a limit as $n \to \infty$, at a set of points having a limit point inside $D$. Then $f_n(z)$ tends to a limit in any region bounded by a contour, interior to $D$, the limit being an analytic function of $z$.

**Proof** See [Tit 91](Theorem 5.21).

**Theorem 2.1.1.2 (Phragmen–Lindelöf).** Let $D$ be the open strip in $\mathbb{C}$ defined by

$$D = \{ z ; z \in \mathbb{C}, a < \Re z < b \}$$

and $\overline{D}$ the closure of $D$. Let $f$ be a complex function which is analytic on $D$, and bounded and continuous on $\overline{D}$. It follows that the function

$$y \in [a, b] \mapsto g(y) = \log \left( \sup_{x \in \mathbb{R}} |f(x + iy)| \right)$$

is convex. In particular,

$$\sup_{z \in \overline{D}} |f(z)| = \max \left\{ \sup_{x \in \mathbb{R}} |f(x + ia)|, \sup_{x \in \mathbb{R}} |f(x + ib)| \right\}.$$

**Proof** Vide [Rob 81](Proposition 5.3.5).

**Theorem 2.1.1.3** Suppose that $\gamma$ is the boundary of an unbounded region $\Omega$, $f \in H(\Omega)$, $f$ is continuous on $\Omega \cup \gamma$, and there are constants $B < \infty$ and $M < \infty$, such that, $|f| \leq M$ on $\Gamma$ and $|f| \leq B$ in $\Omega$. Then, we actually have $|F| \leq M$ in $\Omega$.

**Proof** See problem 11 on page 264 in [Rud 87].
2.1.2 Measure Theoretic Preliminaries

Let \( \Omega \) be an abstract set, \( \mathcal{C} \) a \( \sigma \)-ring of subsets of \( \Omega \), and \( m \) defined on \( \Omega \) be a \( \sigma \)-finite measure. In this section, we study the notion of measurability for vector valued functions \( f(\sigma) \) on \( \Omega \), taking values in a Banach space \( X \), relative to the measure \( m \). There are several notions of measurability for vector valued functions.

The following definitions have been taken from [Hil 57].

**Definition 2.1.2.1**  
1. \( f(\sigma) \) is said to be finitely-valued if it is constant on each of a finite number of disjoint measurable sets \( E_j \) and equal to 0 on \( \Omega \setminus \bigcup E_j \).

2. It is a simple function if it is finitely-valued and if the set for which \( \|f(\sigma)\| > 0 \) is of finite measure.

3. \( f(\sigma) \) is a countably-valued function if it assumes at most a countable set of values in \( X \), assuming each value different from 0 on a measurable subset.

**Definition 2.1.2.2** \( f(\sigma) \) is said to be separably-valued if its range, \( f(\Omega) \) is separable. It is almost separably-valued if there exists a \( m \)-null set \( E_0 \in \mathcal{C} \) such that \( f(\Omega \setminus E_0) \) is separable.

**Definition 2.1.2.3**  
1. \( f(\sigma) \) is said to be weakly measurable in \( \Omega \) if the numerical functions \( x^*(f(\sigma)) \) are measurable for each \( x^* \in X^* \).
2. \( f(\sigma) \) is strongly measurable if there exists a sequence of countably-valued functions converging almost everywhere in \( \Omega \) to \( f(\sigma) \).

Note that if \( m(\Omega) < \infty \), then we may replace "countably-valued" in part (2) by "simple".

A subset \( \Lambda \subseteq X^* \) is said to be determining for \( X \) if \( \|x\| = \sup\{ |x^*(x)| ; x^* \in \Lambda \} \) for all \( x \in X \).

**Theorem 2.1.2.4** If \( f(\sigma) \) is weakly measurable and if there exists a denumerable set \( \Lambda \) which is determining for \( X \), then the numerically valued function \( \|f(\sigma)\| \) is measurable.

**Proof** Refer to [Hil 57](Theorem 3.5.4).

**Theorem 2.1.2.5** A vector valued function on \( \Omega \) taking values in \( X \) is strongly measurable if and only if it is weakly measurable and almost separably valued.

**Proof** Theorem 3.5.3 in [Hil 57].

**Corollary 2.1.2.6** If \( X \) is separable, then strong and weak measurability are equivalent notions.

**Theorem 2.1.2.7**

1. If \( f(\sigma) \) and \( g(\sigma) \) are strongly measurable functions on \( \Omega \) taking values in \( X \), and \( \gamma_1, \gamma_2 \) are constants, then \( \gamma_1 f(\sigma) + \gamma_2 g(\sigma) \) is strongly measurable.

2. If \( h(\sigma) \) is a finite numerically valued function which is measurable, then \( h(\sigma) f(\sigma) \) is strongly measurable if \( f(\sigma) \) has this property.
3. If \( f(\sigma) \) is the limit almost everywhere of a sequence of strongly measurable functions, then \( f(\sigma) \) is strongly measurable.

4. The same conclusion is valid if in (3) the word "limit" (that is, strong limit) is replaced by "weak limit".

5. The conclusion is also valid if the "limit almost everywhere" is replaced by the "limit in measure".

Proof See theorem 3.5.4 in [Hil 57].

Next, we introduce the Bochner integral. The results listed in this part of the section have been taken from [Hil 57], chapter 3.

Definition 2.1.2.8 A countably valued function \( f(\sigma) \) from \( \Omega \) to \( X \) is Bochner integrable, if and only if, \( \|f(\sigma)\| \) is Lebesgue integrable. By definition, the Bochner integral of \( f(\sigma) \) on \( E \in \mathcal{C} \), denoted by \( (B) \int_E x(\sigma)dm \) is given by

\[
(B) \int_E f(\sigma)dm = \sum_{k=1}^{\infty} x_km(E_k \cap E),
\]

where \( f(\sigma) = x_k \) on \( E_k \in \mathcal{C} (k = 1, 2, \ldots) \). This integral is well defined for all \( E \in \mathcal{C} \) and for \( \Omega \) itself. This follows from the fact that \( \|f(\sigma)\| \) is integrable.

