Chapter 4

Ergodic Properties of Spectra of Evolution Groups

4.1 Arveson Spectrum

Here we introduce the notion of Arveson spectrum.

Let $X$ be a Banach space and $X_*$ a linear subspace of the dual $X^*$ of $X$ such that, $\|x\| = \sup\{|\rho(x)| : \rho \in X_*, \|\rho\| \leq 1\}$ for every $x \in X$. Let $B(X)$, $(B_w(X))$ denote the algebra of all bounded ($\sigma(X, X_*)$–continuous) linear operators on $X$. As usual, denote the convolution group algebra of the additive group of real numbers $\mathbb{R}$, by $L^1(\mathbb{R})$. A representation of $\mathbb{R}$ on $X$ is a homomorphism $t \mapsto V_t$ of $\mathbb{R}$ into the group of all invertible elements of $B_w(X)$ such that, $\sup_t \|V_t\| < \infty$ and for each $x \in X$, the map $t \mapsto V_t x$ is $\sigma(X, X_*)$–continuous. Now, if for every $x \in X$, there is an unique vector $y$ defined by

$$\int_{-\infty}^{\infty} f(t)\rho(V_t x)dt = \rho(y); \quad \rho \in X_*, f \in L^1(\mathbb{R}),$$

then we obtain an operator $\Gamma(f)$ defined by $\Gamma(f)x = y$. Therefore, we have a representation $\Gamma$, of $L^1(\mathbb{R})$ in $B(X)$, associated with $V$. 

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Definition 4.1.0.34  The Arveson spectrum $SpV$ of $V$ is a subset of the dual group $\hat{\mathbb{R}}$ of $\mathbb{R}$ defined as

$$SpV = \{\sigma \in \hat{\mathbb{R}}|\hat{f}(\sigma) = 0, \forall f \in \ker\Gamma\},$$

where $\hat{f}$ is the Fourier transform of $f$.

If $\mathcal{A}$ is a $C^*$-algebra and $\tau_t$ a strongly continuous, one-parameter group of automorphisms of the $C^*$-algebra, then the Bochner integral

$$\int_{-\infty}^{\infty} f(t)\tau_t(A)dt = \Gamma(f)A; \quad A \in \mathcal{A}, \ f \in L^1(\mathbb{R}),$$

defines a representation of $L^1(\mathbb{R})$ into the bounded operators on $\mathcal{A}$. Now, on applying the foregoing definition in this case, the Arveson spectrum $Sp(\tau)$ of $\tau$ is given by

$$Sp(\tau) = \{s \in \mathbb{R} : \hat{f}(s) = 0, \forall f \in \ker\Gamma\}.$$

It can be shown that $s \in Sp(\tau)$, if and only if, $|\hat{f}(s)| \leq \|\Gamma(f)\|$, for all $f \in L^1(\mathbb{R})$ (Proposition 8.1.9 in [Ped 79]).

Our aim is to show that the Arveson spectrum of the evolution group $\tau_t(\omega)$ is almost surely constant. To this end, we have the following theorem.

**Theorem 4.1.0.35** Let $\tau_t(\omega)$ be the strongly continuous, one-parameter group of automorphisms of $\mathcal{A}$, which determines the evolution of the spin system. Then, the Arveson spectrum $Sp(\tau(\omega))$ of $\tau_t(\omega)$ is almost surely constant.

**Proof** For $s \in \mathbb{R}$, let $T_s = \{\omega : \|\Gamma(\omega)(f)\| \geq |\hat{f}(s)| \forall f \in L^1(\mathbb{R})\}$, where

$$\Gamma(\omega)(f)(A) = \int_{-\infty}^{\infty} f(t)\tau_t(\omega)(A)dt, \quad \forall A \in \mathcal{A}.$$
We show that $T_s$ is a measurable subset of $\Omega$. Since $L^1(\mathbb{R})$ is separable, there exists a countable dense set $F = \{f_n | n \in \mathbb{Z}^+\}$ in $L^1(\mathbb{R})$. Hence, for each $f \in L^1(\mathbb{R})$, there exists a sequence $f_{n_k}$ in $F$, converging to $f$ in the $L^1$-norm. Therefore,

$$||\Gamma(\omega)(f_{n_k})|| - ||\Gamma(\omega)(f)|| \leq ||\Gamma(\omega)(f_{n_k}) - \Gamma(\omega)(f)||$$

$$\leq \sup_{||A||=1} ||\Gamma(\omega)(f_{n_k} - f)(A)||$$

$$\leq \sup_{||A||=1} \left| \int_{-\infty}^{\infty} (f_{n_k} - f)(t)\tau_t(\omega)(A)dt \right|$$

$$\leq \sup_{||A||=1} \left( \int_{-\infty}^{\infty} |(f_{n_k} - f)(t)|\tau_t(\omega)(A)dt \right)$$

$$\leq \sup_{||A||=1} \left( ||A|| \int_{-\infty}^{\infty} |(f_{n_k} - f)(t)|dt \right)$$

$$\leq \int_{-\infty}^{\infty} |(f_{n_k} - f)(t)|dt$$

$$\leq ||f_{n_k} - f||_1$$

Therefore, $||\Gamma(\omega)(f_{n_k})||$ converges to $||\Gamma(\omega)(f)||$, for $f_{n_k}$ converging to $f$, in the $L^1$-norm. In view of this, and the fact that $F$ is dense in $T_s$, we have

$$T_s = \bigcap_{n=1}^{\infty} T_s^n,$$

where $T_s^n = \{\omega | ||\Gamma(\omega)(f_n)|| \geq |\hat{f}_n(s)|\}$. In order to show that each of these $T_s^n$'s is a measurable subset of $\Omega$, it is sufficient to establish the measurability of the function $\omega \mapsto ||\Gamma(\omega)(f_n)||$, for all $n \in \mathbb{Z}^+$. On appealing to proposition 3.7.0.29, we conclude that for $f \in L^1(\mathbb{R})$ and $A \in \mathcal{A}$, $(t, \omega) \mapsto f(t)\tau_t(\omega)(A)$
is strongly, jointly measurable in $t$ and $\omega$. Moreover,

$$\int_{\mathbb{R}\times\Omega} \|f(t)\tau_t(\omega)(A)\|\,d(\mu \times P)(t,\omega) = \int_{\mathbb{R}\times\Omega} |f(t)|\|\tau_t(\omega)(A)\|\,d(\mu \times P)(t,\omega)$$

$$= \int_{\mathbb{R}} \int_{\Omega} \|A\||f(t)|\,d\mu(t)\,dP(\omega) < \infty.$$ 

Hence, it follows from theorem 3.7.4 in [Hil 57] that, $(t,\omega) \mapsto f(t)\tau_t(\omega)(A)$ is Bochner integrable on $\mathbb{R}\times\Omega$. Therefore, as a consequence of the analogue of Fubini's theorem for vector valued functions (Proposition 3.7.13, [Hil 57]), the map $\omega \mapsto \Gamma(\omega)(f)(A)$ is strongly measurable in $\omega$. Hence, $\omega \mapsto \|\Gamma(\omega)(f)(A)\|$ is a measurable, real valued function on $\Omega$. Thus it readily follows that for $f \in L^1(\mathbb{R})$, $\omega \mapsto \|\Gamma(\omega)(f)(A)\|$ is measurable for all $A \in \mathcal{A}$. Now, $\mathcal{A}$ being a separable $C^*$-algebra, we have for $c \in \mathbb{R}$ and $f \in L^1(\mathbb{R})$,

$$\{\omega | \|\Gamma(\omega)(f)\| \leq c\} = \bigcap_{n \in \mathbb{Z}^+} \{\omega \in \Omega | \|\Gamma(\omega)(f)(A_n)\| \leq c; \|A_n\| \leq 1\},$$

where $\mathcal{U}_0 = \{A_n \in \mathcal{A} | n \in \mathbb{Z}^+\}$ is a dense subset of the closed unit ball in $\mathcal{A}$. This identity, coupled with the fact that $\omega \mapsto \|\Gamma(\omega)(f)(A_n)\|$ is a measurable function of $\omega$ for all $n \in \mathbb{Z}^+$, permits us to conclude that the set $\{\omega | \|\Gamma(\omega)(f)\| \leq c\}$ is a measurable subset of $\Omega$. Since $c$ is arbitrary, the function $\omega \mapsto \|\Gamma(\omega)(f)\|$ is a measurable function of $\omega$. Thus, $\omega \mapsto \|\Gamma(\omega)(f)\|$ is measurable for all $f \in L^1(\mathbb{R})$. Therefore, $\omega \mapsto \|\Gamma(\omega)f_n\|$ is measurable $\forall n \in \mathbb{Z}^+$. Hence, each of these $T^*_n$'s is a measurable subset of $\Omega$. This proves conclusively that the set $T^*_n$'s is a measurable subset of $\Omega$. Now, using the fact that the action of the measure preserving group of automorphisms is ergodic, we show that $T^*_n$ has a measure either zero or one. It follows from
the properties of the Bochner integral [Hil 57] (Chapter 3) and the fact that
\( \alpha_s \) is a \(^\ast\)-automorphism of the \( C^* \)-algebra \( A \) that, for \( f \in L^1(\mathbb{R}) \),
\[
\| \Gamma(\omega)(f) \| = \sup_{\| A \|=1} \| \Gamma(\omega)(f)(A) \|
\]
\[
= \sup_{\| A \|=1} \left\| \int_{-\infty}^{\infty} f(t) \tau_t(\omega)(A)dt \right\|
\]
\[
= \sup_{\| A \|=1} \left\| \alpha_s \left( \int_{-\infty}^{\infty} f(t) \tau_t(\omega)(A)dt \right) \right\|
\]
\[
= \sup_{\| A \|=1} \left\| \int_{-\infty}^{\infty} f(t) \tau_t(T_{-a}\omega)(\alpha_a(A))dt \right\|
\]
for all \( a \in \mathbb{Z}^\nu \). The last equality follows from proposition 3.7.0.31. Consequently, we have
\[
\| \Gamma(\omega)(f) \| = \sup_{\| A \|=1} \| \Gamma(T_{-a}\omega)(f)(\alpha_a(A)) \|
\]
\[
= \| \Gamma(T_{-a}\omega)(f) \|
\]
for all \( a \in \mathbb{Z}^\nu \) and \( f \in L^1(\mathbb{R}) \). Therefore, for all \( a \in \mathbb{Z}^\nu \), \( \| \Gamma(\omega)(f) \| = \| \Gamma(T_{-a}\omega)(f) \| \), for \( f \in L^1(\mathbb{R}) \). Hence, as the action of the measure preserving group of automorphisms is assumed to be ergodic, it is clear from the above equality that \( T_s \) is an invariant measurable subset of \( \Omega \) and therefore, the set \( T_s \) has measure either zero or one. Hence, \( s \) lies in the Arveson spectrum of \( \tau_t(\omega) \) with probability either zero or one. Thus, one concludes that the Arveson spectrum \( Sp(\tau(\omega)) \) of \( \tau_t(\omega) \) is almost surely constant. \( \Delta \)

### 4.2 KMS States

In this section we analyse the KMS states of the spin system on a lattice with random interactions. The following definition of a KMS state has been
Definition 4.2.0.36 Let \((A, \tau)\) be a C*-dynamical system, or a \(W^*\)-dynamical system and \(\rho\) a state over \(A\) which is assumed to be normal in the \(W^*\) case. Then, \(\rho\) is said to be a \((\tau, \beta)\)-KMS state if, for \(\beta > 0\) and any pair \(A, B \in A\), there exists a complex function \(F_{A,B}\) which is analytic on the open strip \(0 < \Im z < \beta\), uniformly bounded and continuous on the closed strip \(0 \leq \Im z \leq \beta\) such that,

\[
F_{A,B}(t) = \rho(A \tau_t(B)) \quad \text{and} \quad F_{A,B}(t + i\beta) = \rho(\tau_t(B)A).
\]

If \(\beta < 0\), then \(\rho\) is a \((\tau, \beta)\)-KMS state if, there exists a complex function \(F_{A,B}\) which is analytic on the open strip \(\beta < \Im z < 0\), uniformly bounded and continuous for \(\beta \leq \Im z \leq 0\) such that,

\[
F_{A,B}(t) = \rho(A \tau_t(B)) \quad \text{and} \quad F_{A,B}(t + i\beta) = \rho(\tau_t(B)A).
\]

4.2.1 Construction of a Family of KMS States

We know from theorem 3.7.0.28 that, for almost every \(\omega \in \Omega\), there exists a strongly continuous one-parameter group of *-automorphisms \(\tau_t(\omega)\), which determines the evolution of the spin system. Now, for \(\omega \in \Omega\), and \(\beta \in \mathbb{R}\setminus\{0\}\), the local Gibbs state associated with the interaction \(\Phi(,\omega)\) is given by

\[
\rho_A(\omega)(A) = \frac{Tr(e^{-\beta H(\lambda,\omega)}A)}{Tr(e^{-\beta H(\lambda,\omega)})}, \quad \forall A \in \mathcal{A}_A.
\]

Although \(\rho_A(\omega)\) is defined on \(\mathcal{A}_A\), it has an extension as a state to the whole of \(A\). The extension is by no means unique. It follows from proposition
3.4.0.13 in chapter 3, which can be adopted to $\rho_\Lambda(\omega)$ with $\tau^\Lambda(\omega)$ as the local automorphism group that, these states are $(\tau^\Lambda(\omega), \beta)$-KMS states of the finite spin system confined to the region $\Lambda$, where $\tau^\Lambda(\omega)$ are the local automorphism groups. Next, for $\omega \in \Omega$, let

$$O_\omega = \{ T_a \omega \mid a \in \mathbb{Z}^\nu \}.$$ 

Clearly, any two $O_\omega$'s corresponding to distinct $\omega$'s are either disjoint or identical and the $O_\omega$'s form a partition of $\Omega$. Therefore, using the axiom of choice, we pick a subset $\Omega' \subseteq \Omega$, and write the space $\Omega$ as

$$\Omega = \bigcup_{\omega \in \Omega'} O_\omega,$$

where the $O_\omega$'s in the union are pairwise disjoint. Next, for each $\omega \in \Omega'$, we establish the existence of the thermodynamic limit $\rho(T_{-a} \omega)$ of the local Gibbs states $\rho_\Lambda(T_{-a} \omega)$, for all $a \in \mathbb{Z}^\nu$. To this end, we argue as follows.

