Chapter 5

Summary

This final chapter of the thesis is devoted to a discussion on the results obtained in chapters (3) and (4) and their implications. Some of the open problems which remain unresolved are identified. Before proceeding any further it is worth recalling the aims and objectives of the work undertaken here.

The purpose of this investigation was to understand and explain the behaviour of a quantum spin glass through its dynamics. Spin glasses have always been something of a mystery. They are among the least understood systems even in equilibrium statistical mechanics. In particular, their low temperature regime and critical behaviour are extremely complex.

Traditionally, quantum spin glasses have been studied as systems of quantum spins interacting through random interactions. These models are essentially Ising-type models with random coupling. The coupling coefficients are assumed to be independent, identically distributed random variables. Extensive investigations on the existence of the thermodynamic limit have been made by van Hemmen et al [Ent 83, Hem 83]. The almost sure existence of
the free energy of an infinite spin system on a lattice with random interactions has been established. This is a generalization of the result of Khanin and Sinai [Sin 79] in the classical case. An alternate model of a quantum spin glass can be based on the realization that, the magnetic ions in a spin glass are randomly distributed at lattice sites. The spins therefore may be considered to be located at the vertices of an infinite connected graph with countably infinite number of vertices. Here, one caricatures a quantum spin glass as a quantum spin system on an infinite connected graph with countably infinite number of vertices. This model may be regarded as a quantum analogue of the systems studied by Preston and others [Pre 74]. But, inspite of the fact that a quantum spin glass admits a natural dynamics, this aspect has not been investigated.

In this thesis, we have attempted the study of the dynamics of a quantum spin glass with the help of both these models, namely, a quantum spin system on an infinite connected graph having countably infinite number of vertices with deterministic interactions of the nearest neighbour type and a quantum spin system on an infinite lattice \( \mathbb{Z}^d \) with random interactions. The problem to which we have addressed ourselves is that of explaining the behaviour of a quantum spin glass through the dynamics of these spin systems and the associated KMS states.

In the case of the quantum spin system on an infinite graph, the global dynamics has been established. This was achieved by constructing a strongly continuous, one-parameter group of \( \star \)-automorphisms \( \tau_t \) of the quasi-local
algebra $\mathcal{A}$ associated with the spin system. As expected, the existence of an equilibrium state which is by no means unique, has been established. The equilibrium state $\rho$ was obtained as the thermodynamic limit of the local Gibbs states $\rho_\Lambda$. It was also shown that $\rho$ satisfies the Kubo-Martin-Schwinger (KMS) condition with respect to the time evolution group $\tau_t$.

However, all attempts to establish the maximum entropy principle for the infinite spin system were thwarted due to the absence of spatial homogeneity. In fact, one failed to establish the existence of mean entropy and free energy for the infinite system. The problem of establishing the existence of mean entropy and free energy for the infinite system as well as that of establishing the maximum entropy principle remains open.

The other model studied was a quantum spin system on an infinite lattice $\mathbb{Z}^\nu$, with random interactions. Here we have established the existence of a family of strongly continuous, one-parameter groups of $\ast$-automorphisms $\{\tau_t(\omega)\}$ of the quasi-local algebra $\mathcal{A}$ associated with the spin system, where $\omega$ lives in a probability space $(\Omega, S, P)$. These automorphism groups $\tau_t(\omega)$ determine the evolution of the infinite spin system. The joint measurability of the map $(t, \omega) \mapsto \tau_t(\omega)(A)$ for all $A \in \mathcal{A}$, has been proved. Some interesting algebraic properties of the generator $\delta(\omega)$ of these automorphism groups have been derived. The notion of ergodicity of a measure preserving group of automorphisms of $\Omega$, is used to prove the almost sure independence of the Arveson spectrum $Sp(\tau(\omega))$ of the evolution group $\tau_t(\omega)$. Next, the existence of a family of $(\tau(\omega), \beta)$-KMS states $\{\rho(\omega)\}$ has been established.
for all $\beta \in IR \setminus \{0\}$. They have been shown to satisfy the condition $\rho(\omega)(A) = \rho(T_{-\alpha}(a\omega))(A)$ for all $A \in A$ and all $a \in \mathbb{Z}$, where $\alpha$ is the action of the lattice $\mathbb{Z}$ on the quasi-local algebra $A$. We assume that there exists one such family of $(\tau(\omega), \beta)$-KMS states $\{\rho(\omega)\}$, where $\omega \mapsto \rho(\omega)(A)$ is measurable for all $A \in A$. It has been shown that the spin system on an infinite lattice with random interactions exhibits a phase structure. In fact, it has been established that there exists an unique KMS state $\rho(\omega)$, above a certain a critical temperature $T_c$ almost surely independent of $\omega$. There is a close connection between the Arveson spectrum of $\tau(\omega)$, and the spectrum of the generator of the unitary group $U_t(\omega)$ which implements $\tau(\omega)$ in the cyclic representation $\pi_\omega$ induced by the $(\tau(\omega), \beta)$-KMS state $\rho(\omega)$. This fact has been exploited to prove the almost sure independence of the spectrum of the generator of $U_t(\omega)$.

Now, the cyclic representations $\pi_\omega$ induced by the $(\tau(\omega), \beta)$-KMS states $\rho(\omega)$, which satisfy the conditions mentioned above gives rise to an ensemble of von Neumann algebras $\{\pi_\omega(A)''\}$, where each of these von Neumann algebras acts on a separable Hilbert space $H_\omega$. As these von Neumann algebras correspond to distinct realizations of the quasi-local algebra $A$, they are treated as distinct objects. This establishes a need to invoke the theory of measurable fields of von Neumann algebras. Using the cyclicity of $\pi_\omega$, we have constructed a collection of measurable vector fields $F$, which endows the field of separable Hilbert spaces $\omega \mapsto H_\omega$ with a measurable structure. Equipped with this structure, we have shown that for each $t \in IR$, $\omega \mapsto U_t(\omega)$
is a measurable field of unitary operators. Further, the joint measurability of \((t, \omega) \mapsto \langle U_t(\omega), \xi(\omega), \eta(\omega) \rangle_\omega\) for all \(\xi, \eta \in \mathcal{F}\) is established. We have also derived some interesting ergodic properties of the spectra of generators \(H(\omega)\) of the unitary groups \(U_t(\omega)\).

In the final part of the thesis we have constructed a direct integral \(\mathcal{M}\) from the measurable field of von Neumann algebras \(\omega \mapsto \pi_\omega(\mathcal{A})''\). The existence of a strongly continuous, one-parameter group of unitaries \(U_t\) on the direct integral Hilbert space \(\mathcal{H}\) constructed from the measurable field of Hilbert spaces \(\omega \mapsto \mathcal{H}_\omega\), has been established. This group of unitaries in turn gives rise to a \(\sigma\)-weakly continuous group of automorphisms \(\tau_t\) of \(\mathcal{M}\). From the measurable field of KMS states \(\omega \mapsto \rho(\omega)\), which are extensions of the KMS states \(\rho(\omega)\) to the von Neumann algebras \(\{\pi_\omega(\mathcal{A})''\}\), a faithful normal \((\tau, \beta)\)-KMS state \(\hat{\rho}\) of \(\mathcal{M}\) has been constructed.

The problem that remains to be resolved in this particular model is that of establishing that the transport coefficients of the spin system are almost surely constant. One would expect this to be generally true on physical grounds.

The other problem that remains open is that of establishing a connection between the spectra of the generator of \(U_t(\omega)\) and that of the generator of the unitary group \(U_t\) on the direct integral Hilbert space \(\mathcal{H}\).