Definition 2.1.2.9 A function \( f(\sigma) \) from \( \Omega \) to \( X \) is Bochner integrable if, and only if, there exists a sequence of countably valued Bochner integrable functions \( \{f_n(\sigma)\} \) converging almost everywhere to \( f(\sigma) \) and such that

\[
\lim_{n \to \infty} \int_{\Omega} \|f(\sigma) - f_n(\sigma)\| dm = 0.
\]
By definition,

\[ (B) \int_E f(\sigma) dm = \lim_{n \to \infty} (B) \int_E f_n(\sigma) dm, \]

for each \( E \in \mathcal{C} \) and \( E = \Omega \).

**Theorem 2.1.2.10** A necessary and sufficient condition for \( f(\sigma) \) from \( \Omega \) to \( X \) be Bochner integrable is that, \( f(\sigma) \) be strongly measurable and that

\[ \int_{\Omega} \| f(\sigma) \| dm < \infty. \]

We shall denote the class of all Bochner integrable functions relative to \( m \), by \( B(\Omega, X, m) \). Some interesting properties of the Bochner integral have been listed below

**Proposition 2.1.2.11** If \( f_1(\sigma) \) and \( f_2(\sigma) \in B(\Omega, X, m) \) and \( \gamma_1, \gamma_2 \) are constants, then \( \gamma_1 f_1(\sigma) + \gamma_2 f_2(\sigma) \in B(\Omega, X, m) \) and

\[ \int_E (\gamma_1 f_1(\sigma) + \gamma_2 f_2(\sigma)) dm = \gamma_1 \int_E f_1(\sigma) dm + \gamma_2 \int_E f_2(\sigma) dm, \]

for all \( E \in \mathcal{C} \) and \( E = \Omega \).

**Proposition 2.1.2.12** If \( f(\sigma) \in B(\Omega, X, m) \), then

\[ \| \int_E f(\sigma) dm \| \leq \int_E \| f(\sigma) \| dm, \]

for all \( E \in \mathcal{C} \) and \( E = \Omega \).

**Proposition 2.1.2.13** Let \( T \) be a closed linear transformation from \( X \) to \( Y \). If \( f(\sigma) \in B(\Omega, X, m) \) and \( T(f(\sigma)) \in B(\Omega, Y, m) \), then

\[ T \left( \int_E f(\sigma) dm \right) = \int_E T(f(\sigma)) dm \]

for all \( E \in \mathcal{C} \) and \( E = \Omega \).
If, in particular, \( T \) is a bounded, linear transformation from \( X \) to \( Y \), then the theorem applies if only \( f(\sigma) \in B(\Omega, X, m) \).

The last result in this section is an analogue of Fubini's theorem for Bochner integrals.

Suppose that \( S \) and \( T \) are abstract sets possessing \( \sigma \)-rings of subsets \( \mathcal{C} \) and \( \mathcal{F} \), with \( \sigma \)-finite measures \( m \) and \( n \) defined on \( \mathcal{C} \) and \( \mathcal{F} \), respectively. We denote the \( \sigma \)-ring of subsets of \( S \times T \) generated by the class of measurable rectangles by \( \mathcal{C} \times \mathcal{F} \). Finally we denote the product measure by \( m \times n \).

**Theorem 2.1.2.14** If a function \( f(\sigma, \tau) \) on \( S \times T \) taking values in \( X \), is Bochner integrable, then the functions \( g(\sigma) = \int_T f(\sigma, \tau) dn \) and \( h(\tau) = \int_S f(\sigma, \tau) dm \) are defined almost everywhere in \( S \) and \( T \) respectively, Bochner integrable on \( S \) and \( T \) respectively, and

\[
\int_{S \times T} f(\sigma, \tau) d(m \times n) = \int_S g(\sigma) dm = \int_T h(\tau) dn.
\]

**Proof** Vide [Hil 57], theorem 3.7.13. \( \triangle \)

In the next section we collect some standard results pertaining to the spectral theory of self adjoint operators. We also include some important results which arise in the theory of one-parameter groups of unitaries.

### 2.2 Operator Theoretic Preliminaries

**Theorem 2.2.0.15 (Spectral Theorem).** Let \( A \) be a self adjoint operator on a Hilbert space \( \mathcal{H} \) with inner product \( \langle ., . \rangle \). Then, there exists an unique
spectral family $E(\lambda)$ on $\mathcal{H}$ such that,

$$A\phi = \int_{\mathbb{R}} \lambda \, dE(\lambda).$$

and the domain of $A$,

$$D(A) = \{ \phi \in \mathcal{H} : \int_{\mathbb{R}} \lambda^2 d(E(\lambda)\phi, \phi) < \infty \}.$$

**Proposition 2.2.0.16** Let $A$ be a self adjoint operator with spectral family $E(\lambda)$, then $s \in \sigma(A)$ if and only if, $E(s + \epsilon) - E(s - \epsilon) \neq 0$ for every $\epsilon > 0$.

**Proof** See [Wei 80](Theorem 7.22). \(\triangle\)

We denote the essential spectrum of a self adjoint operator $A$ by $\sigma_e(A)$ and the discrete spectrum by $\sigma_d(A)$.

**Proposition 2.2.0.17** Let $A$ be as in the above proposition, then $s \in \sigma_e(A)$ if and only if, for every $\epsilon > 0$, we have $\dim(R(E(s + \epsilon) - E(s - \epsilon))) = \infty$.

**Proof** Vide [Wei 80](Theorem 7.24). \(\triangle\)

**Proposition 2.2.0.18** Let $A$ be a self adjoint operator on a Hilbert space $\mathcal{H}$, and $R(A, z)$ denote the resolvent $(A - zI)^{-1}$ at $z$. Then for $\phi, \psi \in \mathcal{H}$, we have

$$\langle E(\lambda)\phi, \psi \rangle = \lim_{\delta \to 0^+} \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\lambda + \delta} \langle ((s + i\epsilon - A)^{-1} - (s + i\epsilon - A)^{-1}) \phi, \psi \rangle ds.$$

**Proof** Refer to theorem 7.17 in [Wei 80]. \(\triangle\)
Proposition 2.2.0.19 Let $A$ be a self adjoint operator on a Hilbert space $\mathcal{H}$ and $\phi, \psi \in \mathcal{H}$. If $\Re z > 0$, then
\[
\langle (R(A, z))\phi, \psi \rangle = i \int_0^\infty e^{itz} \langle e^{-iAt}\phi, \psi \rangle dt
\]
and if $\Re z < 0$,
\[
\langle (R(A, z))\phi, \psi \rangle = -i \int_0^\infty e^{-itz} \langle e^{iAt}\phi, \psi \rangle dt,
\]
where the integral is a Riemann integral.