Since the quasi-local algebra $\mathcal{A}$ is a separable $C^*$-algebra, the collection of states $\mathcal{E}_\mathcal{A}$ of $\mathcal{A}$ is weak*-compact. Therefore, for each $\omega \in \Omega'$ there exists a state $\rho(\omega)$, and a sequence $\{\Lambda_n\}$ of finite subsets of $\mathbb{Z}^\nu$ depending on $\omega$ such that, $\rho(\omega)$ is the weak*-limit of a sequence of extensions $\hat{\rho}_{\Lambda_n}(\omega)$ of $\rho_{\Lambda_n}(\omega)$. That is, for each $\omega \in \Omega'$, there exists a sequence $\{\Lambda_n\}$ of finite subsets of $\mathbb{Z}^\nu$ such that,

$$\lim_{n \to \infty} \hat{\rho}_{\Lambda_n}(\omega)(A) = \rho(\omega)(A); \quad \forall A \in \mathcal{A}.$$ 

In particular,

$$\lim_{n \to \infty} \rho_{\Lambda_n}(\omega)(A) = \rho(\omega)(A),$$

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for all $A \in \mathcal{A}_{\Lambda_0}$ and all finite $\Lambda_0 \subseteq \mathbb{Z}^\nu$. Therefore for each $\omega \in \Omega'$, $\rho(\omega)$ is a weak*-limit point of the net of extensions of $\rho_\Lambda(\omega)$'s to $\mathcal{A}$. Hence, it follows from definition 3.4.0.12, which can be easily adopted to $\rho(\omega)$ with $\rho_\Lambda(\omega)$ as the local Gibbs states, that for each $\omega \in \Omega'$, the state $\rho(\omega)$ is the thermodynamic limit of the local Gibbs states $\{\rho_\Lambda(\omega)\}$. Next, for each $\omega \in \Omega'$ and all $a \in \mathbb{Z}^\nu$, define

$$
\rho(T_{-a}\omega)(A) = \rho(\omega)(\alpha_{-a}(A))
$$

Now, keeping in mind the identity

$$
H(\Lambda_n,\omega) = \alpha_{-a}(H(\Lambda_n+a,T_{-a}\omega)) = V_{\Lambda_n+a}(-a)H(\Lambda_n+a,T_{-a}\omega)V_{\Lambda_n+a}(-a)^{-1},
$$

it follows from function calculus and the invariance property of the trace, that, for each $\omega \in \Omega'$ and all $a \in \mathbb{Z}^\nu$,

$$
\begin{align*}
\rho_{\Lambda_n+a}(T_{-a}\omega)(A) &= \frac{Tr(e^{-\beta H(\Lambda_n+a,T_{-a}\omega)}A)}{Tr(e^{-\beta H(\Lambda_n,a,T_{-a}\omega)})} \\
&= \frac{Tr(V_{\Lambda_n+a}(-a)e^{-\beta H(\Lambda_n+a,T_{-a}\omega)}AV_{\Lambda_n+a}(-a)^{-1})}{Tr(V_{\Lambda_n+a}(-a)e^{-\beta H(\Lambda_n+a,T_{-a}\omega)}V_{\Lambda_n+a}(-a)^{-1})} \\
&= \frac{Tr(\alpha_{-a}(e^{-\beta H(\Lambda_n+a,T_{-a}\omega)})\alpha_{-a}(A))}{Tr(\alpha_{-a}(e^{-\beta H(\Lambda_n+a,T_{-a}\omega})))} \\
&= \frac{Tr(e^{-\beta H(\Lambda_n,\omega)}\alpha_{-a}(A))}{Tr(e^{-\beta H(\Lambda_n,\omega)})} \\
&= \rho_{\Lambda_n}(\omega)(\alpha_{-a}(A)),
\end{align*}
$$

for all $A \in \mathcal{A}_{\Lambda_0}$ and $\Lambda_n \supseteq \Lambda_0$. Hence, for each $\omega \in \Omega'$ and all $a \in \mathbb{Z}^\nu$,

$$
\begin{align*}
\rho(T_{-a}\omega)(A) &= \rho(\omega)(\alpha_{-a}(A)) \\
&= \lim_{n \to \infty} \rho_{\Lambda_n}(\omega)(\alpha_{-a}(A)) \\
&= \lim_{n \to \infty} \rho_{\Lambda_n+a}(T_{-a}\omega)(A),
\end{align*}
$$

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for all \( A \in \mathcal{A}_{\Lambda_0} \) and all finite \( \Lambda_0 \subseteq \mathbb{Z}^\nu \). If for each \( \omega \in \Omega' \) and all \( a \in \mathbb{Z}^\nu \), we define

\[ \hat{\rho}_{\Lambda_{n+a}}(T_{-a}\omega)(A) = \hat{\rho}_{\Lambda_n}(\omega)(\alpha_{-a}(A)); \quad \forall A \in \mathcal{A}, \]

then the states \( \rho(T_{-a}\omega) \) are the weak*-limits of the sequence of extensions \( \{\hat{\rho}_{\Lambda_{n+a}}(T_{-a}\omega)\} \) of the local Gibbs states \( \rho_{\Lambda_{n+a}}(T_{-a}\omega) \), for each \( \omega \in \Omega' \) and all \( a \in \mathbb{Z}^\nu \) such that, \( \rho(T_{-a}\omega)(A) = \rho(\omega)(\alpha_{-a}(A)) \), for all \( A \in \mathcal{A} \). Thus, for each \( \omega \in \Omega' \) and all \( a \in \mathbb{Z}^\nu \), \( \rho(T_{-a}\omega) \) is a weak*-limit point of the net of extensions of \( \rho_{\Lambda}(T_{-a}\omega) \)'s to \( \mathcal{A} \). Hence, for each \( \omega \in \Omega' \) and all \( a \in \mathbb{Z}^\nu \), \( \rho(T_{-a}\omega) \) is the thermodynamic limit of the local Gibbs states \( \rho_{\Lambda}(T_{-a}\omega) \). Since the union of \( O_\omega \)'s, where \( \omega \in \Omega' \), exhausts all the points in \( \Omega \) i.e., \( \Omega = \bigcup_{\omega \in \Omega'} O_\omega \), we have succeeded in establishing the existence of the thermodynamic limit \( \rho(\omega) \) of the local Gibbs states \( \rho_{\Lambda}(\omega) \), for all \( \omega \in \Omega \). It is clear from the above construction that, for all \( \omega \in \Omega \), these states satisfy, \( \rho(\omega)(A) = \rho(T_{-a}\omega)(\alpha_{a}(A)) \), for all \( a \in \mathbb{Z}^\nu \) and \( A \in \mathcal{A} \).

Now, each of these states \( \rho(\omega) \), is a thermodynamic limit of the local Gibbs states \( \rho_{\Lambda}(\omega) \). Hence, each \( \rho(\omega) \) is a weak*-limit point of the extensions of the local Gibbs states \( \rho_{\Lambda}(\omega) \), with \( \tau_\omega \) as the evolution group. Therefore, it follows that \( \rho(\omega) \) is a \((\tau(\omega), \beta)\)-KMS state [Rob 81] (Proposition 5.3.25). Thus, we have succeeded in establishing the existence of a family of \((\tau(\omega), \beta)\)-KMS states \( \{\rho(\omega)\} \), for each \( \beta \in \mathbb{R} \setminus \{0\} \), obtained as the thermodynamic limits of the local Gibbs states \( \rho_{\Lambda}(\omega) \), and satisfying, \( \rho(\omega)(A) = \rho(T_{-a}\omega)(\alpha_{a}(A)) \) for all \( a \in \mathbb{Z}^\nu \).
4.2.2 Uniqueness of KMS States

Next, we shall demonstrate that a quantum spin system on an infinite lattice with random interactions exhibits a phase structure. To this end, we have the following theorem which establishes that there is an unique KMS state \( \rho(\omega) \), associated with the evolution group \( \tau_t(\omega) \), above a certain critical temperature \( T_c \) almost surely independent of \( \omega \). In order to demonstrate this, we use the fact that the function \( \omega \mapsto \sup_{\mathbf{X} \in \mathbb{Z}^\mathbb{R}} (\sum_{\mathbf{X} \in \mathbb{Z}^\mathbb{R}} \Phi(\mathbf{X}, T_\omega)) \) is almost surely constant.

**Definition 4.2.2.1** Let \( \mathcal{N} \) be the matrix algebra of all \( n \times n \) matrices over \( \mathbb{C} \) and \( \{E_{p,q}\} \) be the finite collection of matrices in \( \mathcal{N} \) such that, \( E_{p,q} \) is the matrix with all entries zero except in position \( (p, q) \)–where the entry is 1. The \( E_{p,q} \)'s are such that, \( E_{p,q}^* = E_{p,q} \), \( E_{p,q}E_{r,s} = 0 \) if \( q \neq r \), \( E_{p,q}E_{q,r} = E_{p,r} \) and \( \sum_p E_{p,p} = I \). These \( E_{p,q} \)'s are called matrix units in \( \mathcal{N} \).

The following theorem establishes that there is an unique KMS state \( \rho(\omega) \), above a certain critical temperature \( T_c \) almost surely independent of \( \omega \).

**Theorem 4.2.2.2** If \( \Phi \) is a finite range random interaction of the quantum spin system on an infinite lattice \( \mathbb{Z}^\mathbb{R} \), satisfying the assumptions of theorem 3.7.0.28, then there is an unique KMS state \( \rho(\omega) \), associated with the evolution group \( \tau_t(\omega) \), above a certain critical temperature \( T_c \) almost surely independent of \( \omega \).

**Proof** Since we have established the existence of \( (\tau(\omega), \beta) \)-KMS states for all \( \beta \in \mathcal{R} \setminus \{0\} \), the aim of this theorem is to show that there is an unique
KMS state $\rho(\omega)$, above a critical temperature $T_c$ almost surely independent of $\omega$. The proof of the theorem goes along the lines of the discussion preceding proposition 6.2.45, in [Rob 81]. The $(\tau(\omega), \beta)$–KMS condition will play a crucial role in establishing the above fact. For $x \in \mathbb{Z}^\nu$, let $e(i_x, j_x); i_x, j_x = 0, 1$, be a set of matrix units for $A_x$. Let $A \subseteq \mathcal{A}$, where $x \notin \Lambda$. Now, the $(\tau(\omega), \beta)$–KMS condition and the identity

$$e(i_x, j_x) = \frac{1}{2} \sum_{k_x=0}^{1} e(i_x, k_x) e(k_x, j_x),$$

yield

$$\rho(\omega)(e(i_x, j_x)A)$$

$$= \frac{1}{2} \sum_{k_x=0}^{1} \{\rho(\omega)(e(k_x, j_x)Ae(i_x, k_x))$$

$$+ \rho(\omega)(e(k_x, j_x)A(\tau_t(\omega) - I)(e(i_x, k_x)))\}_{|t=\beta}$$

$$= \frac{1}{2} \delta_{i_x,j_x} \rho(\omega)(A) + \frac{1}{2} \sum_{k_x=0}^{1} \{\rho(\omega)(e(k_x, j_x)A(\tau_t(\omega) - I)(e(i_x, k_x)))\}_{|t=\beta}.$$

In view of the fact that the local elements are dense in the quasi-local algebra $A$, it is enough to establish that $\rho(\omega)(A)$ can be uniquely determined for all $A \subseteq \mathcal{A}$, and all $\Lambda \subseteq \mathbb{Z}^\nu$. It is seen from the proof of theorem 3.7.0.28, and the remark following the proof, that the local elements of $A$ are analytic with respect to the generator $\mathfrak{F}(\omega)$ of $\tau_t(\omega)$, with radius of analyticity

$$r_\omega \geq \left(2 \sup_{a \in \mathbb{Z}^\nu} \left(\sum_{X \in \mathcal{A}} \|\Phi(X, T_a \omega)\| \right) e^{\Delta \omega}\right)^{-1},$$

where

$$\omega \mapsto \left(2 \sup_{a \in \mathbb{Z}^\nu} \left(\sum_{X \in \mathcal{A}} \|\Phi(X, T_a \omega)\| \right) e^{\Delta \omega}\right)^{-1}.$$
is almost surely constant. Therefore, the second term on the right hand side of the last equation can be expressed as a power series in $\beta$ without a constant term, for sufficiently small $\beta$ almost surely independent of $\omega$.

Besides, $A \in \mathcal{A}_N$ can be expressed as a linear combination of matrix units

$$e(I_A, J_A) = \prod_{i=1}^{n} e(i_{z_i}, j_{z_i}),$$

where $\Lambda = \{x_1, \ldots, x_n\}$; $I_A = \{i_{x_1}, \ldots, i_{x_n}\}$, and $J_A = \{j_{x_1}, \ldots, j_{x_n}\}$. Thus, it suffices to consider only the special choices $A = e(I_A, J_A)$; $I_A \in \{0, 1\}^\Lambda$ and $\Lambda \subseteq \mathcal{Z}^\nu$. On adopting the assumptions of theorem 3.7.0.28, one has

$$(\tau_{i\beta}(\omega) - I)(e(i_x, k_x))$$

$$= \sum_{n=1}^{\infty} \frac{(-\beta)^n}{n!} \sum_{X_1 \cap S_0 \neq \emptyset} \cdots \sum_{X_n \cap S_{n-1} \neq \emptyset} [\Phi(X_n, \omega) \cdots [\Phi(X_1, \omega), e(i_x, k_x)]],$$

where $S_0 = x$ and $S_j = X_j \cup X_{j-1} \cup \cdots \cup X_1 \cup x$. Now, if

$$B = \sum_{I_A, J_A} B(I_A, J_A)e(I_A, J_A)$$

is the decomposition of $B \in \mathcal{A}_N$, for some $\Lambda \subseteq \mathcal{Z}^\nu$, then the complex coefficients $B(I_A, J_A)$ satisfy, $\|B(I_A, J_A)\| \leq \|B\|$.

Hence,

$$e(k_x, j_x)e(I_A, J_A)[\Phi(X_n, \omega), \cdots [\Phi(X_1, \omega), e(i_x, k_x)]]$$

$$= \sum_{I_{S_n}^{\mathcal{Z}}, J_{S_n}^{\mathcal{Z}}} \gamma_{n, \omega}(I_{S_n}^{\mathcal{Z}}, J_{S_n}^{\mathcal{Z}}) e(I_{S_n}^{\mathcal{Z}}, J_{S_n}^{\mathcal{Z}}),$$

where $S_n^{\mathcal{Z}} = x \cup \Lambda \cup S_n$, and there are at most $2^{2|S_n|}$ nonzero coefficients $\gamma_{n, \omega}$, which satisfy

$$|\gamma_{n, \omega}(I_{S_n}^{\mathcal{Z}}, J_{S_n}^{\mathcal{Z}})| \leq 2^n \prod_{i=1}^{n} \|\Phi(X_i, \omega)\|.$$
This perturbation expansion can be combined with the previous identity for \( \rho(\omega) \), evaluated with \( A = e(I_A, J_A) \), to obtain a linear equation involving the family \( \{ \rho(\omega)(e(I_A, J_A)); \Lambda \subseteq \mathbb{Z}^n \} \). To this end, let \( \mathcal{X} \) be the Banach space of bounded complex functions \( f \), on the pairs \( \{ I_A, J_A \} \), where \( I_A, J_A \subseteq \{0, 1\} \), \( \Lambda \subseteq \mathbb{Z}^n \), and \( f(I_\emptyset, J_\emptyset) \in \mathcal{C} \). The space \( \mathcal{X} \) is equipped with the usual operations of addition and scalar multiplication, together with the supremum norm. If \( \rho(\omega) \) denotes the family \( \{ \rho(\omega)(e(I_A, J_A)); \Lambda \subseteq \mathbb{Z}^n \} \), where we take \( e(I_\emptyset, J_\emptyset) = I \), it follows that \( \rho(\omega) \in \mathcal{X} \) and \( \| \rho(\omega) \| = 1 \). The foregoing identity and perturbation expansion yield the equation

\[
\rho(\omega) = \eta + L_\beta(\omega)\rho(\omega),
\]

where \( \eta, K, \) and \( L_\beta(\omega) \) are defined as follows: \( \eta \in \mathcal{X} \) and

\[
\eta(I_A, J_A) = \begin{cases} 
1 & \text{if } \Lambda = \emptyset \\
\frac{1}{2} \delta_{i_1,j_1} & \text{if } \Lambda = \{x\} \\
0 & \text{otherwise},
\end{cases}
\]

\( K \) is a linear operator with action \( Kf(I_A, J_A) = \frac{1}{2} \delta_{i_1,j_1} f(I_{A'}, J_{A'}) \), if \( \Lambda = \{x_1, x_2, \ldots, x_n\}, \Lambda' = \{x_2, \ldots, x_n\} \) and \( n \geq 2 \), and \( (Kf)(I_A, J_A) = 0 \), if \( |\Lambda| < 2 \). \( L_\beta(\omega) \) is a linear operator such that,

\[
(L_\beta(\omega)f)(I_A, J_A) = \frac{1}{2} \sum_{K_x = 0}^{1} \sum_{n=1}^{\infty} \frac{(-\beta)^n}{n!} \sum_{X_1 \cap S_0 \neq \emptyset} \cdots \sum_{X_n \cap S_{n-1} \neq \emptyset} \gamma_{n,\omega}(I_{S_0}, J_{S_0}) f(I_{S_0}, J_{S_0}),
\]

where \( \gamma_{n,\omega}'s \) arising from the perturbation expansion are associated with a fixed splitting \( \Lambda = \{x\} \cup \Lambda' \), where \( \Lambda' = \Lambda \setminus \{x\} \). Thus, the above equation has the form \( (I - K - L_\beta(\omega))\rho(\omega) = \eta \). Hence, \( \rho(\omega) \) is uniquely determined if, \( \| K + L_\beta(\omega) \| < 1 \). But \( \| K \| = \frac{1}{2} \), and so uniqueness will follow if \( \| L_\beta(\omega) \| < 96 \).
\[ \frac{1}{2}. \] This involves estimating the norm of \( L_\beta(\omega) \). To this end, we establish the following.

\[
\|L_\beta(\omega)\| \leq 2^2 e \sum_{n=1}^{\infty} \left( 2^{2|\Delta_\omega|+1} |\beta| e^{2|\Delta_\omega|} \left( \sup_{\alpha \in \mathbb{Z}^n} \left( \sum_{x \in \mathcal{X}} \| \Phi(x, T_\alpha \omega) \| \right) \right) \right)^n < \infty,
\]

whenever

\[
2^{2|\Delta_\omega|+1} e^{2|\Delta_\omega|} \left| \beta \right| \left( \sup_{\alpha \in \mathbb{Z}^n} \left( \sum_{x \in \mathcal{X}} \| \Phi(x, T_\alpha \omega) \| \right) \right) < 1.
\]

The estimation procedure will be much like the one employed in the construction of global dynamics. We have

\[
|(L_\beta(\omega) f)(I_\Lambda, J_\Lambda)| \leq \frac{1}{2} \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \frac{|\beta|^n}{n!} \sum_{X_1 \cap S_0 \neq \emptyset} \sum_{X_n \cap S_{n-1} \neq \emptyset} \sum_{s_{j_1} \in S_{j_1}} \cdots \sum_{s_{j_k} \in S_{j_k}} |\gamma_{n, \omega}(I_{S_{j_k}^n}, J_{S_{j_k}^n})| |f(I_{S_{j_k}^n}, J_{S_{j_k}^n})| \\
\leq 2 \cdot \frac{1}{2} \sum_{n=1}^{\infty} \frac{|\beta|^n}{n!} \sum_{X_1 \cap S_0 \neq \emptyset} \sum_{X_n \cap S_{n-1} \neq \emptyset} 2^{2|S_n|} (2^n \| \Phi(x_n, \omega) \| \cdots \| \Phi(x_1, \omega) \|) \|f\|_\infty
\]

The last inequality follows from the remark made earlier, regarding the norms of the complex coefficients \( \gamma_{n, \omega} \). Since \( \Phi(\cdot, \omega) \) has a finite range \( \Delta_\omega \), \( \Phi(x, \omega) = 0 \) whenever \( |x| > |\Delta_\omega| \). Therefore,

\[
|S_j| = |X_j \cup X_{j-1} \cup \cdots \cup X_1 \cup x| \\
\leq |X_j| + |X_{j-1}| + \cdots + |X_1| + |x| \\
\leq (|\Delta_\omega| + |\Delta_\omega| \cdots + |\Delta_\omega| + 1) \\
\leq (j|\Delta_\omega| + 1).
\]

We also have

\[
\sup_{x \in \mathbb{Z}^n} \left( \sum_{x \in \mathcal{X}} \| \Phi(x, T_\omega \omega) \| \right) < \infty.
\]

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Thus,

\[
\| (L_\beta(\omega)f) \|
\leq \sum_{n=1}^{\infty} 2^{2(n|\Delta_\omega|+1)} n! \sum_{x_1 \in S_0} \sum_{x_1 \in x_1} \cdots \sum_{x_n \in S_{n-1}} \sum_{x_n \in x_n} \alpha_{-x_n}(\Phi(X_n, \omega))
\]
\[
\cdots \| \alpha_{-x_1}(\Phi(X_1, \omega)) \| \| f \|_\infty
\]
\[
\leq 2^2 \sum_{n=1}^{\infty} 2^{n(2|\Delta_\omega|+1)} n! \sum_{x_1 \in S_0} \sum_{x_1 \in x_1} \cdots \sum_{x_n \in S_{n-1}} \sum_{x_n \in x_n} \Phi(X_n - x_n, T_{x_n}\omega)
\]
\[
\cdots \| \Phi(X_1 - x_1, T_{x_1}\omega) \| \| f \|_\infty
\]
\[
\leq 2^2 \sum_{n=1}^{\infty} 2^{n(2|\Delta_\omega|+1)} n! \sum_{x_1 \in S_0} \sum_{x_1 \in x_1} \cdots \sum_{x_n \in S_{n-1}} \sum_{x_n \in x_n} \left( \sup_{X \in \mathcal{X}} \left( \sum_{Y \in \mathcal{Y}} \| \Phi(Y, T_x \omega) \| \right) \right) \| f \|_\infty
\]
\[
\leq 2^2 \sum_{n=1}^{\infty} 2^{n(2|\Delta_\omega|+1)} n! \sum_{x_1 \in S_0} \sum_{x_1 \in x_1} \cdots \sum_{x_n \in S_{n-1}} \sum_{x_n \in x_n} \left( \sup_{X \in \mathcal{X}} \left( \sum_{Y \in \mathcal{Y}} \| \Phi(X, T_a \omega) \| \right) \right) \| f \|_\infty
\]
\[
\leq 2^2 \sum_{n=1}^{\infty} 2^{n(2|\Delta_\omega|+1)} n! \sum_{x_1 \in S_0} \sum_{x_1 \in x_1} \cdots \sum_{x_n \in S_{n-1}} \sum_{x_n \in x_n} \left( \sup_{X \in \mathcal{X}} \left( \sum_{Y \in \mathcal{Y}} \| \Phi(Y, T_x \omega) \| \right) \right) \| f \|_\infty
\]
\[
\leq 2^2 \sum_{n=1}^{\infty} 2^{n(2|\Delta_\omega|+1)} \beta^n \left( \sum_{X \in \mathcal{X}} \| \Phi(X, T_a \omega) \| \right) \| f \|_\infty
\]
\[
\leq 2^2 \sum_{n=1}^{\infty} 2^{n(2|\Delta_\omega|+1)} \beta^n \left( \sum_{X \in \mathcal{X}} \| \Phi(X, T_a \omega) \| \right) \| f \|_\infty
\]
\[
\leq 2^2 e^1 \sum_{n=1}^{\infty} \left( 2^{2|\Delta_\omega|+1} \beta^n \right) \left( \sup_{X \in \mathcal{X}} \left( \sum_{X \in \mathcal{X}} \| \Phi(X, T_a \omega) \| \right) \right) \| f \|_\infty
\]
\[
\leq 2^2 e^1 \sum_{n=1}^{\infty} \left( 2^{2|\Delta_\omega|+1} \beta^n \right) \left( \sup_{X \in \mathcal{X}} \left( \sum_{X \in \mathcal{X}} \| \Phi(X, T_a \omega) \| \right) \right) \| f \|_\infty
\]

All these inequalities have been obtained by employing the estimation procedure used in the construction of dynamics. Hence, from the above estimate we have

\[
\| L_\beta(\omega) \| \leq 2^2 e \sum_{n=1}^{\infty} \left( 2^{2|\Delta_\omega|+1} \beta^n \right) \left( \sup_{X \in \mathcal{X}} \left( \sum_{X \in \mathcal{X}} \| \Phi(X, T_a \omega) \| \right) \right) \| f \|_\infty
\]
Now, we know that the KMS state $\rho(\omega)$ is unique whenever $\|L_\beta(\omega)\| < \frac{1}{2}$.

Therefore, $\rho(\omega)$ is unique whenever

$$2^2 e^{2|\Delta_\omega| |1 + e|\Delta_\omega| - 1} \frac{\beta (\sup_{\omega \in \mathbb{Z}^\prime} (\sum_{X \neq 0} \|\Phi(X, T_\omega)\|))}{1 - 2^2 |\Delta_\omega| e |\Delta_\omega| |\beta| (\sup_{\omega \in \mathbb{Z}^\prime} (\sum_{X \neq 0} \|\Phi(X, T_\omega)\|))} < \frac{1}{2},$$

i.e., whenever

$$|\beta| < \left(2^2 |\Delta_\omega| + 1 + e |\Delta_\omega| (1 + 2^3 e)\right)^{-1} \frac{\left(\sup_{\omega \in \mathbb{Z}^\prime} \left(\sum_{X \neq 0} \|\Phi(X, T_\omega)\|\right)\right)^{-1}}{\beta \left(\sup_{\omega \in \mathbb{Z}^\prime} \left(\sum_{X \neq 0} \|\Phi(X, T_\omega)\|\right)\right)^{-1}}.$$

Next, by lemma 3.6.0.23, we have $\omega \mapsto |\Delta_\omega|$ is almost surely constant. Moreover, by lemma 3.6.0.24 $\omega \mapsto (\sup_{\omega \in \mathbb{Z}^\prime} (\sum_{X \neq 0} \|\Phi(X, T_\omega)\|))$ is also almost surely constant. Hence,

$$\omega \mapsto \left(2^2 |\Delta_\omega| + 1 + e |\Delta_\omega| (1 + 2^3 e)\right)^{-1} \frac{\left(\sup_{\omega \in \mathbb{Z}^\prime} \left(\sum_{X \neq 0} \|\Phi(X, T_\omega)\|\right)\right)^{-1}}{(\sup_{\omega \in \mathbb{Z}^\prime} \left(\sum_{X \neq 0} \|\Phi(X, T_\omega)\|\right))^{-1}}$$

is almost surely constant. Therefore, there exists a critical temperature $T_c$ almost surely independent of $\omega$ such that, for temperatures $T > T_c$, there exists an unique KMS state $\rho(\omega)$ associated with $\tau_t(\omega)$.

It is worth noting that the estimate on $\beta$ can be improved upon in several ways. Since $\rho(\omega)$ is an unique KMS state with respect to $\tau_t(\omega)$, above a certain critical temperature $T_c$ almost surely independent of $\omega$, it follows from theorem 5.3.30 in [Rob 81] that, $\rho(\omega)$ is an extremal KMS state and hence, a factor state. As the quasi-local algebra is norm asymptotically abelian, it also follows that $\rho(\omega)$ is strongly clustering with respect to the group $\mathbb{Z}^\prime$ of lattice translations.

Next, as $\rho(\omega)$ is an unique KMS state associated with the evolution group $\tau_t(\omega)$, one can easily conclude that the net of local Gibbs states $\rho_A(\omega)$ must
converge in the weak*—topology to \( \rho(\omega) \), as \( \Lambda \to \infty \). This is a trivial consequence of the fact that each weak*—limit point of the local Gibbs states \( \rho_A(\omega) \) is a \((\tau(\omega), \beta)\)—KMS state, and hence, by uniqueness of the KMS state \( \rho(\omega) \), it must be equal to \( \rho(\omega) \). Next, since \( \rho_A(\omega)(A) = \rho_{A+a}(T_{-a}\omega)(\alpha_a(A)) \), for \( A \in \mathcal{A}_A \), we have \( \rho(\omega)(A) = \rho(T_{-a}\omega)(\alpha_a(A)) \), for all \( A \in \mathcal{A}_{\Lambda_0} \) and all \( \Lambda_0 \subseteq \mathcal{Z}^\nu \). Since the local elements are norm dense in \( \mathcal{A} \), we have \( \rho(\omega)(A) = \rho(T_{-a}\omega)(\alpha_a(A)) \), for all \( A \in \mathcal{A} \). Let \( \{\Lambda_n\} \) be a sequence of finite subsets increasing to \( \mathcal{Z}^\nu \). Since \( \rho(\omega) \) is an unique KMS state, above a critical temperature \( T_c \) almost surely independent of \( \omega \), we have for all \( A \in \mathcal{A}_{\Lambda_0} \) and all \( \Lambda_0 \subseteq \mathcal{Z}^\nu \)

\[
\rho(\omega)(A) = \lim_{n \to \infty} \rho_{\Lambda_n}(\omega)(A),
\]

for almost every \( \omega \in \Omega \), where

\[
\rho_{\Lambda_n}(\omega)(A) = \frac{\text{Tr}(e^{-\beta H(\Lambda_n, \omega)} A)}{\text{Tr}(e^{-\beta H(\Lambda_n, \omega)})}.
\]

However, as \( \omega \mapsto H(\Lambda_n, \omega) \) is strongly measurable and \( \text{Tr}(e^{-\beta H(\Lambda_n, \omega)}) \neq 0 \), for all \( n \in \mathcal{Z}^+ \), it is clear that \( \omega \mapsto \rho_{\Lambda_n}(\omega)(A) \) is a scalar valued measurable function for \( A \in \mathcal{A}_{\Lambda_0} \) and all finite \( \Lambda_0 \subseteq \mathcal{Z}^\nu \). Since the local elements are dense in \( \mathcal{A} \), it is readily seen that \( \omega \mapsto \rho(\omega)(A) \) is measurable for all \( A \in \mathcal{A} \).

It has been established in section 4.2.1 that for \( \beta \in \mathbb{R} \setminus \{0\} \), there exists a family of states \( \{\rho(\omega)\} \) on \( \mathcal{A} \), satisfying \( \rho(\omega)(A) = \rho(T_{-a}\omega)(\alpha_a(A)) \), for all \( A \in \mathcal{A} \), where the \( \rho_\omega \)'s are obtained as the thermodynamic limit of the local Gibbs states \( \rho_A(\omega) \). It is also seen that \( \rho(\omega) \) is a \((\tau(\omega), \beta)\)—KMS state with respect to the evolution group \( \tau_t(\omega) \). Next, let us assume that for
\[ \beta \in \mathbb{R} \setminus \{0\}, \text{ there exists one such family of } (\tau(\omega), \beta)\text{-KMS states } \rho(\omega), \text{ for which the function } \omega \mapsto \rho(\omega)(A) \text{ is measurable for all } A \in \mathcal{A}. \text{ Henceforth, we shall denote this family of states satisfying the above conditions by } \{\rho(\omega)\}. \]

It may be noted from the discussion following the proof of theorem 4.2.2.2 that, above the critical temperature \( T_c \), almost surely independent of \( \omega \), there exists a family of unique KMS states, for which these conditions hold.

Next, we prove the following theorem.

**Theorem 4.2.2.3** If \( \{\rho(\omega)\}_{\omega \in \Omega} \) be the family of \( (\tau(\omega), \beta)\)–KMS states on \( \mathcal{A} \) satisfying the conditions mentioned above and \( \beta > 0 \), then for any pair \( A, B \in \mathcal{A} \), we have the following:

1. Both \( \omega \mapsto \rho(\omega)(A\tau_t(\omega)(B)) \) and \( \omega \mapsto \rho(\omega)(\tau_t(\omega)(B)A) \) are jointly measurable functions of \( t \) and \( \omega \).