Proof Vide [Dun 63](Chapter XII, Section VI, Theorem 1) \(\Delta\)

Definition 2.2.0.20 An operator function $U(t)$ on a Hilbert space $\mathcal{H}$, satisfying

1. For each $t \in \mathbb{R}$, $U(t)$ is a unitary operator and $U(t+s) = U(t)U(s)$ for all $t \in \mathbb{R}$.

2. If $\phi \in \mathcal{H}$ and $t \to t_0$, then $U(t)\phi \to U(t_0)\phi$,

is called a strongly continuous, one-parameter group of unitary operators.

Theorem 2.2.0.21 (Stone’s Theorem). Let $U(t)$ be a strongly continuous, one-parameter unitary group on a Hilbert space $\mathcal{H}$. Then there is a self adjoint operator $A$ on $\mathcal{H}$ such that $U(t) = e^{itA}$.

Proof See [Sim 80](Theorem VIII.8). \(\Delta\)

Definition 2.2.0.22 If $U(t)$ is a strongly continuous, one-parameter unitary group, then the self adjoint operator $A$ with $U(t) = e^{itA}$, is called the infinitesimal generator of $U(t)$. 

20
It is worth noting that if $U(t)$ is weakly continuous, then it is strongly continuous.

In the next section we collect a number of results in the theory of operator algebras which are relevant to the study of Quantum Statistical Mechanics.

2.3 Operator Algebraic Preliminaries

2.3.1 Standard Results in the Theory of $C^*$-Algebras and von Neumann Algebras

Definition 2.3.1.1 A normed algebra $A$ with an involution which is complete and has the property $\|A^*\| = \|A\|$, is called a Banach $*$-algebra.

Definition 2.3.1.2 A $C^*$-algebra is a Banach $*$-algebra with the property

$$\|A^*A\| = \|A\|^2.$$  

Definition 2.3.1.3 A linear functional $\rho$ over a $C^*$-algebra $A$ is defined to be positive if,

$$\rho(A^*A) \geq 0,$$

for all $A \in A$. A positive linear functional $\rho$ over a $C^*$-algebra $A$ with $\|\rho\| = 1$ is called a state.

Note that the set of states $E_A$ of the $C^*$-algebra is weak*-compact if, and only if, $A$ contains an identity.

Definition 2.3.1.4 A von Neumann algebra on $\mathcal{H}$ is a $*$-algebra $\mathcal{M}$ of $\mathcal{L}(\mathcal{H})$ such that

$$\mathcal{M} = \mathcal{M}'',$

21
where $\mathcal{M}'$ denotes the commutant of $\mathcal{M}$ and $\mathcal{M}''$ denotes the commutant of $\mathcal{M}'$. The center $\mathcal{E}(\mathcal{M})$ of a von Neumann algebra is defined by

$$\mathcal{E}(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'.$$

A von Neumann algebra is called a factor, if it has a trivial center, i.e. if $\mathcal{E}(\mathcal{M}) = \mathcal{C}I$.

**Definition 2.3.1.5** Let $\mathcal{M}$ be a von Neumann algebra and $\rho$ a positive linear functional on $\mathcal{M}$. If $\rho(\limsup_{\alpha} A_\alpha) = \limsup_{\alpha} \rho(A_\alpha)$ for all increasing nets $\{A_\alpha\}$ in $\mathcal{M}_+$ with an upper bound, then $\rho$ is defined to be normal.

**Definition 2.3.1.6** A von Neumann algebra $\mathcal{M}$ is said to be $\sigma$-finite if all collections of mutually orthogonal projections have at most a countable cardinality.

**Definition 2.3.1.7** A representation of a C*-algebra $\mathcal{A}$ is defined to be a pair $(\mathcal{H}, \pi)$, where $\mathcal{H}$ is a complex Hilbert space and $\pi$ a *-morphism of $\mathcal{A}$ into $\mathcal{L}(\mathcal{H})$. The representation is said to be faithful if and only if, $\pi$ is a *-isomorphism between $\mathcal{A}$ and $\pi(\mathcal{A})$.

It is worth mentioning that if $\pi$ is a representation of a C*-algebra $\mathcal{A}$, then $\pi$ is continuous and $\|\pi(A)\| \leq \|A\|$ for all $A \in \mathcal{A}$. The equality holds only in the case of a faithful representation.

**Definition 2.3.1.8** A vector $\Theta$ in a Hilbert space $\mathcal{H}$ is said to be cyclic for a set of bounded operators $\mathcal{N}$, if the set $\{A\Theta | A \in \mathcal{M}\}$ is dense in $\mathcal{H}$.
Definition 2.3.1.9 A cyclic representation of a C*-algebra $\mathcal{A}$, is defined to be a triple $(\mathcal{H}, \pi, \Theta)$, where $(\mathcal{H}, \pi)$ is a representation of $\mathcal{A}$, and $\Theta$ is a vector in $\mathcal{H}$ which is cyclic for $\pi$ in $\mathcal{H}$.

Theorem 2.3.1.10 Let $\rho$ be a state over the C*-algebra $\mathcal{A}$. It follows that there exists a cyclic representation $(\mathcal{H}_\rho, \pi_\rho, \Theta_\rho)$ of $\mathcal{A}$ such that,

$$
\rho(A) = (\Theta, \pi_\rho(A)\Theta),
$$

for all $A \in \mathcal{A}$ and consequently, $\|\Theta_\rho\|^2 = \|\rho\| = 1$. Moreover, the representation is unique up to unitary equivalence.

Proof Refer to theorem 2.3.16 in [Rob 87].

Definition 2.3.1.11 A state $\rho$ of a C*-algebra is called a primary state, or a factor state, if $\pi_\rho(A)^\prime$ is a factor, where $\pi_\rho$ is the associated cyclic representation.

Definition 2.3.1.12 Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$. A subset $\mathcal{R} \subseteq \mathcal{H}$ is separating for $\mathcal{M}$ if for any $A \in \mathcal{M}$, $A\xi = 0$ for all $\xi \in \mathcal{R}$ implies $A = 0$.

Definition 2.3.1.13 A subset $\mathcal{R} \subseteq \mathcal{H}$ is cyclic for $\mathcal{M}$ if $[\mathcal{M}\mathcal{R}] = \mathcal{H}$, where $[\mathcal{M}\mathcal{R}]$ denotes the closure of the linear span of elements of the form $A\xi$, where $A \in \mathcal{M}$ and $\xi \in \mathcal{H}$.

Proposition 2.3.1.14 Let $\mathcal{M}$ be a von Neumann algebra on $\mathcal{H}$ and $\mathcal{R} \subseteq \mathcal{H}$ a subset. The following conditions are equivalent:
1. \( \mathcal{R} \) is cyclic for \( \mathcal{M} \);

2. \( \mathcal{R} \) is separating for \( \mathcal{M}' \).

**Proof** Vide [Rob 87](Proposition 2.5.3).

Next, a directed set \( J \) is said to possess an orthogonality relation if there exists a relation \( \perp \), between pairs of elements of \( J \) such that,

1. if \( a \in J \) then there is a \( b \in J \) with \( a \perp b \);

2. if \( a \leq b \) and \( b \perp \gamma \) then \( a \perp \gamma \);

3. if \( \alpha \perp b \) and \( \alpha \perp \gamma \) then there exists a \( \delta \in J \) such that, \( \alpha \perp \delta \) and \( \delta \geq b, \gamma \).

**Remark** If \( \sigma \) is an automorphism of a \( C^* \)-algebra which satisfies \( \sigma^2 = 1 \), then each element \( A \in \mathcal{A} \), has an unique decomposition into odd and even parts with respect to \( \sigma \). This decomposition is defined by

\[
A = A^+ + A^-; \quad A^\pm = A \pm \sigma(A) \frac{1}{2}.
\]

It follows that \( \sigma(A^\pm) = \pm A \), the even elements of \( \mathcal{A} \) form a \( C^* \)-subalgebra \( \mathcal{A}^e \) of \( \mathcal{A} \) and the odd elements \( \mathcal{A}^o \) form a Banach space.

**Definition 2.3.1.15** A quasi-local algebra is a \( C^* \)-algebra \( \mathcal{A} \) and a net \( \{ \mathcal{A}_\alpha \}_{\alpha \in J} \) of \( C^* \)-subalgebras such that, index set \( J \) has an orthogonality relation and the following properties are valid:

1. if \( \alpha \geq \beta \) then \( \mathcal{A}_\alpha \supseteq \mathcal{A}_\beta \);

2. \( \mathcal{A} = \overline{\bigcup_\alpha \mathcal{A}_\alpha} \), where the bar denotes the uniform closure;
3. The algebras $A_{\alpha}$ have a common identity $I$:

4. there exists an automorphism $\sigma$ such that $\sigma^2 = 1$, $\sigma(A_{\alpha}) = A_{\alpha}$ and

$$[A_{\alpha}, A_{\beta}] = \{0\}, [A_{\alpha}, A_{\beta}] = \{0\}, \{A_{\alpha}, A_{\beta}\} = \{0\}$$

whenever $\alpha \perp \beta$, where $A_{\alpha} \subseteq A_{\alpha}$ and $A_{\alpha} \subseteq A_{\alpha}$ are odd and even elements with respect to $\sigma$.

We have used the notation $\{A, B\} = AB + BA$. One case covered by this definition is $\sigma = 1$ and then $A_{\alpha} = A_{\alpha}$ and the condition (4) simplifies to

$$[A_{\alpha}, A_{\beta}] = \{0\}$$

whenever $\alpha \perp \beta$.

**Proposition 2.3.1.16** Let $A$, $\{A_{\alpha}\}_{\alpha \in I}$ be a quasi-local algebra and assume that each $A_{\alpha}$ is simple. It follows that $A$ is simple.

**Proof** See corollary 2.6.19 in [Rob 87].

**Definition 2.3.1.17** A $C^*$-algebra $A$ with unit $I$, is said to be uniformly matricial if there is a sequence $\{A_n\}$ of $C^*$-subalgebras of $A$ and a sequence $\{n_j\}$ of positive integers such that, $A_j$ is $*$-isomorphic to the algebra of all $n_j \times n_j$ complex matrices,

$$I \in A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots,$$

and $A$ is the norm-closure of $\bigcup A_j$. We then describe $A$ in more detail, as uniformly matricial of type $\{n_j\}$, and refer to the sequence $\{A_j\}$ as a generating nest of type $\{n_j\}$ for $A$. 

25
Proposition 2.3.1.18 There is a uniformly matricial C*-algebra of type \{n_j\}, if and only if, the sequence \{n_j\} of positive integers is strictly increasing and \(n_j\) divides \(n_{j+1}(j = 1, 2, \ldots)\). When these conditions are satisfied, all uniformly matricial algebras of type \(\{n_j\}\) are \(*\)-isomorphic and are simple C*-algebras.

Proof See proposition 10.4.18 in [Kad 86].

Definition 2.3.1.19 Let \(\{A_j : j \in J\}\) be a family of C*-algebras (with unit \(I_j\) in \(A_j\)), in which the index set \(J\) is directed by a binary relation \(\leq\). Suppose that, whenever \(j, k \in J\) and \(j \leq k\), there is specified, a \(*\)-isomorphism \(\Phi_{k,j}\) from \(A_j\) into \(A_k\) (with \(\Phi_{k,j}(I_j) = I_k\)); and finally, suppose that \(\Phi_{l,k} \circ \Phi_{k,j} = \Phi_{l,j}\) whenever \(j, k, l \in J\) and \(j \leq k \leq l\). In these circumstances, we say that the C*-algebras \(\{A_j : j \in J\}\), together with the \(*\)-isomorphisms \(\{\Phi_{j,k} : j, k \in J, j \leq k\}\), constitute a directed system of C*-algebras. Note that \(\Phi_{j,j}\) is the identity mapping on \(A_j\).

Proposition 2.3.1.20 Suppose that the C*-algebras \(\{A_j : j \in J\}\), and the \(*\)-isomorphisms \(\Phi_{j,k} : A_j \mapsto A_k\) \((j, k \in J; j \leq k)\), together form a directed system.