2. In particular, if \( \rho(\omega) \) is the unique KMS state with respect to the evolution group \( \tau_t(\omega) \), at some inverse temperature \( \beta > 0 \) almost surely independent of \( \omega \), then both \( \rho(\omega)(A\tau_t(\omega)(B)) \) and \( \rho(\omega)(\tau_t(\omega)(B)A) \) are strongly, jointly measurable. Moreover, there exists a function \( F_{A,B}(z,\omega) \) such that, for a fixed \( \omega \), \( F_{A,B}(z,\omega) \) is analytic in the strip \( 0 < \Im z < \beta \), continuous and uniformly bounded in the closed strip \( 0 \leq \Im z \leq \beta \), and

\[
F_{A,B}(t,\omega) = \rho(\omega)(A\tau_t(\omega)(B)) \quad \text{and} \quad F_{A,B}(t+i\beta,\omega) = \rho(\omega)(\tau_t(\omega)(B)A).
\]

Besides, \( F_{A,B}(z,\omega) \) is measurable in \( \omega \) for each \( z \) in the open strip \( 0 < \Im z < \beta \).
Proof On appealing to theorem 3.7.0.29, we have for $A, B \in \mathcal{A}$, $\omega \mapsto A\tau_t(\omega)(B)$ is strongly, jointly measurable in $t$ and $\omega$. It follows from the definition of strong measurability that, there exists a sequence of countably valued functions $g_n(t, \omega)$ on $\mathbb{R} \times \Omega$, converging almost everywhere to $A\tau_t(\omega)(B)$. Therefore, for almost every $(t, \omega) \in \mathbb{R} \times \Omega$, 

$$\rho(\omega)(A\tau_t(\omega)(B)) = \lim_{n \to \infty} \rho(\omega)(g_n(t, \omega)).$$

In the sequel, we shall establish that for each $n \in \mathbb{Z}^+$, $\rho(\omega)(g_n(t, \omega))$ is measurable on the product space $\mathbb{R} \times \Omega$. Let $g_n(t, \omega)$ take nonzero constant values $A_1,n, A_2,n, \ldots, A_k,n, \ldots$ on measurable subsets $E_1,n, E_2,n, \ldots, E_k,n, \ldots$, of $\mathbb{R} \times \Omega$. There is no loss of generality in assuming that $\rho(\omega)(g_n(t, \omega))$ takes real values. This is because the $A_k,n$'s can always be written as linear combinations of self adjoint elements in $\mathcal{A}$, and $\rho(\omega)$ being a state, it takes real values on self adjoint elements of $\mathcal{A}$. Now, for $c \in \mathbb{R}$,

$$\{(t, \omega) \in \mathbb{R} \times \Omega | \rho(\omega)(g_n(t, \omega)) < c\} = \{(t, \omega) \in E_0,n | \rho(\omega)(g_n(t, \omega)) < c\} \bigcup \left\{ \bigcup_{k=1}^{\infty} \{(t, \omega) \in E_k,n | \rho(\omega)(g_n(t, \omega)) < c\} \right\},$$

$$\{(t, \omega) \in E_0,n | \rho(\omega)(0) < c\} \bigcup \left\{ \bigcup_{k=1}^{\infty} \{(t, \omega) \in E_k,n | \rho(\omega)(A_k,n) < c\} \right\},$$

where $E_{0,n}$ is the set on which $g_n$ takes the value zero(0). Therefore, it is evident from the measurability of the function $\omega \mapsto \rho(\omega)(A)$, for all $A \in \mathcal{A}$ that, the two sets on the right hand side of the equality are measurable subsets of $\mathbb{R} \times \Omega$. Hence, as $c$ is arbitrary, $\rho(\omega)(g_n(t, \omega))$, is a jointly measurable function of $t$ and $\omega$ for each $n \in \mathbb{Z}^+$. Since $\rho(\omega)(A\tau_t(\omega)(B))$ is the limit almost everywhere of $\rho(\omega)(g_n(t, \omega))$ on $\mathbb{R} \times \Omega$, we conclude that $\rho(\omega)(A\tau_t(\omega)(B))$
is a jointly measurable function of $t$ and $\omega$. Similarly, it can be shown that the function $(t, \omega) \mapsto \rho(\omega)(\tau_t(\omega)(B)A)$ is jointly measurable in $t$ and $\omega$. This proves (1) conclusively.

Now, in order to prove (2), choose a sequence of finite subsets $\{\Lambda_n\}$ which increases to $\mathbb{Z}^\nu$. It has been shown in the discussion following the proof of theorem 4.2.2.2 that, if $\rho(\omega)$ is the unique KMS state with respect to the evolution group $\tau_t(\omega)$ at some inverse temperature $\beta$ almost surely independent of $\omega$, then for all $A \in \mathcal{A}_{\Lambda_0}$ and all $\Lambda_0 \subseteq \mathbb{Z}^\nu$,

$$\lim_{n \to \infty} \rho_{\Lambda_n}(\omega)(A) = \rho(\omega)(A)$$

almost everywhere. It was also established in this discussion that $\omega \mapsto \rho(\omega)(A)$ is measurable for all $A \in \mathcal{A}$. Since $\omega \mapsto \rho(\omega)(A)$ is measurable, the joint measurability in $t$ and $\omega$, of both $\rho(\omega)(A\tau_t(\omega)(B)))$ and $\rho(\omega)(\tau_t(\omega)(B)A)$ can be proved along the lines of (1). Next, for $A, B \in \mathcal{A}_{\Lambda_0}$, let

$$\tau_{\Lambda_n}(\omega)(B) = e^{iH(\Lambda_n,\omega)z}B e^{-iH(\Lambda_n,\omega)z},$$

where $\Lambda_n \supseteq \Lambda_0$. Also for $\Lambda_n \supseteq \Lambda_0$, define

$$F_{A, B}^{\Lambda_n}(z, \omega) = \frac{Tr(e^{-\beta H(\Lambda_n,\omega)}A \tau_{\Lambda_n}(\omega)(B))}{Tr(e^{-\beta H(\Lambda_n,\omega)})}.$$

Clearly, $\{F_{A, B}^{\Lambda_n}\}$ is a sequence of entire functions, which is uniformly bounded on the strip $0 \leq \Re z \leq \beta$ such that,

$$F_{A, B}^{\Lambda_n}(t, \omega) = \rho_{\Lambda_n}(\omega)(A \tau_t^{\Lambda_n}(\omega)(B))$$

and

$$F_{A, B}^{\Lambda_n}(t+i\beta, \omega) = \rho_{\Lambda_n}(\omega)(\tau_t^{\Lambda_n}(\omega)(B)A).$$

(See the proof of theorem 3.4.0.13 which can be adopted to $\rho^\Lambda(\omega)$ with $\tau_t^\Lambda(\omega)$ as the local automorphism group). On mimicking the proof of proposition (3.4.0.14)
in chapter 3, we have for $A, B \in \mathcal{A}_{\Lambda_0}$

$$\lim_{n \to \infty} \rho_{\Lambda_n}(\omega)(A_{\tau_t}(\omega)(B)) = \rho(\omega)(A_{\tau_t}(\omega)(B)),$$

where the limit exists almost everywhere in $\omega$, for all real $t$ and uniformly in $t$ in a ball around zero. Hence, as a consequence of Vitali’s theorem, see [Tit 91], for almost every $\omega \in \Omega$ the sequence $F_{A,B}^n(z, \omega)$ converges uniformly on every compact subset of the strip to $F_{A,B}(z, \omega)$, which for a fixed $\omega$ is analytic in the open strip $0 < \Im z < \beta_1$, continuous and uniformly bounded in the closed strip $0 \leq \Im z \leq \beta$ such that,

$$F_{A,B}(t) = \rho(\omega)(A_{\tau_t}(B)) \text{ and } F_{A,B}(t + i\beta) = \rho(\omega)(\tau_t(B)A).$$

This proves the existence of $F_{A,B}(z, \omega)$ satisfying the conditions in (2) for $A, B \in \mathcal{A}_{\Lambda_0}$, where $\Lambda_0 \subseteq \mathbb{Z}^\nu$.

Now, it follows from the strong measurability of $\omega \mapsto H(\Lambda, \omega)$ for finite $\Lambda \subseteq \mathbb{Z}^\nu$ that, both $A_{\tau}^{\Lambda_{\tau_n}}(\omega)B$ and $e^{-\beta H(\Lambda_{\tau_n}, \omega)}$ are strongly measurable in $\omega$ for each $z$ in the open strip and $n \in \mathbb{Z}^+$. Therefore, for each $z$ in the open strip, $F_{A,B}^n(z, \omega)$ is a scalar valued measurable function of $\omega$. This is in view of the fact that, the trace, denoted by $\text{Tr}$, is a continuous linear functional and hence, $F_{A,B}^n(z, \omega)$ is a ratio of two measurable functions with $\text{Tr}(e^{-\beta H(\Lambda_{\tau_n}, \omega)}) \neq 0$ for all $n \in \mathbb{Z}^+$. It has been seen from Vitali’s theorem that, for almost every $\omega \in \Omega$ the sequence $F_{A,B}^n(z, \omega)$ converges uniformly on every compact subset of the strip to a function $F_{A,B}(z, \omega)$. Hence for each $z$ in the open strip, $F_{A,B}^n(z, \omega)$ converges to $F_{A,B}(z, \omega)$ almost everywhere. Therefore, we conclude that for each $z$ in the open strip, $F_{A,B}(z, \omega)$ is a measurable function of $\omega$. This proves the measurability of $F_{A,B}(z, \omega)$ in $\omega$ for
each \( z \) in the open strip, for \( A, B \in A_{\Lambda_0} \) and \( \Lambda_0 \subseteq \mathbb{Z}^n \).

Next, for \( A, B \in A \), let \( A_n \to A \) and \( B_n \to B \) be sequences of local elements converging to \( A \) and \( B \) respectively. Therefore, it follows from what was established earlier that, there exists a sequence of scalar valued functions \( F_{A_n,B_n}(z,\omega) \) such that, for each \( z \) in the open strip, \( F_{A_n,B_n}(z,\omega) \) is measurable and for a fixed \( \omega \), \( F_{A_n,B_n}(z,\omega) \) is analytic in the open strip, uniformly bounded and continuous on the closed strip. Moreover, \( F_{A_n,B_n}(t,\omega) = \rho(\omega)(A_n \tau_t(\omega)(B_n)) \) and \( F_{A_n,B_n}(t+i\beta,\omega) = \rho(\omega)(\tau_t(\omega)(B_n)A_n) \). Now, there is a version of the Phragmen—Lindelöf theorem [Rob 81] (Vol 2, Proposition 5.3.5, Pg 81) which states that, the supremum of the modulus of a function which is bounded and analytic on the strip, is the supremum of the modulii of its boundary values. Since \( A_n \to A \) and \( B_n \to B \) in the norm, the sequence \( F_{A_n,B_n}(z,\omega) \to \rho(\omega)(A \tau_t(\omega)(B)) \) and \( F_{A_n,B_n}(t+i\beta,\omega) \to \rho(\omega)(\tau_t(\omega)(B)A) \). The convergence being uniform in \( t \). Thus, since the sequence \( F_{A_n,B_n}(z,\omega) \) converges uniformly on the boundary of the strip \( 0 \leq \Im z \leq \beta \), it converges uniformly throughout the closed strip, to say, \( F_{A,B}(z,\omega) \). \( F_{A,B}(z,\omega) \) being analytic in the open strip and uniformly bounded and continuous in the closed strip, such that

\[
F_{A,B}(t) = \rho(\omega)(A \tau_t(B)) \quad \text{and} \quad F_{A,B}(t+i\beta) = \rho(\omega)(\tau_t(B)A),
\]

for a fixed \( \omega \). Also for each \( z \) in the open strip, \( F_{A,B}(z,\omega) \) is the limit of the sequence of measurable functions \( F_{A_n,B_n}(z,\omega) \) for almost every \( \omega \in \Omega \). Hence, for each \( z \) in the open strip, \( F_{A,B}(z,\omega) \) is a measurable function of \( \omega \).
The theorem can be established along the same lines in the case of $\beta < 0$, by considering the closed strip $\beta \leq \Im z \leq 0$.

### 4.2.3 Representations Associated with the KMS States

In this subsection, we aim to study the cyclic representations $\pi_\omega$ associated with the states $\rho(\omega)$. These states are thermodynamic limits of the local Gibbs states $\rho_\Lambda(\omega)$, and satisfy the following conditions: $\rho(\omega)(A) = \rho(T_{-\omega})(a A(a))$, for all $A \in \mathcal{A}$ and $a \in \mathbb{Z}^n$, and $\omega \mapsto \rho(\omega)(A)$ is measurable, for all $A \in \mathcal{A}$. It is also seen that $\rho(\omega)$ is a $(\tau(\omega), \beta)$--KMS state, where $\tau_t(\omega)$ is the evolution group. We shall exploit the quasi--local structure of the $C^*$--algebra to demonstrate some interesting features of the representations $\pi_\omega$, and establish the separability of the Hilbert space $\mathcal{H}_\omega$. Algebraic properties of the group of unitaries $U_t(\omega)$, which implements the evolution group $\tau_t(\omega)$ of the spin system have also been derived.

Now, associated with every $\rho(\omega)$, we have a cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \Theta_\omega)$ of the quasi--local algebra $\mathcal{A}$, obtained through the G.N.S construction. The idea behind this construction is to convert the $C^*$--algebra $\mathcal{A}$ into a pre--Hilbert space by introducing a positive semi--definite scalar product on $\mathcal{A}$. In the process, we end up with a pre--Hilbert space of equivalence classes $\psi_A(\omega), \psi_B(\omega)$, defined by $\psi_A(\omega) = \hat{A}(\omega); \hat{A}(\omega) = A + J$, where $J \in \mathcal{J}_\omega$, and $\mathcal{J}_\omega = \{A \in \mathcal{A}| \rho(\omega)(A^*A) = 0\}$, with the scalar product given by

$$\langle \psi_A(\omega), \psi_B(\omega) \rangle_\omega = \rho(\omega)(A^*B).$$
Before we complete this pre-Hilbert space to give us the Hilbert space $\mathcal{H}_\omega$, we define the representation $\pi_\omega$ by specifying the action of the representative $\pi_\omega(A)$ on the pre-Hilbert space as follows:

$$\pi_\omega(A)(\psi_B(\omega)) = \psi_{AB}(\omega).$$

The cyclic vector is defined by $\Theta_\omega = \psi_I(\omega)$. Note that, $\{\pi_\omega(A)\Theta_\omega; A \in \mathcal{A}\}$ is exactly the dense set of equivalence classes $\{\psi_A; A \in \mathcal{A}\}$, and hence, $\Theta_\omega$ is cyclic for $(\mathcal{H}_\omega, \pi_\omega)$. Since $\mathcal{A}$ is simple, $\pi(\omega)$ is a faithful representation of $\mathcal{A}$.

Moreover, $\mathcal{A}$ being a uniformly matricial $C^*$-algebra (or UHF algebra), each of these states $\rho(\omega)$, is a locally normal state. Therefore, it follows from the remarks made on the characterization of locally normal states of an abstract $C^*$-algebra in [Em 71] (Page 283), that, the Hilbert space $\mathcal{H}_\omega$ associated with the representation $\pi_\omega$, is a separable Hilbert space. Since $\mathcal{A}$ is simple, and $(\mathcal{H}_\omega, \pi_\omega, \Theta_\omega)$ is a cyclic representation of $\mathcal{A}$ induced by the KMS state $\rho(\omega)$, it follows from [Win 70] (Section 5, Page 253) that, the vector $\Theta_\omega$ is cyclic and separating for the von Neumann algebra $\pi_\omega(\mathcal{A})''$.

Next, every element $a \in \mathcal{Z}$, induces an isomorphism $D_a : \mathcal{H}_\omega \to \mathcal{H}_{T_{-a}\omega}$, as follows: Define $D_a(\psi_A(\omega)) = \psi_{aA}(T_{-a}\omega)$. Note that, $D_a$ is defined on a dense subspace of $\mathcal{H}_\omega$, and

$$\langle D_a(\psi_A(\omega)), D_a(\psi_B(\omega)) \rangle_{T_{-a}\omega} = \langle \psi_{aA}(T_{-a}\omega), \psi_{aB}(T_{-a}\omega) \rangle_{T_{-a}\omega}$$

$$= \rho(T_{-a}\omega)((\alpha_a(A))^*(\alpha_a(B)))$$

$$= \rho(T_{-a}\omega)(\alpha_a(A^*B))$$

$$= \rho(\omega)(A^*B)$$

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Hence, for each \( a \in \mathbb{Z}^\nu \), \( D_{-a} : \mathcal{H}_\omega \to \mathcal{H}_{T^{-a}\omega} \) preserves the inner product on a dense subspace \( \mathcal{V}_\omega = \{ \psi_A(\omega); A \in \mathcal{A} \} \) of \( \mathcal{H}_\omega \). Besides, it is clear that \( D_{-a} \) maps \( \mathcal{V}_\omega \), onto a dense subspace \( \mathcal{V}_{T^{-a}\omega} \) of the separable Hilbert space \( \mathcal{H}_{T^{-a}\omega} \). Therefore, for each \( a \in \mathbb{Z}^\nu \), \( D_{-a} \) can be extended to an isomorphism between the Hilbert spaces \( \mathcal{H}_\omega \) and \( \mathcal{H}_{T^{-a}\omega} \).

It is worth noting that for \( \omega \in \Omega \) and \( a \in \mathbb{Z}^\nu \),

\[
(D^{-1}_{-a}(\pi_{T^{-a}\omega}(\alpha_a(A))))D_{-a})(\psi_B(\omega)) = D^{-1}_{-a}(\pi_{T^{-a}\omega}(\alpha_a(A)))(\psi_{\alpha_a(B)}(T^{-a}\omega))
\]

\[
= D^{-1}_{-a}(\psi(\alpha_a(A))(\alpha_a(B))(T^{-a}\omega))
\]

\[
= D^{-1}_{-a}(\psi_{\alpha_a(AB)}(T^{-a}\omega))
\]

\[
= \psi_{AB}(\omega)
\]

\[
= \pi_\omega(A)(\psi_B(\omega)),
\]

for all \( A \in \mathcal{A} \). Since the \( \psi(\omega) \)'s are dense in \( \mathcal{H}_\omega \), we have

\[
\pi_\omega(A) = D^{-1}_{-a}(\pi_{T^{-a}\omega}(\alpha_a(A)))D_{-a} \quad \forall A \in \mathcal{A}.
\]

Thus, \( D_{-a} \) exhibits an interesting intertwining property which establishes some sort of equivalence between the representations \( \pi_\omega \) and \( \pi_{T^{-a}\omega} \). This equivalence is reminiscent of the notion of unitary equivalence between representations. It follows readily from the identity

\[
\pi_{T^{-a}\omega}(\tau_1(T^{-a}\omega)(\alpha_a(A))) = \pi_{T^{-a}\omega}(\alpha_a(\tau_1(\omega)(A))),
\]

where \( \tau_1(\omega) \) is the evolution group, and the intertwining property of \( D_{-a} \) that,

\[
D^{-1}_{-a}(\pi_{T^{-a}\omega}(\tau_1(T^{-a}\omega)(\alpha_a(A))))D_{-a} = \pi_\omega(\tau_1(\omega)(A)).
\]
Note that for \( a \in \mathbb{Z}^\nu \) and \( \omega \in \Omega \),

\[
\begin{align*}
D_{-a}(\Theta_\omega) &= D_{-a}(\psi_I(\omega)) \\
&= \psi_{\alpha_a(I)}(T_{-a}\omega) \\
&= \psi_I(T_{-a}\omega) \\
&= \Theta_{T_{-a}\omega}.
\end{align*}
\]

In the final part of this section, we derive an interesting ergodic property of the spectrum of the generators of the unitary groups \( U_t(\omega) \), which implement the evolution groups \( \tau_t(\omega) \) in the representation \( \pi_\omega \).

Since \( \rho(\omega) \) is a \((\tau(\omega), \beta)\)-KMS state, we have \( \rho(\omega)(\tau_t(\omega)(A)) = \rho(\omega)(A) \) for all \( A \in \mathcal{A} \). It follows from the uniqueness of the cyclic representation \((\pi_\omega, \mathcal{H}_\omega, \Theta_\omega)\) that, there exists an unitary operator \( U_t(\omega) : \mathcal{H}_\omega \to \mathcal{H}_\omega \) such that,

\[
U_t(\omega)(\pi_\omega(A))U_t(\omega)^{-1} = \pi_\omega(\tau_t(\omega)(A)) \quad \text{and} \quad U_t(\omega)\Theta_\omega = \Theta_\omega,
\]

for all \( t \in \mathbb{R} \). Here \( U_t(\omega)^{-1} \) denotes the inverse of \( U_t(\omega) \).

**Proposition 4.2.3.1** Let \( U_t(\omega) \) be the strongly continuous, one-parameter group of unitary operators implementing the evolution group \( \tau_t(\omega) \) in the representation \( \pi_\omega \) on \( \mathcal{H}_\omega \). Then, we have

\[
U_t(\omega) = D_{-a}^{-1}(U_t(T_{-a}\omega))D_{-a}.
\]

**Proof** Since

\[
\pi_\omega(\tau_t(\omega)(A)) = D_{-a}^{-1}(\pi_{T_{-a}\omega}(\tau_t(T_{-a}\omega)(\alpha_a(A))))D_{-a},
\]

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we have

\[
\langle U_t(\omega)(\pi_\omega(A)\Theta_\omega), \pi_\omega(B)\Theta_\omega\rangle_\omega
\]

\[
= \langle \pi_\omega(\tau_t(\omega)(A))\Theta_\omega, \pi_\omega(B)\Theta_\omega\rangle_\omega
\]

\[
= \langle D^{-1}_a(\pi_{T_{-a}\omega}(\tau_t(T_{-a}\omega)(\alpha_\omega(A))))D_{-a}\Theta_\omega, D^{-1}_a(\pi_{T_{-a}\omega}(\alpha_\omega(B)))D_{-a}\Theta_\omega\rangle_\omega
\]

\[
= \langle D^{-1}_a(\pi_{T_{-a}\omega}(\tau_t(T_{-a}\omega)(\alpha_\omega(A)))\Theta_{T_{-a}\omega}), D^{-1}_a(\pi_{T_{-a}\omega}(\alpha_\omega(B)))\Theta_{T_{-a}\omega}\rangle_\omega
\]

\[
= \langle D^{-1}_a(U_t(T_{-a}\omega))\pi_{T_{-a}\omega}(\alpha_\omega(A))\Theta_{T_{-a}\omega}, D^{-1}_a(\pi_{T_{-a}\omega}(\alpha_\omega(B)))\Theta_{T_{-a}\omega}\rangle_\omega
\]

\[
= \langle (D^{-1}_a(U_t(T_{-a}\omega)))D_{-a}\pi_\omega(A)(D^{-1}_a\Theta_{T_{-a}\omega}), \pi_\omega(B)(D^{-1}_a\Theta_{T_{-a}\omega})\rangle_\omega
\]

\[
= \langle (D^{-1}_a(U_t(T_{-a}\omega)))D_{-a}\pi_\omega(A)\Theta_\omega, \pi_\omega(B)\Theta_\omega\rangle_\omega.
\]

Therefore,

\[
\langle U_t(\omega)(\pi_\omega(A)\Theta_\omega), \pi_\omega(B)\Theta_\omega\rangle_\omega = \langle (D^{-1}_a(U_t(T_{-a}\omega)))D_{-a}\pi_\omega(A)\Theta_\omega, \pi_\omega(B)\Theta_\omega\rangle_\omega.
\]

Since \(\Theta_\omega\) is a cyclic vector for \(\pi_\omega(A)\), the above equality implies that

\[
U_t(\omega) = D^{-1}_a(U_t(T_{-a}\omega))D_{-a}.
\]

\(\triangle\)

By virtue of the above proposition we have the following corollary.

**Corollary 4.2.3.2** Let \(H(\omega)\) be the generator of the strongly continuous, one-parameter group of unitaries \(U_t(\omega)\), which implement the evolution group \(\tau_t(\omega)\). If \(E_\lambda(\omega)\) are the spectral projections associated with \(H(\omega)\), then the spectral projections \(E_\lambda(T_{-a}\omega)\) associated with the generator \(H(T_{-a}\omega)\) of the unitary group \(U_t(T_{-a}\omega)\) can be expressed as \(E_\lambda(T_{-a}\omega) = D_{-a}(E_\lambda(\omega))D^{-1}_{-a}\).
Proof We know from Stone's theorem ([Sim 80](Theorem VIII.8)) that, the spectral family $E_\lambda(T_{-\omega})$ associated with the unitary group $U_t(T_{-\omega})$ is unique. Hence the proof follows from proposition 4.2.3.1.

Next, we shall show that the spectrum of the generator of the unitary group $U_t(\omega)$, is almost surely independent of $\omega$. To this end, we have the following proposition.

**Proposition 4.2.3.3** Let $H(\omega)$ be the generator of the strongly continuous, one-parameter group of unitaries $U_t(\omega)$. Then the spectrum $\sigma(H(\omega))$ of the generator $H(\omega)$ is almost surely independent of $\omega$.

**Proof** Let $\pi_\omega$ denote the representation associated with the $(\tau(\omega), \beta)$–KMS state $\rho(\omega)$, with cyclic vector $\Theta_\omega$. The unitary group $U_t(\omega)$ with generator $H(\omega)$ implements $\tau_t(\omega)$ in this representation $\pi_\omega$. Now, for $f \in L^1(\mathbb{R})$, we have

\[
\Psi_\omega(f) \phi = \int_{-\infty}^{\infty} f(t) U_t(\omega) \phi dt = 0, \quad \forall \phi \in \mathcal{H}_\omega
\]

\[
\Leftrightarrow \int_{-\infty}^{\infty} f(t) U_t(\omega)(\pi_\omega(A) \Theta_\omega) dt = 0, \quad \forall A \in \mathcal{A}
\]

\[
\Leftrightarrow \int_{-\infty}^{\infty} f(t) \pi_\omega(\tau_t(\omega)(A)) \Theta_\omega dt = 0, \quad \forall A \in \mathcal{A}
\]

\[
\Leftrightarrow \left( \int_{-\infty}^{\infty} f(t) \pi_\omega(\tau_t(\omega)(A)) dt \right) \Theta_\omega = 0, \quad \forall A \in \mathcal{A}
\]

\[
\Leftrightarrow \pi_\omega \left( \int_{-\infty}^{\infty} f(t) \tau_t(\omega)(A) dt \right) \Theta_\omega = 0, \quad \forall A \in \mathcal{A}
\]

\[
\Leftrightarrow \int_{-\infty}^{\infty} f(t) \tau_t(\omega)(A) dt = 0, \quad \forall A \in \mathcal{A}
\]
The first step follows from the fact that $\Theta_\omega$ is a cyclic vector for $\pi_\omega(\mathcal{A})$. The second follows from the definition of $U_t(\omega)$. Since $\rho(\omega)$ is a KMS state, the separating character of the cyclic vector $\Theta_\omega$ for $\pi_\omega(\mathcal{A})''$, accounts for the penultimate step. We arrive at the final step by virtue of the fact that the representation $\pi_\omega$ is faithful. Now, $\sigma(H(\omega)) = \{s \in \mathbb{R} | \hat{f}(s) = 0, \forall f \in \ker \Phi(\omega)\}$, vide [Bri 77] (Chapter 1, Definition 1.4). Therefore, from the above derivation we have $\sigma(H(\omega)) = \{s \in \mathbb{R} | \hat{f}(s) = 0, \forall f \in \ker \Gamma(\omega)\}$, where $\Gamma(\omega)$ is as defined in theorem 4.1.0.35. Hence, the proof follows from theorem (4.1.0.35).

4.3 Direct Integral von Neumann Algebra.

In the previous section, we saw how each of the $\rho(\omega)$'s gave rise to a representation $\pi_\omega(\mathcal{A})$ of the quasi-local algebra $\mathcal{A}$, on a separable Hilbert space $\mathcal{H}_\omega$. It is seen that, these states are $(\tau(\omega), \beta)$-KMS states with respect to the evolution groups $\tau_t(\omega)$. Now, the representations $\pi_\omega$ associated with these states in turn give rise to an ensemble of von Neumann algebras $\{\pi_\omega(\mathcal{A})\}''_{\omega \in \Omega}$. As these von Neumann algebras correspond to distinct realizations of the quasi-local algebra $\mathcal{A}$, one has to treat them as distinct objects. Therefore, one is obliged to invoke the theory of measurable field of von Neumann algebras. The assumption that the action of the measure preserving group of automorphisms is ergodic, allows us to derive some interesting results concerning the spectra of the generators of the unitaries $U_t(\omega)$, which implement the evolution groups $\tau_t(\omega)$ in the representation $\pi_\omega$. Moreover, the evolution group
$\tau_t(\omega)$ can be extended to a $\sigma$-weakly continuous group of automorphisms $\bar{\tau}_t(\omega)$ of the von Neumann algebra $\pi_\omega(A)''$ such that,

$$\bar{\tau}_t(\omega)(S_\omega) = U_t(\omega)S_\omega U_t(\omega)^{-1}, \forall S_\omega \in \pi_\omega(A)''.$$ 

Since

$$\bar{\tau}_t(\pi_\omega(A)) = U_t(\omega)\pi_\omega(A)U_t(\omega)^{-1} = \pi_\omega(\tau_t(A)),$$

the restriction of $\bar{\tau}_t$ to $\pi_\omega(A)$ is $\tau_t(\omega)$. In the sequel, we impose a measurable structure on the field of Hilbert spaces $\omega \mapsto \mathcal{H}_\omega$, and construct a direct integral Hilbert space $H = \int_\Omega \mathcal{H}_\omega dP(\omega)$. We also demonstrate that $\omega \mapsto \pi_\omega(A)''$, is a measurable field of von Neumann algebras and establish the existence of the associated direct integral von Neumann algebra $\int_\Omega \pi_\omega(A)''dP(\omega)$. Further, we establish the existence of a strongly continuous, one-parameter group of unitaries $U_t$ acting on the direct integral Hilbert space $\int_\Omega \mathcal{H}_\omega dP(\omega)$ such that, $U_tSU_t^{-1} \in \int_\Omega \pi_\omega(A)''dP(\omega)$, for all $S \in \int_\Omega \pi_\omega(A)''dP(\omega)$. Finally, we construct a faithful, normal state $\rho$ of the direct integral von Neumann algebra $\int_\Omega \pi_\omega(A)''dP(\omega)$, which satisfies the Kubo–Martin–Schwinger condition with respect to the $\sigma$-weakly continuous group of automorphisms $\bar{\tau}_t(S) = U_tSU_t^{-1}$, for every decomposable operator $S \in \int_\Omega \pi_\omega(A)''dP(\omega)$.

4.3.1 Measurable Field of Hilbert spaces

We begin with the following proposition.
Proposition 4.3.1.1 The field of separable Hilbert spaces $\omega \mapsto \mathcal{H}_\omega$ is a measurable field of Hilbert spaces.