1. There is a C*-algebra \(A\) and for each \(j\) in \(J\), a \(*\)-isomorphism \(\phi_j\), from \(A_j\) into \(A\) (carrying the unit of \(A_j\) into that of \(A\)), such that \(\phi_j = \phi_k \circ \Phi_{k,j}\) when \(j \leq k\) and \(\cup \{\phi_j(A_j); j \in J\}\) is everywhere dense in \(A\).
2. The $C^*$-algebra $A$ occurring in (1) is uniquely determined, up to $^*$-isomorphism; if $C$ is a $C^*$-algebra, $\psi_j : A_j \to C$ is a $^*$-isomorphism (for each $j$ in $J$) and conditions analogous to those in (1) are satisfied, then there is a $^*$-isomorphism $\Psi$ from $A$ onto $C$, such that $\psi_j = \Psi \circ \phi_j$, for each $j$ in $J$.

Proof Refer to proposition 11.4.1 in [Kad 86].

The $C^*$-algebra $A$ occurring in the above proposition is called the inductive limit of the directed system $\{A_j; j \in J\}$.

Proposition 2.3.1.21 If $A$ is the inductive limit of a directed system of simple $C^*$-algebras, then $A$ is simple.

Proof Vide proposition 11.4.2 on [Kad 86].

Definition 2.3.1.22 A one-parameter family $t \in \mathbb{R} \mapsto \tau_t$ of automorphisms of the $C^*$-algebra $A$, is called a strongly continuous group of automorphisms of $A$, if,

1. $\tau_{t_1+t_2} = \tau_{t_1} \circ \tau_{t_2}$, $t_1, t_2 \in \mathbb{R}$, and $\tau_0 = 1$;

2. $t \mapsto \tau_t(A)$ is continuous in norm for all $A \in A$.

Definition 2.3.1.23 A one-parameter family $t \in \mathbb{R} \mapsto \tau_t$ of automorphisms of a von Neumann algebra $M$ is called a weakly continuous group of automorphisms of $M$ if

1. $\tau_{t_1+t_2} = \tau_{t_1} \circ \tau_{t_2}$, $t_1, t_2 \in \mathbb{R}$, and $\tau_0 = 1$;
2. $t \mapsto \tau_t(A)$ is weakly continuous for all $A \in \mathcal{M}$.

Definition 2.3.1.24 A derivation $\delta$ of a $C^*$-algebra $A$ is a linear operator from a $*$-subalgebra $D(\delta)$, the domain of $\delta$, into $A$ with the following properties:

1. $\delta(A)^* = \delta(A^*)$, $A \in D(\delta)$;

2. $\delta(AB) = \delta(A)B + A\delta(B)$; $A, B \in D(\delta)$.

Definition 2.3.1.25 Let $S_n$ be a sequence of operators on a Banach space $X$ and let $G(S_n) \subseteq X \times X$ be their graphs. Define

$$G = \lim_{n \to \infty} G(S_n)$$

as the set of pairs $(A, B) \in X \times X$ such that there exists a sequence $(A_n, B_n) \in X \times X$ with $A_n \in D(S_n)$, $B_n = S_n A_n$, and

$$A = \lim_{n \to \infty} A_n, \quad B = \lim_{n \to \infty} B_n.$$ 

Define $D(G)$ as the set of $A \in X$ such that, there exists $B \in X$ with $(A, B) \in G$ and similarly, $R(G)$ is the set of $B \in X$ such that, $(A, B) \in G$ for some $A \in X$. If $G$ is the graph of an operator $S$, then $S$ is called the graph limit of $S_n$. Then clearly $D(S) = D(G)$ and $R(S) = R(G)$.

Definition 2.3.1.26 Let $(A, G, \tau)$ be a $C^*$-dynamical system. We say that the system is asymptotically abelian if there is a net $g_\alpha$ in $G$, such that

$$\lim_{\alpha} \| A\tau_{g_\alpha}(B) - \tau_{g_\alpha}(B)A \| \to 0.$$
Furthermore, the states $\rho$ for which there exists a net $g_{\alpha}$ in $G$ such that,

$$\lim_{\alpha} |\rho(A_{\tau_{\alpha}}(B)) - \rho(A)\rho(\tau_{\alpha}(B))| = 0$$

are called strongly clustering states.

### 2.3.2 KMS States and Associated Representations

**Definition 2.3.2.1** Let $(A, \tau)$ be a $C^*$-dynamical system, or a $W^*$-dynamical system and $\rho$ a state over $A$ which is assumed to be normal in the $W^*$ case. Then, $\rho$ is said to be a $(\tau, \beta)$-KMS state if, for $\beta > 0$ and any pair $A, B \in A$, there exists a complex function $F_{A,B}$ which is analytic on the open strip $0 < \Im z < \beta$, uniformly bounded and continuous on the closed strip $0 \leq \Im z \leq \beta$ such that,

$$F_{A,B}(t) = \rho(A \tau t B) \quad \text{and} \quad F_{A,B}(t + i\beta) = \rho(\tau_B(A)).$$

If $\beta < 0$, then $\rho$ is said to be a $(\tau, \beta)$-KMS state if there exists a complex function $F_{A,B}$ which is analytic on the open strip $\beta < \Im z < 0$, uniformly bounded and continuous for $\beta \leq \Im z \leq 0$ such that,

$$F_{A,B}(t) = \rho(A \tau t B) \quad \text{and} \quad F_{A,B}(t + i\beta) = \rho(\tau_B(A)).$$

**Proposition 2.3.2.2** Let $\rho$ be a $(\tau, \beta)$-KMS state of the $C^*$-dynamical system $(A, \tau)$ with $\beta \in \mathbb{R} \setminus \{0\}$ and let $\tilde{\rho}$ be the normal extension of $\rho$ to the weak closure $\mathcal{M}_\rho = \pi(A)'$ of $A$ in the cyclic representation $(\mathcal{H}_\rho, \pi_\rho, \Theta_\rho)$. It follows that there exists an unique $\sigma$-weakly continuous group $t \mapsto \tilde{\tau}_t$ of $^*$-automorphisms of $A_\rho$ such that

$$\tilde{\tau}_t(\pi_\rho(A)) = \pi_\rho(\tau_t(A))$$
for all $A \in \mathcal{A}$ and $t \in \mathbb{R}$. Moreover, $\tilde{\rho}$ is a $(\tilde{\tau}, \beta)$-KMS state on $\mathcal{M}_\rho$.

**Proof** Refer to corollary 5.3.4 in [Rob 81].

**Proposition 2.3.2.3** If $\rho$ is the only state satisfying the $(\tau, \beta)$-KMS condition, then $\rho$ is a primary state.

**Proof** Vide corollary 4.14 in [Hug 72].

**Proposition 2.3.2.4** An extremal invariant state $\rho$, which satisfies the $(\tau, \beta)$-KMS condition is primary.

**Proof** See corollary 4.15 in [Hug 72].

Let $(\mathcal{A}, \tau)$ be a $C^*$-dynamical system. If $\mathcal{E}$ be the set of states of the $C^*$-algebra $\mathcal{A}$, then an extremal invariant state is an extreme point of the convex set $\mathcal{E}$, which is invariant under the action of the automorphism group $\tau$.

Some algebraic properties of a KMS state and that of its associated representation are as follows:

1. If $\rho$ is a $(\tau, \beta)$-KMS state, then $\rho(\tau_t A) = \rho(A)$.

2. The sets $I_1 = \{A \in \mathcal{A} | \rho(A^*A) = 0\}$ and $I_2 = \{A \in \mathcal{A} | \rho(AA^*) = 0\}$ are identical and form a two sided ideal.

3. If $(\mathcal{H}_\rho, \pi_\rho, \Theta_\rho)$ is the cyclic representation of $\mathcal{A}$ associated with the state $\rho$, then the von Neumann algebra $\pi_\rho(\mathcal{A})''$ has a cyclic and separating vector in $\Theta_\rho$. 

30
2.3.3 Arveson Spectrum

Let $\mathcal{A}$ be a $C^*$-algebra and $\tau_t$ a strongly continuous, one-parameter group of automorphisms of $\mathcal{A}$. Now, the Bochner integral

$$\int_{-\infty}^{\infty} f(t)\tau_t(A)dt = \Gamma(f)A; \quad A \in \mathcal{A}, f \in L^1(\mathbb{R}),$$

defines a representation $\Gamma$ of $L^1(\mathbb{R})$ into the bounded operators on $\mathcal{A}$. Then the Arveson spectrum $Sp(\tau)$ of $\tau$ is given by

$$Sp(\tau) = \{s \in \mathbb{R} : \hat{f}(s) = 0, \forall f \in \ker \Gamma\}.$$

**Proposition 2.3.3.1** If $\tau$ is a strongly continuous, one-parameter group of automorphisms of a $C^*$-algebra $\mathcal{A}$, then the following statements are equivalent:

1. $s \in Sp(\tau)$.

2. For every $f$ in $L^1(\mathbb{R})$ we have $|\hat{f}(s)| \leq \|\Gamma(f)\|$.

3. If $f \in L^1(\mathbb{R})$ such that $\Gamma(f) = 0$ then $\hat{f}(s) = 0$.

**Proof** Refer to proposition 8.1.9 in [Ped 79].

The last section deals with the theory of direct integrals and decompositions.

2.4 Standard Results in the Theory of Direct Integrals and Decompositions

All the results listed here can be found in [Dix 81].
2.4.1 Measurable Vector Fields

Let $\Omega$ be a Borel space and $\mu$ a finite measure on $\Omega$. A mapping $\omega \mapsto \mathcal{H}_{\omega}$ on $\Omega$, such that $\mathcal{H}_{\omega}$ is a Hilbert space for every $\omega \in \Omega$, with inner product $\langle.,.\rangle_{\omega}$, is called a field of complex Hilbert spaces. Now let $\mathcal{F}$ be the collection of all mappings $\omega \mapsto x(\omega)$ such that, $x(\omega) \in \mathcal{H}_{\omega}$. Such a mapping is called a vector field. It is clearly seen that $\mathcal{F}$ is a complex vector space.

**Definition 2.4.1.1** Let $\omega \mapsto \mathcal{H}_{\omega}$ be a field of complex Hilbert spaces over $\Omega$ and $\mathcal{F}$ the vector space of vector fields. We say that $\omega \mapsto \mathcal{H}_{\omega}$ is a $\mu$-measurable field of Hilbert spaces if there is given a subspace $\mathcal{K}$ of $\mathcal{F}$ having the following properties:

1. For every $x \in \mathcal{K}$, the function $\omega \mapsto \|x(\omega)\|$ is $\mu$-measurable;

2. If $y \in \mathcal{F}$ is such that, for every $x \in \mathcal{K}$, the function $\omega \mapsto \langle x(\omega), y(\omega) \rangle_{\omega}$ is $\mu$-measurable, then, $y \in \mathcal{K}$;

3. There exists a sequence $\{x_1, x_2, \ldots\}$ of elements of $\mathcal{K}$, such that, for every $\omega \in \Omega$, the $x_n(\omega)$'s form a total sequence in $\mathcal{H}_{\omega}$.

The vector fields belonging to $\mathcal{K}$ are then called $\mu$-measurable vector fields. A sequence $\{x_1, x_2, \ldots\}$ of $\mu$-measurable vector fields possessing property (3) is called a fundamental sequence of $\mu$-measurable vector fields. In fact property (3) implies that the $\mathcal{H}_{\omega}$'s are separable.

Hence, it is easily seen that if $x$ and $y$ are measurable vector fields then, $\omega \mapsto \langle x(\omega), y(\omega) \rangle_{\omega}$ is a measurable function of $\omega$. By property (2) of the
above definition, the product of a measurable vector field with a complex valued measurable function is a measurable vector field. The same property also implies that the weak limit of a sequence of measurable vector fields which converges at each point of $\Omega$ is a measurable vector field.

**Proposition 2.4.1.2** Let $\Omega$ be a Borel space, $\mu$ a finite measure and $\omega \mapsto \mathcal{H}_\omega$ a measurable field of Hilbert spaces.

1. The set $\Omega_p$ of all $\omega \in \Omega$ such that the dimension $d(\omega)$ of $\mathcal{H}_\omega$ is equal to $p$ is measurable.

2. There exists a sequence $\{y_1, y_2, \ldots\}$ of measurable vector fields possessing the following properties:

   (a) if $d(\omega) = \aleph$, $\{y_1(\omega), y_2(\omega), \ldots\}$ is an orthonormal basis of $\mathcal{H}_\omega$;

   (b) if $d(\omega) < \aleph$, $\{y_1(\omega), y_2(\omega), \ldots, y_{d(\omega)}(\omega)\}$ is an orthonormal basis of $\mathcal{H}_\omega$, and $y_i(\omega) = 0$ for all $i > d(\omega)$.