Proof To this end, recall that the family of states $\{\rho(\omega)\}$ on $\mathcal{A}$, is such that $\rho(\omega)$ is a $(\tau(\omega), \theta)$-KMS state with respect to the evolution group $\tau_t(\omega)$ and $\omega \mapsto \rho(\omega)(A)$ is a measurable function of $\omega$ for all $A \in \mathcal{A}$. Since $(\mathcal{H}_\omega, \pi_\omega, \Theta_\omega)$ is a cyclic representation of $\mathcal{A}$ induced by $\rho(\omega)$, we have $\rho(\omega)(A) = (\pi_\omega(A)\Theta_\omega, \Theta_\omega)$. $\mathcal{A}$, being an uniformly matricial $C^*$-algebra, there exists a sequence of elements $\{A_n\}$ in $\mathcal{A}$ such that, $\mathcal{A}_0 = \{A_n | n \in \mathbb{Z}^+\}$ is dense in $\mathcal{A}$ and hence, for each $\omega \in \Omega$, $\pi_\omega(A_0)$ is operator-norm dense in $\pi_\omega(A)$. Since $\Theta_\omega$ is a cyclic vector associated with the representation $\pi_\omega$ on $\mathcal{H}_\omega$, $\{\pi_\omega(A)\Theta_\omega | A \in \mathcal{A}\}$ is dense in $\mathcal{H}_\omega$ for $\omega \in \Omega$. As $\pi_\omega(A_0)$ is operator-norm dense in $\pi_\omega(A)$, it is easily seen that the sequence of vector fields $\omega \mapsto \pi_\omega(A_i)\Theta_\omega; i = 1, 2, \ldots$, is a total sequence in $\mathcal{H}_\omega$ for all $\omega \in \Omega$. Moreover, in view of the assumption that the map $\omega \mapsto \rho(\omega)(A)$ is measurable for all $A \in \mathcal{A}$, it is readily seen that the function $\omega \mapsto (\pi_\omega(A_i)\Theta_\omega, \pi_\omega(A_j)\Theta_\omega)_{\omega}$ is measurable for $i, j = 1, 2, \ldots$. Therefore, it follows from [Dix 81] (Part II, Chapter 1, Prop 4) that, there exists exactly one measurable vector field structure on the $\mathcal{H}_\omega$'s given by a collection of vector fields $\mathcal{F}$ such that, the vector fields $\omega \mapsto \pi_\omega(A_i)\Theta_\omega$, are measurable with respect to this collection $\mathcal{F}$. Therefore the field of Hilbert spaces $\omega \mapsto \mathcal{H}_\omega$ is a measurable field of Hilbert spaces. \triangle

The above fact allows us to define the direct integral Hilbert space $\mathcal{H}$, of all square integrable vector fields in $\mathcal{F}$ over $\Omega$, from the measurable field of
Hilbert spaces $\omega \mapsto \mathcal{H}_\omega$. Here the inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}_\omega$ is given by

$$\langle \xi, \eta \rangle = \int_\Omega \langle \xi(\omega), \eta(\omega) \rangle dP(\omega),$$

for all square integrable vector fields $\xi, \eta \in \mathcal{F}$. We denote the same by $\int_\Omega \mathcal{H}_\omega dP(\omega)$. It is worth mentioning that the Hilbert space $\mathcal{H}$ is a separable Hilbert space. This follows from the fact that $P$ is the completion of a probability measure defined on the Borel sigma algebra generated by the topology of a complete separable metric space $\Omega$.

Next, we aim to show the following.

**Proposition 4.3.1.2** If $U_t(\omega)$ is the strongly continuous, one-parameter group of unitaries implementing the evolution group $\tau_t(\omega)$ in the representation $\pi_\omega$ then, for each $t \in \mathbb{R}$, $\omega \mapsto U_t(\omega)$ is a measurable field of unitary operators. In fact $(t, \omega) \mapsto \langle U_t(\omega)(\pi_\omega(A_i)\Theta_\omega), \pi_\omega(A_j)\Theta_\omega \rangle$ is jointly measurable in $t$ and $\omega$ for $i, j = 1, 2, \ldots$.

**Proof** It is clear from the proof of proposition 4.3.1.1 and definition 1 in [Dix 81] (Part II, Chapter 1) that, $\{x_i\}$, where $x_i(\omega) = \pi_\omega(A_i)\Theta_\omega$, is a fundamental sequence of measurable vector fields with values in $\mathcal{H}_\omega$. Therefore, it is easily seen from proposition 1 (Chapter 2, Part (II)) in [Dix 81] that, the above proposition will follow if one can show that $\omega \mapsto \langle U_t(\omega)(\pi_\omega(A_i)\Theta_\omega), \pi_\omega(A_j)\Theta_\omega \rangle$, is a measurable scalar valued function for $i, j = 1, 2, \ldots$. Since $U_t(\omega)$ implements $\tau_t(\omega)$, we have $U_t(\omega)(\pi_\omega(A_i))\Theta_\omega = \pi_\omega(\tau_t(\omega)(A_i))\Theta_\omega$. Now $(\pi_\omega, \mathcal{H}_\omega, \Theta_\omega)$ being a cyclic representation of $\mathcal{A}$ associated with the state $\rho(\omega)$, we have $\rho(\omega)(A) = \langle \pi_\omega(A)\Theta_\omega, \Theta_\omega \rangle_\omega$ for all
Therefore, \( \langle U_t(\omega)(\pi_\omega(A_i)\Theta_\omega), \pi_\omega(A_j)\Theta_\omega \rangle_\omega = \rho(\omega)(A_i^*\tau_t(\omega)(A_i)) \), for \( i, j = 1, 2, \ldots \). Hence, the proof of the proposition follows from (1) in theorem 4.2.2.3. In fact, as a consequence of (1) in theorem 4.2.2.3, we have actually shown that the map \( (t, \omega) \mapsto \langle U_t(\omega)(\pi_\omega(A_i)\Theta_\omega), \pi_\omega(A_j)\Theta_\omega \rangle_\omega \) is jointly measurable in \( t \) and \( \omega \) for \( i, j = 1, 2, \ldots \).

The following proposition is a consequence of the preceding proposition.

**Proposition 4.3.1.3** Let \( \{\xi(\omega)\} \) and \( \{\eta(\omega)\} \) be two measurable vector fields in \( \mathcal{F} \). Then the map \( (t, \omega) \mapsto \langle U_t(\omega)(\xi(\omega), \eta(\omega))_\omega \rangle \), is a jointly measurable, scalar valued function of \( t \) and \( \omega \), for all measurable vector fields \( \xi \) and \( \eta \) in \( \mathcal{F} \).

**Proof** It is seen that \( \{x_i\} \), where \( x_i(\omega) = \pi_\omega(A_i)\Theta_\omega \), is a fundamental sequence of measurable vector fields. Therefore, it follows from [Dix 81] (Problem 3, Chapter 1, Part II) that, for any measurable vector field \( \xi \) in \( \mathcal{F} \), there exists a sequence of vector fields \( \xi_n \), of the form \( \xi_n(\omega) = \sum_{i=1}^{n} f_i(\omega)\pi_\omega(A_i)\Theta_\omega \), converging to \( \xi \) almost everywhere, where the \( f_i \)'s are complex valued measurable functions on \( \Omega \). Clearly, these vector fields are measurable with respect to \( \mathcal{F} \). It is readily seen from proposition 4.3.1.2 that, for any two complex valued measurable functions \( f \) and \( g \) on \( \Omega \), the scalar valued function \( (t, \omega) \mapsto \langle U_t(\omega)(f(\omega)\pi_\omega(A_i)\Theta_\omega), g(\omega)\pi_\omega(A_j)\Theta_\omega \rangle_\omega \), is jointly measurable in \( t \) and \( \omega \), for all \( i, j = 1, 2, \ldots \). Hence, for any two vector fields \( \xi \) and \( \eta \) of the form 

\[
\xi(\omega) = \sum_{i=1}^{n} f_i(\omega)\pi_\omega(A_i)\Theta_\omega \quad \text{and} \quad \eta(\omega) = \sum_{k=1}^{m} g_k(\omega)\pi_\omega(A_k)\Theta_\omega
\]

respectively, the scalar valued function \( (t, \omega) \mapsto \langle U_t(\omega)(\xi(\omega), \eta(\omega))_\omega \rangle \), is jointly measurable, where the \( f_i \)'s and \( g_k \)'s are complex valued measurable functions on \( \Omega \). Next,
we shall show that for all measurable vector fields $\xi, \eta$ in $\mathcal{F}$, the function $(t, \omega) \mapsto \langle U_t(\omega)\xi(\omega), \eta(\omega) \rangle_\omega$ is jointly measurable. To this end, let $\{\xi_n\}$ and $\{\eta_n\}$ be sequences of vector fields of the form $\xi_n(\omega) = \sum_{i=1}^n f_i(\omega)\pi_\omega(A_i)\Theta_\omega$ and $\eta_n(\omega) = \sum_{j=1}^n g_j(\omega)\pi_\omega(A_j)\Theta_\omega$ respectively, converging to $\xi$ and $\eta$ almost everywhere, where the $f_i$'s and $g_j$'s are complex valued measurable functions on $\Omega$.

Therefore, for almost every $(t, \omega)$ in $\mathbb{R} \times \Omega$,

$$\lim_{n \to \infty} \langle U_t(\omega)\xi_n(\omega), \eta_n(\omega) \rangle_\omega = \langle U_t(\omega)\xi(\omega), \eta(\omega) \rangle_\omega.$$ 

Since $(t, \omega) \mapsto \langle U_t(\omega)\xi_n(\omega), \eta_n(\omega) \rangle_\omega$ is jointly measurable in $t$ and $\omega$, for all $n \in \mathbb{Z}^+$, the measurability of $(t, \omega) \mapsto \langle U_t(\omega)\xi(\omega), \eta(\omega) \rangle_\omega$ follows easily. \(\triangle\)

Proposition 4.3.1.2 yields the following corollary.

**Corollary 4.3.1.4**

1. For $\lambda \in \mathbb{R}$, if $E_\lambda(\omega)$ are the spectral projections of the generator $H(\omega)$ of $U_t(\omega)$, then $\omega \mapsto E_\lambda(\omega)$ is a measurable field of orthogonal projections.

2. For each $z$ in $\mathcal{C}$, $\omega \mapsto \langle (R(H(\omega), z))\xi(\omega), \eta(\omega) \rangle_\omega$, is a measurable field of resolvent operators, where $R(H(\omega), z)$ stands for the resolvent $(H(\omega) - zI)^{-1}$.

**Proof** Let $\xi$ and $\eta$ be two measurable vector fields in $\mathcal{F}$. Then (2) follows from the fact that, if $\Im z > 0$,

$$\langle (R(H(\omega), z))\xi(\omega), \eta(\omega) \rangle_\omega = i \int_0^\infty e^{ist} \langle e^{-iH(\omega)t}\xi(\omega), \eta(\omega) \rangle_\omega dt$$

and if $\Im z < 0$,

$$\langle (R(H(\omega), z))\xi(\omega), \eta(\omega) \rangle_\omega = -i \int_0^\infty e^{-ist} \langle e^{iH(\omega)t}\xi(\omega), \eta(\omega) \rangle_\omega dt,$$
where $H(\omega)$ is the generator of $U_t(\omega)$ and the integral is a Riemann integral. Now, (1) follows from (2) if we notice that

$$\langle E_{\lambda}(\omega)\xi(\omega), \eta(\omega) \rangle_{\omega}$$

$$= \lim_{\delta \to 0^+} \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\lambda+\delta} \left( (t - i\epsilon - H(\omega))^{-1} - (t + i\epsilon - H(\omega))^{-1} \right) \xi(\omega), \eta(\omega)_{\omega} dt.$$

It was established in corollary 4.2.3.2 that, if $E_{\lambda}(\omega)$ are the spectral projections associated with the generator $H(\omega)$ of $U_t(\omega)$, then, for all $a \in \mathbb{Z}^\nu$, we have $D_{-a}(E_{\lambda}(\omega))D_{-a}^{-1} = E_{\lambda}(T_{-a}\omega)$. Here $D_{-a} : \mathcal{H}_\omega \to \mathcal{H}_{T_{-a}\omega}$ is the isomorphism constructed in subsection 4.2.3. In view of this fact, we have the following proposition.

**Proposition 4.3.1.5** Let $E_{\lambda}(\omega)$ be the spectral projection associated with the generator $H(\omega)$ of $U_t(\omega)$. Then, $\dim(E_{\lambda}(\omega))$ is almost surely constant.

**Proof** We know from proposition 1 in [Dix 81] (Chapter 1, Part II) that, there exists a measurable field of orthonormal bases $\{\xi_n\}$ in the collection of measurable vector fields $\mathcal{F}$. Hence,

$$\dim(E_{\lambda}(\omega)) = \sum_{i=1}^{\infty} \langle E_{\lambda}(\omega)\xi_i(\omega), \xi_i(\omega) \rangle_{\omega}.$$

Using the fact that $E_{\lambda}(T_{-a}\omega) = D_{-a}(E_{\lambda}(\omega))D_{-a}^{-1}$, it is evident that $\dim(E_{\lambda}(\omega))$ is an invariant function of $\omega$. Besides, the measurability of the function $\omega \mapsto \dim(E_{\lambda}(\omega))$ follows from the corollary to proposition 4.3.1.2. Hence, by the ergodicity of the measure preserving group of automorphisms, this invariant measurable function is almost surely constant. \(\triangle\)
The fact that the action of the measure preserving group of automorphisms is ergodic, allows us to derive some interesting results pertaining to the spectra of the generators $H(\omega)$ of the unitary groups $U_t(\omega)$. In this connection we have the following proposition.

**Proposition 4.3.1.6** The discrete and essential spectrum of the generator $H(\omega)$ of the unitary group $U_t(\omega)$, are almost surely independent of $\omega$.

**Proof** For each $\lambda \in \mathbb{R}$ the map $\omega \mapsto E_\lambda(\omega)$ is a measurable field of orthogonal projections associated with the generators $H(\omega)$ of $U_t(\omega)$. Now $\lambda$ is in the essential spectrum $\sigma_{\text{ess}}(H(\omega))$ of $H(\omega)$, if and only if,

$$\dim(E_\nu(\omega) - E_\mu(\omega)) = \infty,$$

for $\mu < \lambda < \nu$. Clearly, the function $\omega \mapsto \dim(E_\nu(\omega) - E_\mu(\omega))$ is an invariant measurable function. So, by ergodicity of the action of the measure preserving group of automorphisms, it is almost surely independent of $\omega$. Hence the essential spectrum $\sigma_{\text{ess}}(H(\omega))$ of $H(\omega)$ is almost surely independent of $\omega$. Now, as the discrete spectrum $\sigma_{\text{dis}}(H(\omega))$ of $H(\omega)$ is such that $\sigma_{\text{dis}}(H(\omega)) = \sigma(H(\omega)) \cap \{\sigma_{\text{ess}}(H(\omega))\}^c$, it follows from proposition 4.2.3.3, that the discrete spectrum is also surely independent of $\omega$. △

**4.3.2 Measurable Field of von Neumann Algebras.**

We begin with the definition of a measurable field of von Neumann algebras.

**Definition 4.3.2.1** Let $\omega \mapsto \mathcal{H}_\omega$ be a measurable field of complex Hilbert spaces over $\Omega$, and for each $\omega \in \Omega$, $\mathcal{A}(\omega)$ be a von Neumann algebra acting
on $\mathcal{H}_\omega$. The field of von Neumann algebras, $\omega \mapsto \mathcal{A}(\omega)$ over $\Omega$, is said to be measurable if, there exists a sequence $\omega \mapsto T_1(\omega), \omega \mapsto T_2(\omega), \ldots$ of measurable field of operators such that, $\mathcal{A}(\omega)$ is the von Neumann algebra generated by the $T_i(\omega)$'s almost everywhere.

On appealing to proposition 4.3.1.1 we demonstrate the following fact.

**Proposition 4.3.2.2** The field of von Neumann algebras $\omega \mapsto \pi_\omega(\mathcal{A})''$ is a measurable field of von Neumann algebras.

**Proof** Let $\mathcal{A}_0$ be defined as in proposition 4.3.1.1. It has been shown that $\omega \mapsto \mathcal{H}_\omega$ is a measurable field of Hilbert spaces on which the $\pi_\omega(\mathcal{A})''$'s act. Consider the sequence $\omega \mapsto \pi_\omega(A_1), \omega \mapsto \pi_\omega(A_2), \ldots$, of fields of operators on $\mathcal{H}_\omega$, where $A_i \in \mathcal{A}_0$. As observed earlier in proposition 4.3.1.1, the vector fields $\{x_i\}$, where $x_i(\omega) = \pi_\omega(A_i)\Theta_\omega$, form a fundamental sequence of measurable vector fields. Therefore, it follows from the measurability of $\omega \mapsto (\pi_\omega(A)\Theta_\omega, \Theta_\omega)_\omega$ for all $A \in \mathcal{A}$, and proposition 1 (Part II, Chapter 2) in [Dix 81] that, $\omega \mapsto \pi_\omega(A_i)$ are measurable fields of operators with respect to $\mathcal{F}$. We have to show that for almost every $\omega \in \Omega$, the von Neumann algebra $\pi_\omega(\mathcal{A})''$ is generated by the $\pi_\omega(A_i)$'s. By definition, this amounts to showing that for almost every $\omega \in \Omega$, $\pi_\omega(\mathcal{A})''$ is the smallest von Neumann algebra containing $\pi_\omega(\mathcal{A}_0)$. i.e., showing that for almost every $\omega \in \Omega$, $\pi_\omega(\mathcal{A})''$ is the smallest von Neumann algebra containing $\pi_\omega(\mathcal{A}_0) \cup \pi_\omega(\mathcal{A}_0)^*$, since the von Neumann algebras containing $\pi_\omega(\mathcal{A}_0)$ are just those containing $\pi_\omega(\mathcal{A}_0) \cup \pi_\omega(\mathcal{A}_0)^*$. This is tantamount to showing that for
almost every \( \omega \in \Omega \), \( \pi_\omega(\mathcal{A})'' \) is the smallest von Neumann algebra containing the \(*\)-algebra \( \mathcal{G}(\pi_\omega(\mathcal{A}_0)) \) generated by \( \pi_\omega(\mathcal{A}_0) \) (smallest \(*\)-algebra containing \( \pi_\omega(\mathcal{A}_0) \)). This follows from the fact that the smallest von Neumann algebra containing \( \pi_\omega(\mathcal{A}_0) \cup \pi_\omega(\mathcal{A}_0)^* \) is precisely the smallest von Neumann algebra containing \( \mathcal{G}(\pi_\omega(\mathcal{A}_0)) \). Since \( \mathcal{G}(\pi_\omega(\mathcal{A}_0))'' \) is the smallest von Neumann algebra containing \( \mathcal{G}(\pi_\omega(\mathcal{A}_0)) \), it amounts to showing that,

\[
\pi_\omega(\mathcal{A})'' = \mathcal{G}(\pi_\omega(\mathcal{A}_0))''.
\]

Since \( \{\pi_\omega(A)\Theta_\omega | A \in \mathcal{A}_0\} \) is dense in \( \mathcal{H}_\omega \), for all \( \omega \in \Omega \), it follows from theorem (10) in [Em 72] (Page 116) that,

\[
\mathcal{G}(\pi_\omega(\mathcal{A}_0))'' = \overline{\mathcal{G}(\pi_\omega(\mathcal{A}_0))}^w,
\]

where for any subset \( \mathcal{N} \subseteq \mathcal{H} \), \( \overline{\mathcal{N}} \) denotes the closure with respect to the operator norm topology and \( \overline{\mathcal{N}}^w \) with respect to the weak operator topology.

Next, \( \pi_\omega(\mathcal{A}_0) \) is operator-norm dense in \( \pi_\omega(\mathcal{A}) \). Therefore,

\[
\overline{\mathcal{G}(\pi_\omega(\mathcal{A}_0))} = \pi_\omega(\mathcal{A}).
\]

Further,

\[
\pi_\omega(\mathcal{A}) = \overline{\mathcal{G}(\pi_\omega(\mathcal{A}_0))} \subseteq \overline{\mathcal{G}(\pi_\omega(\mathcal{A}_0))}^w.
\]

This implies that,

\[
\overline{\pi_\omega(\mathcal{A})}^w \subseteq \overline{\mathcal{G}(\pi_\omega(\mathcal{A}_0))}^w.
\]

Hence,

\[
\overline{\pi_\omega(\mathcal{A})}^w = \overline{\mathcal{G}(\pi_\omega(\mathcal{A}_0))}^w = \mathcal{G}(\pi_\omega(\mathcal{A}_0))''.
\]
Now, the von Neumann algebra $\pi_\omega(A)''$ is the weak closure of $\pi_\omega(A)$. Hence, we conclude that $\pi_\omega(A)''$ is the smallest von Neumann algebra containing $G(\pi_\omega(A_0))$, for almost every $\omega \in \Omega$. Thus, for almost every $\omega \in \Omega$, $\pi_\omega(A)''$ is generated by the measurable field of operators $\omega \mapsto \pi_\omega(A_i)$. This proves the proposition conclusively.

In proposition 4.3.2.2, we demonstrated that $\omega \mapsto \pi_\omega(A)''$ is a measurable field of von Neumann algebras. Since the quasi-local algebra $A$ is simple, the cyclic representation $\pi_\omega$ is a faithful representation of $A$. Therefore, the measurable fields of operators $\omega \mapsto \pi_\omega(A_i)$ which generate $\pi_\omega(A)''$, are essentially bounded. Thus, they define a sequence of decomposable operators $\int_\Omega \pi_\omega(A_i) dP(\omega)$, on the direct integral Hilbert space $\mathcal{H} = \int_\Omega \mathcal{H}_\omega dP(\omega)$. Therefore, it follows from [Dix 81] ((i) in Prop 1, Chapter 3, Part II) that, the set $M$ of all decomposable operators

$$T = \int_\Omega T(\omega) dP(\omega),$$

where $T(\omega) \in \pi_\omega(A)''$, almost everywhere, is a von Neumann algebra, which by definition is a decomposable von Neumann algebra [Dix 81] (See Part II, Chapter 3, Definition 2), and denoted as

$$M = \int_\Omega \pi_\omega(A)'' dP(\omega).$$

Thus, we have succeeded in constructing a direct integral von Neumann algebra $M$, from the representations $\pi_\omega$ of the quasi-local algebra $A$. This was achieved by putting a measurable structure on the Hilbert fields $\omega \mapsto \mathcal{H}_\omega$, using the cyclicity of the representations $\pi_\omega$. This at the same time allowed
the field of von Neumann algebras $\omega \mapsto \pi_\omega(A)$ to acquire a measurable structure. The direct integral von Neumann algebra $\mathcal{M}$ constructed by us is generated by the set of all diagonalisable operators $\mathcal{N}$ and the countable family $\{\int_\Omega \pi_\omega(A_t) dP(\omega)\}$ of decomposable operators.

4.3.3 Automorphism Group of the Direct Integral von Neumann Algebra

Next, we shall construct a $\sigma$-weakly continuous, one-parameter group of automorphisms $\tilde{\tau}$, of the direct integral von Neumann algebra $\mathcal{M}$. We first construct a strongly continuous, one-parameter group of unitaries $U_t$, on the direct integral Hilbert space $\mathcal{H}$. We know that there exists a strongly continuous, one-parameter group of unitaries $U_t(\omega)$ on the Hilbert space $\mathcal{H}_\omega$, which implements the evolution group $\tau_t(\omega)$. It has already been established in proposition 4.3.1.2 that, for each fixed $t \in \mathbb{R}$, $\omega \mapsto U_t(\omega)$ is a measurable field of unitary operators on $\mathcal{H}_\omega$. Clearly, the measurable field of unitary operators $\omega \mapsto U_t(\omega)$ is essentially bounded for each $t \in \mathbb{R}$. In view of this, the measurable field of unitary operators $\omega \mapsto U_t(\omega)$ defines a one-parameter family of decomposable operators $U_t = \int_\Omega U_t(\omega) dP(\omega)$ on $\mathcal{H}$. Moreover, it has been demonstrated in proposition 4.3.1.3 that, for any two measurable vector fields in $\mathcal{F}$, the scalar valued function $(t, \omega) \mapsto (U_t(\omega)\xi(\omega), \eta(\omega))$ is jointly measurable in $t$ and $\omega$. In the discussion that follows, we demonstrate that for each $t \in \mathbb{R}$, the decomposable operator $U_t$ is an unitary operator on the direct integral Hilbert space $\mathcal{H}$ and that, the one-parameter family of decomposable operators $\{U_t\}$, is indeed a strongly continuous one-parameter
group of unitary operators on the direct integral Hilbert space $\mathcal{H} = \int_{\Omega} \mathcal{H}_\omega$.

To this end, we have the following proposition.

**Proposition 4.3.3.1** $U_t$ is an unitary operator on the direct integral Hilbert space $\mathcal{H}$, of square integrable vector fields, for each $t \in \mathbb{R}$.

**Proof** For square integrable vector fields $\xi, \eta \in \mathcal{F}$, we have

$$\langle U_t \xi, U_t \eta \rangle = \int_{\Omega} \langle U_t(\omega)\xi(\omega), U_t(\omega)\eta(\omega) \rangle_\omega dP(\omega),$$

where, $\langle \cdot, \cdot \rangle$, denotes the inner product on the direct integral Hilbert space $\mathcal{H}$, of all square integrable vector fields. Since $\omega \mapsto U_t(\omega)$ is a measurable field of unitary operators, it follows from proposition 1 (Chapter 2, Part II) in [Dix 81] that, $\omega \mapsto U_t(\omega)^*$ is also a measurable field of unitary operators. $U_t(\omega)$ being an unitary operator, we have

$$\langle U_t(\omega)U_t(\omega)^* \xi(\omega), \eta(\omega) \rangle_\omega = \langle \xi(\omega), \eta(\omega) \rangle_\omega,$$

and

$$\langle U_t(\omega)^* U_t(\omega)\xi(\omega), \eta(\omega) \rangle_\omega = \langle \xi(\omega), \eta(\omega) \rangle_\omega.$$

Now it follows from the properties of decomposable operators [Dix 81] (Proposition 3, Chapter 2, Part II) that,

$$U_t^* = \int_{\omega} U_t(\omega)^* dP(\omega) \quad \text{and} \quad U_t U_t^* = \int_{\omega} U_t(\omega)U_t(\omega)^* dP(\omega).$$

Therefore, for $\xi, \eta \in \mathcal{H}$, we have

$$\langle U_t U_t^* \xi, \eta \rangle = \int_{\Omega} \langle U_t(\omega)U_t(\omega)^* \xi(\omega), \eta(\omega) \rangle_\omega dP(\omega)$$

$$= \int_{\Omega} \langle \xi(\omega), \eta(\omega) \rangle_\omega dP(\omega)$$

$$= \langle \xi, \eta \rangle.$$
Similarly, since
\[ U_t^* U_t = \int_\omega U_t(\omega)^* U_t(\omega) d\rho(\omega), \]
we have
\[
\langle U_t^* U_t \xi, \eta \rangle = \int_\Omega \langle U_t(\omega)^* U_t(\omega) \xi(\omega), \eta(\omega) \rangle_{\omega} d\rho(\omega) \\
= \int_\Omega \langle \xi(\omega), \eta(\omega) \rangle_{\omega} d\rho(\omega) \\
= \langle \xi, \eta \rangle.
\]

The proposition now follows readily from the above equalities. \(\triangle\)

We are now in a position to demonstrate the following.

**Theorem 4.3.3.2** The one-parameter family of unitary operators \(\{U_t\}\) on the direct integral Hilbert space \(\mathcal{H}\), is in fact a strongly continuous, one-parameter group of unitary operators on \(\mathcal{H}\).

**Proof** For \(t_1, t_2 \in \mathbb{R}\), and \(\xi, \eta \in \mathcal{H}\), we have
\[
\langle U_{t_1 + t_2} \xi, \eta \rangle = \int_\Omega \langle U_{t_1 + t_2}(\omega) \xi(\omega), \eta(\omega) \rangle_{\omega} d\rho(\omega).
\]

It follows from the properties of decomposable operators [Dix 81] (Proposition 3, Chapter 2, Part II) that,
\[
U_{t_1} U_{t_2} = \int_\Omega U_{t_1}(\omega) U_{t_2}(\omega) d\rho(\omega).
\]

Therefore, since \(U_t(\omega)\) is an unitary group, we have
\[
\langle U_{t_1 + t_2} \xi, \eta \rangle = \int_\Omega \langle U_{t_1}(\omega) U_{t_2}(\omega) \xi(\omega), \eta(\omega) \rangle_{\omega} d\rho(\omega) \\
= \langle U_{t_1} U_{t_2} \xi, \eta \rangle.
\]

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This shows that the family of unitary operators \( \{ U_t \} \) on \( \mathcal{H} \), is a one-parameter group of unitaries with identity \( \int_{\Omega} U_\omega(\omega) dP(\omega) \).

In order to establish that the one-parameter group of unitaries \( U_t \) on \( \mathcal{H} \), is strongly continuous, it is enough to show that the function \( t \mapsto \langle U_t \xi, \eta \rangle \), is continuous in \( t \), for every square integrable vector field \( \xi, \eta \) in \( \mathcal{F} \). Recall that for each \( t \in \mathbb{R} \), \( \omega \mapsto \langle U_t(\omega)\xi(\omega), \eta(\omega) \rangle_\omega \) has been shown to be a measurable field of unitary operators in proposition 4.3.1.2. We also have

\[
|\langle U_t(\omega)\xi(\omega), \eta(\omega) \rangle_\omega| \leq \|\xi(\omega)||\|\eta(\omega)||
\]

and

\[
\int_{\Omega} \|\xi(\omega)||\|\eta(\omega)||dP(\omega) \leq \left( \int_{\Omega} \|\xi(\omega)\|^2 dP(\omega) \right)^{\frac{1}{2}} \left( \int_{\Omega} \|\eta(\omega)\|^2 dP(\omega) \right)^{\frac{1}{2}} < \infty.
\]

Hence, it follows from the dominated convergence theorem that, for all square integrable vector fields \( \xi, \eta \) in \( \mathcal{F} \),

\[
\lim_{t \to 0} \langle U_t\xi, \eta \rangle = \lim_{t \to 0} \int_{\Omega} \langle U_t(\omega)\xi(\omega), \eta(\omega) \rangle_\omega dP(\omega) = \int_{\Omega} \lim_{t \to 0} \langle U_t(\omega)\xi(\omega), \eta(\omega) \rangle_\omega dP(\omega).
\]

Moreover, \( U_t(\omega) \) is a strongly continuous, one-parameter group of unitary operators. Therefore,

\[
\lim_{t \to 0} \langle U_t\xi, \eta \rangle = \int_{\Omega} \langle \xi(\omega), \eta(\omega) \rangle_\omega dP(\omega) = \langle \xi, \eta \rangle.
\]

Since \( U_t \) has been endowed with a group structure, this proves the strong continuity of the one-parameter group of unitaries \( U_t \), on the direct integral Hilbert space \( \mathcal{H} \), conclusively. \( \triangle \)
Next, for

\[ S = \int_\Omega S(\omega)dP(\omega) \in \mathcal{M}, \]

define

\[ \tilde{\tau}_t(S) = U_tSU_t^{-1}. \]

Now, \( \omega \mapsto U_t(\omega)S(\omega)U_t(\omega)^{-1} \) is an essentially bounded measurable field of operators for each \( t \in \mathbb{R} \). It follows from the properties of decomposable operators that,

\[ U_tSU_t^{-1} = \int_\Omega U_t(\omega)S(\omega)U_t(\omega)^{-1}dP(\omega). \]

Since \( U_t(\omega)S(\omega)U_t(\omega)^{-1} \in \pi_\omega(A)'\), we have \( U_tSU_t^{-1} \in \mathcal{M} \). Thus, it follows from the strong continuity of \( U_t \) that \( \tilde{\tau}_t \) is a \( \sigma \)-weakly continuous, one-parameter group of automorphisms of the decomposable von Neumann algebra \( \mathcal{M} \).