**Proof.** See proposition 1 in [Dix 81] (Chapter 1 of Part II).

**Definition 2.4.1.3** A sequence $\{y_1, y_2, \ldots\}$ of measurable vector fields having the properties listed in (2), of the above proposition, is called a measurable field of orthonormal bases.

**Proposition 2.4.1.4** Let $\{x_1, x_2, \ldots\}$ be a fundamental sequence of measurable fields. For a vector field $x$ over $\Omega$ to be measurable, it is necessary and sufficient that the functions $\omega \mapsto \langle x(\omega), x_i(\omega) \rangle_\omega$ be measurable.
Proof Vide proposition 2 in [Dix 81] (Chapter 1 of Part II).

Proposition 2.4.1.5 Let $\Omega$ be a Borel space, $\mu$ a finite measure on $\Omega$, and $\omega \mapsto \mathcal{H}_\omega$ a field of Hilbert spaces over $\Omega$. Let $\{x_1, x_2, \ldots\}$ be a sequence of vector fields having the following properties:

1. The functions $\omega \mapsto (x_i(\omega), x_j(\omega))_\omega$ are measurable;

2. For every $\omega \in \Omega$, the $x_i(\omega)$ form a total sequence in $\mathcal{H}_\omega$.

Then, there exists exactly one measurable field structure on the $\mathcal{H}_\omega$'s such that the fields $x_i$ are measurable.

Proof Vide proposition 4 in [Dix 81] (Chapter 1 of Part II).

2.4.2 Square Integrable Vector Fields

Let $\omega \mapsto \mathcal{H}_\omega$ be a $\mu$-measurable field of complex Hilbert spaces over $\Omega$. A vector field $x$ is said to be square integrable, if it is measurable and if,

$$\int_\Omega \|x(\omega)\|^2 d\mu(\omega) < \infty.$$ 

The set of square integrable fields is a complex vector space $\mathcal{N}$. For $x, y \in \mathcal{N}$, $(x(\omega), y(\omega))_\omega$ is an integrable function of $\omega$. On putting

$$\langle x, y \rangle = \int_\Omega (x(\omega), y(\omega))_\omega d\mu(\omega),$$

the space $\mathcal{N}$ is endowed with a complex pre-Hilbert space structure. We have for $x \in \mathcal{N}$,

$$\|x\|^2 = \int_\Omega \|x(\omega)\|^2 d\mu(\omega).$$
Thus, the $x \in N$ such that $\|x\| = 0$, are just those $x$'s which vanish almost everywhere. We will identify two elements of $N$ which are equal almost everywhere. In other words, we consider the pre-Hilbert space $\mathcal{H}$ associated with $N$. The elements of $\mathcal{H}$ may be regarded as vector fields. For $x \in \mathcal{H}$ we may therefore speak of the values $x(\omega) \in \mathcal{H}_\omega$. It should be noted that the $x(\omega)$'s are determined to within negligible sets.

**Proposition 2.4.2.1** $\mathcal{H}$ is a Hilbert space.

**Proof** Refer to proposition 5 in [Dix 81] (Chapter 1 of Part II). △

**Definition 2.4.2.2** The space $\mathcal{H}$ is called the direct integral of the $\mathcal{H}_\omega$'s and is denoted by $\int_\Omega \mathcal{H}_\omega d\mu(\omega)$.

**Proposition 2.4.2.3** Let $\{y_1, y_2, \ldots\}$ be a measurable field of orthonormal bases. Let $x$ be a vector field. Then $x \in \mathcal{H}$ if and only if the functions $\omega \mapsto \langle x(\omega), y_i(\omega) \rangle_\omega$ are square integrable and

$$\sum_{i=1}^{\infty} \int_\Omega |\langle x(\omega), y_i(\omega) \rangle_\omega|^2 d\mu(\omega) < \infty.$$  

**Proof** Refer to proposition 6 in [Dix 81] (Chapter 1 of Part II). △

**Proposition 2.4.2.4** Let $\omega \mapsto \mathcal{H}_\omega$ be a $\mu$-measurable field of complex Hilbert spaces over $\Omega$ and $\{x_i\}$ a fundamental sequence of measurable vector fields. For every measurable vector field $\omega \mapsto x(\omega)$, there exists a sequence of vector fields of the form

$$\omega \mapsto \sum_{i=1}^{n} f_i(\omega)x_i(\omega),$$
where the $f_i(\omega)$'s are measurable complex valued functions on $\Omega$, which converge to $x(\omega)$ almost everywhere on $\Omega$.

**Proof** Vide problem 3 in [Dix 81] (Page (176)).  

### 2.4.3 Measurable Fields of Operators

Let $\Omega$ be a Borel space, $\mu$ a finite measure on $\Omega$, and $\omega \mapsto \mathcal{H}_\omega$ a $\mu$-measurable field of complex Hilbert spaces over $\Omega$. For every $\omega \in \Omega$, let $T(\omega)$ be an element of $\mathcal{L}(\mathcal{H}_\omega)$, i.e., a bounded linear operator on $\mathcal{H}_\omega$. Then, the mapping $\omega \mapsto T(\omega)$ is called a field of bounded linear operators over $\Omega$.

**Definition 2.4.3.1** The field of bounded linear operators $\omega \mapsto T(\omega)$ is said to be measurable if, for every measurable vector field $\omega \mapsto x(\omega) \in \mathcal{H}_\omega$, the vector field $\omega \mapsto T(\omega)x(\omega) \in \mathcal{H}_\omega$ is measurable.

**Proposition 2.4.3.2** Let $\{x_1, x_2, \ldots\}$ be a fundamental sequence of measurable vector fields with values in the $\mathcal{H}_\omega$'s. For the field $\omega \mapsto T(\omega)$ to be measurable, it is necessary and sufficient that the functions $\omega \mapsto (T(\omega)x_i(\omega), x_j(\omega))_\omega$ be measurable.

**Proof** Refer to proposition 1 in [Dix 81] (Chapter 2 of Part II).

Let $\omega \mapsto \mathcal{H}_\omega$ be a $\mu$-measurable field of complex Hilbert spaces over $\Omega$. Let

$$\mathcal{H} = \int_\Omega \mathcal{H}_\omega d\mu(\omega).$$

A measurable field of bounded linear operators $\omega \mapsto T(\omega) \in \mathcal{L}(\mathcal{H}_\omega)$ is said to be essentially bounded if the essential supremum $M$ of the function $\omega \mapsto$
\(\|T(\omega)\|\) is finite. If this is the case, for every square integrable vector field \(x\), the vector field \(\omega \mapsto x'(\omega) = T(\omega)x(\omega)\) is also a square integrable vector field and we have \(\|T\| \leq M\). Thus \(x \mapsto x'\) establishes a correspondence \(T : \mathcal{H} \to \mathcal{H}\) such that, \(T\) is a bounded linear operator on \(\mathcal{H}\) with \(\|T\| \leq M\).

**Proposition 2.4.3.3** We have \(\|T\| = M\).

**Proof** Refer to proposition 2 in [Dix 81] (Chapter 2 of PartII). \(\triangle\)

This proposition yields the following corollary.

**Corollary 2.4.3.4** If two essentially bounded measurable fields of bounded linear operators define the same element of \(L(\mathcal{H})\), they are equal almost everywhere.

**Definition 2.4.3.5** An operator \(T \in L(\mathcal{H})\) is said to be decomposable, if it is defined by an essentially bounded measurable field of operators \(\omega \mapsto T(\omega)\). We then write

\[
\int_{\Omega}^{\oplus} T(\omega) d\mu(\omega).
\]

It follows from the corollary that the \(T(\omega)\)'s may be defined upto negligible subsets of \(\Omega\). In particular, given a point \(\omega \in \Omega\) of measure zero, \(T(\omega)\) may be chosen arbitrarily.

**Proposition 2.4.3.6** Let \(T_1, T_2\) be decomposable operators. If

\[
T_1 = \int_{\Omega}^{\oplus} T_1(\omega) d\mu(\omega) \quad \text{and} \quad T_2 = \int_{\Omega}^{\oplus} T_2(\omega) d\mu(\omega),
\]

37
we have

\[ T_1 + T_2 = \int_\Omega (T_1(\omega) + T_2(\omega))d\mu(\omega), \quad T_1 T_2 = \int_\Omega T_1(\omega)T_2(\omega)d\mu(\omega) \]

and

\[ \lambda T_1 = \int_\Omega \lambda T_1(\omega)d\mu(\omega), \quad T_1^* = \int_\Omega T_1^*(\omega)d\mu(\omega). \]

**Proof** Vide proposition 3 in [Dix 81] (Chapter 2 of Part II).

**Proposition 2.4.3.7** Let

\[ T_i = \int_\Omega T_i(\omega)d\mu(\omega) \quad (i = 1, 2, \ldots) \]

and

\[ T = \int_\Omega T(\omega)d\mu(\omega) \]

be decomposable operators.

1. If \( T_i \) converges strongly to \( T \), then there exists a subsequence \( T_{n_k} \) such that, \( T_{n_k}(\omega) \) converges strongly to \( T(\omega) \) almost everywhere.

2. If \( T_i(\omega) \) converges strongly to \( T(\omega) \) almost everywhere and if

\[ \sup_i \| T_i \| < \infty, \] then \( T_i \) converges strongly to \( T \).

**Proof** See proposition 4 in [Dix 81] (Chapter 2, Part II).

Let \( L^\infty(\Omega, \mu) \) be the set of essentially bounded, complex valued measurable functions on \( \Omega \), in which we identify any two functions which are equal almost everywhere. If \( f \in L^\infty(\Omega, \mu) \), then the field of operators \( \omega \mapsto f(\omega)I \in \mathcal{L}(\mathcal{H}_\omega) \) is measurable and essentially bounded. Let \( T_I \) be the corresponding operator of \( \mathcal{H} \).
Definition 2.4.3.8 The operators of the form $T_f$, where $f \in L^\infty(\Omega, \mu)$ are said to be diagonalisable.

If $Z$ denotes all such operators, then, $Z$ is a $*$-algebra of $\mathcal{L}(\mathcal{H})$.

Proposition 2.4.3.9 The algebra $Z$ is an abelian von Neumann algebra and $Z'$ is $\sigma$-finite.

Proof Refer to proposition 7 in [Dix 81] (Chapter 2, Part II).

2.4.4 Measurable Fields of von Neumann Algebras

In this section, $\Omega$ will continue to be a Borel space, $\mu$ a finite measure on $\Omega$ and $\omega \mapsto \mathcal{H}_\omega$ a $\mu$-measurable field of complex Hilbert spaces. For every $\omega \in \Omega$, let $A_\omega$ be a von Neumann algebra on $\mathcal{H}_\omega$. The mapping $\omega \mapsto A_\omega$ is called a field of von Neumann algebras.

Definition 2.4.4.1 A field of von Neumann algebras $\omega \mapsto A_\omega$ over $\Omega$ is said to be measurable, if there exists a sequence $\omega \mapsto T_1(\omega), \omega \mapsto T_2(\omega), \ldots$ of measurable fields of operators such that, $A_\omega$ is the von Neumann algebra generated by the $T_i(\omega)$'s almost everywhere.

Proposition 2.4.4.2 Let $\omega \mapsto A_\omega$ be a measurable field of von Neumann algebras. The set $M$ of decomposable operators

$$\int_\Omega T(\omega) d\mu(\omega),$$

such that $T(\omega) \in A_\omega$ almost everywhere, is a von Neumann algebra on $\mathcal{H}$ such that, $Z \subseteq M \subseteq Z'$. Moreover $M$ is generated by $Z$ and a countable family of elements $\{T_i\}$, where the $T_i(\omega)$'s generate $A_\omega$ for almost every $\omega$. 39
**Proof**  See proposition 1 in [Dix 81] (Chapter 2, Part II).  \(\triangle\)

**Definition 2.4.4.3** A von Neumann algebra \(\mathcal{M}\) on a Hilbert space \(\mathcal{H}\) is said to be decomposable, if it is defined by a measurable field of \(\omega \mapsto \mathcal{A}_\omega\) of von Neumann algebras. We then write

\[
\mathcal{M} = \int_\Omega \mathcal{A}_\omega d\mu(\omega).
\]

The \(\mathcal{A}_\omega\)'s are defined by \(\mathcal{M}\) to within negligible sets.

**Theorem 2.4.4.4** For a von Neumann algebra \(\mathcal{M}\) to be decomposable it is necessary and sufficient that it be the von Neumann algebra generated by \(\mathcal{Z}\) and a countable family of decomposable operators.

**Proof**  Vide theorem 2 in [Dix 81] (Chapter 2, Part II).  \(\triangle\)