4.3.4 Construction of a KMS State of the Direct Integral von Neumann Algebra

Finally, we establish the existence of a faithful, normal \((\tilde{\tau}, \beta)\)-KMS state of \( \mathcal{M} \). Now, the state \( \rho(\omega) \) which can be written as a vector state \( \rho(\omega)(A) = \langle \pi_\omega(A)\Theta_\omega, \Theta_\omega \rangle_\omega \) in the representation \( \pi_\omega \), on a separable Hilbert space \( \mathcal{H} \), is a \((\tau(\omega), \beta)\)-KMS state. Therefore, it follows from corollary 5.3.4, in [Rob 81] and theorem 4.12 in [Hug 72] that, \( \rho(\omega) \) can be easily extended to a faithful, normal \((\tilde{\tau}(\omega), \beta)\)-KMS state \( \tilde{\rho}(\omega) \), of the von Neumann algebra \( \pi_\omega(A)'' \), where \( \tilde{\rho}(\omega)(S) = \langle S\Theta_\omega, \Theta_\omega \rangle_\omega \), for \( S \in \pi_\omega(A)'' \) and \( \tilde{\tau}(\omega) \) is the \( \sigma \)-weakly continuous group of automorphisms of \( \pi_\omega(A)'' \). Clearly, the restriction of \( \tilde{\rho}(\omega) \)
to $\pi_\omega(A)$ gives the state $\rho(\omega)$ on $A$.

Let us now construct a state $\hat{\rho}$ of the von Neumann algebra $\mathcal{M}$ from the field of states $\omega \mapsto \hat{\rho}(\omega)$, on $\pi_\omega(A)''$. Such a field of states on the von Neumann algebras $\pi_\omega(A)''$ is said to be a measurable field if, $\omega \mapsto \hat{\rho}(\omega)(T(\omega))$ is a measurable function of $\omega$ for every measurable field of operators $\omega \mapsto T(\omega)$.

Since, $\omega \mapsto \Theta_\omega$ is a measurable vector field with respect to $\mathcal{F}$, it is clear from the definition of $\hat{\rho}(\omega)$ that, $\omega \mapsto \hat{\rho}(\omega)$ is a measurable field of states on $\pi_\omega(A)''$. Define

$$\hat{\rho} \left( \int_\Omega S(\omega)dP(\omega) \right) = \int_\Omega \hat{\rho}(\omega)(S(\omega))dP(\omega),$$

for all decomposable operators $\omega \mapsto S(\omega)$ in $\mathcal{M}$. Let $\alpha \in \mathcal{C}$, and $\omega \mapsto S(\omega)$, $\omega \mapsto S_1(\omega)$, $\omega \mapsto S_2(\omega)$ define elements in $\mathcal{M}$. It follows from the properties of decomposable operators (Proposition 3, Page 182 in [Dix '81]) that,

$$\hat{\rho} \left( \int_\Omega S_1(\omega)dP(\omega) + \int_\Omega S_2(\omega)dP(\omega) \right)$$

$$= \hat{\rho} \left( \int_\Omega (S_1(\omega) + S_2(\omega))dP(\omega) \right)$$

$$= \int_\Omega \hat{\rho}(\omega)(S_1(\omega) + S_2(\omega))dP(\omega)$$

$$= \int_\Omega \hat{\rho}(\omega)(S_1(\omega))dP(\omega) + \int_\Omega \hat{\rho}(\omega)(S_2(\omega))dP(\omega)$$

$$= \hat{\rho} \left( \int_\Omega S_1(\omega)dP(\omega) \right) + \hat{\rho} \left( \int_\Omega S_2(\omega)dP(\omega) \right)$$

Also,

$$\hat{\rho} \left( \alpha \int_\Omega S(\omega)dP(\omega) \right) = \int_\Omega \hat{\rho}(\omega)(\alpha S(\omega))dP(\omega)$$

$$= \alpha \int_\Omega \hat{\rho}(\omega)S(\omega)dP(\omega)$$

$$= \alpha \hat{\rho} \left( \int_\Omega S(\omega)dP(\omega) \right)$$

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Hence, $\rho$ is a linear functional on $\mathcal{M}$. Since,

$$\rho\left(\int_{\Omega} I_\omega dP(\omega)\right) = \int_{\Omega} \rho(\omega)(I_\omega) dP(\omega) = 1,$$

where $I(\omega)$ is the identity operator on $\mathcal{H}_\omega$, $\rho$ is a state.

**Theorem 4.3.4.1** Let $\rho$ be the state constructed above. Then $\rho$ is a faithful, normal state of the decomposable von Neumann algebra $\mathcal{M}$.

**Proof** Let $\omega \mapsto S(\omega)$ define a decomposable operator in $\mathcal{M}^+$, where $\mathcal{M}^+$ is the set of all positive elements in $\mathcal{M}$. Put $S = \int_{\Omega} S(\omega) dP(\omega)$. Suppose we have $\rho(S) = 0$, then it follows from the definition of $\rho$ that

$$\int_{\Omega} \rho(\omega)(S(\omega)) dP(\omega) = 0.$$

Since $\omega \mapsto \rho(S(\omega))$ is a non negative measurable function of $\omega$, we have $\rho(\omega)(S(\omega)) = 0$, almost everywhere. Therefore, $S(\omega) = 0$ almost everywhere, since the $\rho(\omega)$'s are faithful. Next, we show that the state $\rho$ is a normal state on $\mathcal{M}$. We know that the $\rho(\omega)$'s are normal states. Let $\{S_\lambda\}$ be an increasing net of elements in $\mathcal{M}^+$ with supremum $S \in \mathcal{M}^+$. Let us denote the collection of all diagonalisable operators on $\mathcal{H}$ by $\mathcal{Z}$. Clearly, $\mathcal{Z} \subseteq \mathcal{M} \subseteq \mathcal{Z}'$. Since $\mathcal{Z}'$ is a $\sigma$-finite von Neumann algebra [Dix 81] (See Proposition 7, Chapter 2, Part II) of all decomposable operators on $\mathcal{H}$, it follows that $\mathcal{M}$ is also $\sigma$-finite. Therefore, it follows from the corollary to proposition 1 in [Dix 81] (Part I, Chapter 3) that, one can extract an increasing sequence $S_n = \int_{\Omega} S_n(\omega) dP(\omega)$ from the net $\{S_\lambda\}$ with supremum $S$. Now, there exists an increasing sequence of integers $\{n_k\}$ such that, $S_{n_k}(\omega)$ is an increasing sequence converging strongly to $S(\omega)$ almost...
everywhere. This is a trivial consequence of proposition 4, in [Dix 81] (Chapter 2, Part II). Hence, \( \hat{\rho}(\omega)(S_{n_k}(\omega)) \) is an increasing sequence converging to \( \hat{\rho}(\omega)(S(\omega)) \) almost everywhere. Therefore, it follows from the definition of \( \hat{\rho} \) and the monotone convergence theorem that, \( \hat{\rho}(S_{n_k}) \) is an increasing sequence converging to \( \hat{\rho}(S) \). This proves conclusively that \( \hat{\rho} \) is a normal state on \( \mathcal{M} \).

Now all that remains to be shown is that the state \( \hat{\rho} \) is \((\hat{\tau}, \beta)\)-KMS state. Before we establish this fact, let us give an equivalent definition of a KMS state which we shall have the occasion to use.

**Definition 4.3.4.2** Let \( \hat{\rho} \) be a state on a von Neumann algebra \( R \) and \( \hat{\tau}_t \) a \( \sigma \)-weakly continuous, one-parameter group of automorphisms of the von Neumann algebra \( R \). Then, \( \hat{\rho} \) is said to be a \((\hat{\tau}, \beta)\)-KMS state if,

\[
\int_{-\infty}^{\infty} f_\beta(t) \hat{\rho}(A \hat{\tau}_t B) dt = \int_{-\infty}^{\infty} f_\beta(t) \hat{\rho}(\hat{\tau}_t(B)A) dt,
\]

for all \( A, B \in R \) and all \( f \) infinitely differentiable with compact support. In the above equality, \( f_\gamma(t) = \int_{-\infty}^{\infty} \hat{f}(s)e^{(t+\gamma)s} ds \), for \( \gamma = 0 \) and \( -\beta \).

**Theorem 4.3.4.3** The state \( \hat{\rho} \) constructed above is a \((\hat{\tau}, \beta)\)-KMS state of the direct integral von Neumann algebra \( \mathcal{M} \).

**Proof** For \( S = \int_\Omega S(\omega) dP(\omega), T = \int_\Omega T(\omega) dP(\omega) \in \mathcal{M}, \)

\[
\int_{-\infty}^{\infty} f_\beta(t) \hat{\rho}(T(\hat{\tau}_t(S))) dt
= \int_{-\infty}^{\infty} f_\beta(t) \hat{\rho}(T(U_t S U_t^{-1})) dt
= \int_{-\infty}^{\infty} f_\beta(t) \left( \int_\Omega \hat{\rho}(T(\omega)(U_t(\omega) S(\omega) U_t(\omega)^{-1})) dP(\omega) \right) dt
\]

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\[
= \int_{-\infty}^{\infty} f_{-\beta}(t) \left( \int_{\Omega} (T(\omega)(U(t)(S(\omega)U(t)^{-1})o,\Theta_{\omega}, \Theta_{\omega} dP(\omega)) dt.
\]

It follows readily from the measurable structure imposed on the vector field of Hilbert spaces that, \( \omega \mapsto \Theta_{\omega} \) is a measurable field of vectors. Moreover, both \( \omega \mapsto S(\omega) \) and \( \omega \mapsto T(\omega) \) are essentially bounded measurable fields of operators. Therefore, it follows from the remark made at the end of proposition 1 in [Dix 81] (Part II, Chapter 2) and the definition of measurable fields of operators that, both \( \omega \mapsto S(\omega)\Theta_{\omega} \) and \( \omega \mapsto T(\omega)^*\Theta_{\omega} \) are measurable vector fields. Since

\[
(T(\omega)(U(t)(S(\omega)U(t)^{-1})0,\Theta_{\omega}, \Theta_{\omega}) = (U(t)(S(\omega)Theta), T(\omega)^*Theta_{\omega}),
\]

it follows from proposition 4.3.1.3 that,

\[
(t, \omega) \mapsto (T(\omega)(U(t)(S(\omega)U(t)^{-1})0,\Theta_{\omega}, \Theta_{\omega}),
\]

is jointly measurable in \( t \) and \( \omega \). By the Paley–Weiner theorem, \( f_{-\beta}(t) \) is an integrable function of \( t \). Moreover, the \( S(\omega)'s \), and \( T(\omega)'s \) are essentially bounded in norm. Therefore, invoking Fubini's theorem for scalar valued functions on \( R \times \Omega \) and using the fact that \( \tilde{\rho}(\omega) \) is a \((\tilde{\tau}(\omega), \beta)\)-KMS state on \( \pi_{\omega}(A)^*\), where \( \tilde{\tau}_{\omega}(A) = U(t(\omega)AU(t(\omega)^{-1}, \) we have

\[
\int_{-\infty}^{\infty} f_{-\beta}(t)\tilde{\rho}(T\tilde{\tau}(S))
\]

\[
= \int_{\Omega} \left( \int_{-\infty}^{\infty} f_{-\beta}(t)\tilde{\rho}(T(\omega)(U(t)(S(\omega)U(t)^{-1})\Theta_{\omega}, \Theta_{\omega}) dt \right) dP(\omega)
\]

\[
= \int_{\Omega} \left( \int_{-\infty}^{\infty} f_{-\beta}(t)\tilde{\rho}(\omega)(T(\omega)\tilde{\tau}(\omega)(S(\omega))) dt \right) dP(\omega)
\]

\[
= \int_{\Omega} \left( \int_{-\infty}^{\infty} f_{\omega}(t)\tilde{\rho}(\omega)(\tilde{\tau}(\omega)(S(\omega))) T(\omega) dt \right) dP(\omega)
\]

\[
= \int_{\Omega} \left( \int_{-\infty}^{\infty} f_{\omega}(t)((U(t(\omega)S(\omega)U(t(\omega)^{-1})T(\omega)\Theta_{\omega}, \Theta_{\omega}) dt \right) dP(\omega).
\]
Arguing as above, one can show that both $\omega \mapsto T(\omega)\Theta_\omega$ and $\omega \mapsto S(\omega)^*\Theta_\omega$ are measurable vector fields. Since

$$\langle(U_t(\omega)S(\omega)U_t(\omega)^{-1})T(\omega)\Theta_\omega, \Theta_\omega\rangle_\omega = \langle U_t(\omega)^{-1}(T(\omega)\Theta_\omega), S(\omega)^*\Theta_\omega\rangle_\omega,$$

it follows from proposition 4.3.1.3 that,

$$(t, \omega) \mapsto \langle U_t(\omega)S(\omega)U_t(\omega)^{-1}T(\omega)\Theta_\omega, \Theta_\omega\rangle_\omega,$$

is jointly measurable. Again, by the Paley–Weiner theorem, $f_0(t)$ is an integrable function of $t$. Hence, on applying Fubini's theorem a second time, we get

$$\int_{-\infty}^{\infty} f_{\beta}(t) \hat{\rho}(T\tau_t(S)) = \int_{-\infty}^{\infty} f_0(t) \left( \int_{\Omega} \langle(U_t(\omega)S(\omega)U_t(\omega)^{-1})T(\omega)\Theta_\omega, \Theta_\omega\rangle dP(\omega) \right) dt
= \int_{-\infty}^{\infty} f_0(t) \left( \int_{\Omega} \hat{\rho}(\omega)(U_t(\omega)S(\omega)U_t(\omega)^{-1})T(\omega)) dP(\omega) \right) dt
= \int_{-\infty}^{\infty} f_0(t) \hat{\rho}(U_tSU_t^{-1}) dt
= \int_{-\infty}^{\infty} f_0(t) \hat{\rho}(\tau_t(S)T) dt$$

This proves conclusively that, $\hat{\rho}$ is a $(\tau, \beta)$-KMS state on the direct integral von Neumann algebra $M$. \[ \triangle \]

Since the family of KMS states $\{\rho(\omega)\}$ of $A$ is not unique, the $(\tau, \beta)$-KMS state $\hat{\rho}$ is by no means unique. However, in view of theorem 4.2.2.2, above the critical temperature $T_c$, there is an unique family of KMS states $\{\rho(\omega)\}$, which determines the KMS state $\hat{\rho}$ on $M$ as constructed above.