Chapter 3

Dynamics of a Quantum Spin Glass

In this chapter we give a detailed account of the models of a quantum spin glass investigated by us. In the sequel, we give a description of the models and establish the existence of global dynamics among other things. Traditionally, quantum spin glasses have been studied as systems of quantum spins interacting through random interactions. These models are essentially Ising-type models with random coupling. Generally, the coupling coefficients are assumed to be independent, identically distributed random variables. An alternate model of a quantum spin glass can be based on the realization that the magnetic ions are randomly distributed at lattice sites. The spins therefore, may be considered to be located at the vertices of an infinite graph in a lattice. There is no translation invariance in such a system, the lattice itself plays no significant role. Therefore, one can caricature a quantum spin glass as a quantum spin system with spins located at the vertices of an infinite connected graph with countably infinite number of vertices. In such a system, it is not necessary to consider random interactions. The study is
restricted to deterministic interactions of the nearest neighbour type, with
two spins defined as neighbours if an edge connects the two. This model
may be regarded as a quantum analogue of the systems studied by Preston
[Pre 74] and others. In the sequel, we establish the existence of the global
dynamics of this infinite system of quantum spins, discuss the equilibrium
state and establish the Kubo-Martin-Schwinger (KMS) condition. However,
our attempts to establish the maximum entropy principle failed on account
of absence of spatial homogeneity.

As expected, the thermodynamic limit of the local Gibbs states exists. Thus,
an equilibrium state at a fixed inverse temperature $\beta$, exists for a quantum
spin system on an infinite graph. But this state is by no means unique. It
is shown that it satisfies the Kubo–Martin–Schwinger condition. We would
like to point out that these equilibrium states which arise as thermodynamic
limits of the the local Gibbs state are known to exist in the case of quan-
tum spin systems, where the spins are located at each point of a countably
infinite set $L$. In such cases, there is no additional structure imposed on the
set $L$. However, in order to construct the dynamics for such spin systems,
one has to put stringent conditions on the nature of the interactions between
spins. In fact, in many cases, the interaction potentials are assumed to be of
exponential nature. Whereas, in the case of a quantum spin system on an
infinite graph, because of the additional structure, one does not have to be
very restrictive regarding the class of interaction potentials.
3.1 A Quantum Spin System on an Infinite Graph

Definition 3.1.0.5 A graph is said to be simple if it has no loops or multiple edges. Such a graph is said to have a finite valency if there exists an \( \alpha \in \mathbb{Z}^+ \) such that, at most \( \alpha \) edges are incident on any vertex. Here \( \mathbb{Z}^+ \) denotes the set of all positive integers.

Definition 3.1.0.6 A non empty finite subset \( S \subseteq V \) is said to be a simplex of the graph \( G(V,E) \) if, for every \( v_1, v_2 \in S \), there exists an edge connecting the two. A subset \( S \subseteq V \) is said to be a \( n \)-simplex \( (n \geq 0) \) if \( S \) is a simplex of the graph \( G \) and \( |S| = n + 1 \). Here \( |.| \) denotes the cardinality of the set.

Lemma 3.1.0.7 It is easily seen that, given a simple graph \( G(V,E) \) with finite valency \( \alpha \in \mathbb{Z}^+ \) and \( v \in V \), there is no \( n \)-simplex for \( n > \alpha \) and there are at most only a finite number of simplexes containing \( v \).

Consider a quantum spin system on an infinite connected graph \( G(V,E) \), where \( V \) is the set of countably infinite number of vertices and \( E \) the collection of edges. The graph is assumed to be simple and has finite valency, say, \( \alpha \in \mathbb{Z}^+ \). By a connected graph we mean that there is a path connecting any two vertices of the graph. A quantum spin is assumed to be located at each of these vertices. Two spins interact if they are connected by an edge. A quasi-local UHF algebra constructed over finite subsets of the vertices of the graph is associated with this spin system. Explicitly, one can order the collection of all finite subsets of vertices by inclusion. With each vertex \( v \)
of the graph $G(V,E)$, one can associate a two dimensional complex Hilbert space $H_v$. Then, with each finite $^1 \Lambda \subseteq V$, we associate the tensor product space

$$\mathcal{H}_\Lambda = \bigotimes_{v \in \Lambda} H_v.$$ 

We then define the local $\mathcal{C}^*$-algebra $\mathcal{A}_\Lambda$ for each finite subset $\Lambda \subseteq V$ by $\mathcal{A}_\Lambda = \mathcal{L}(H_\Lambda)$, where $\mathcal{L}(H_\Lambda)$ denotes the space of all bounded linear operators on $H_\Lambda$. Now, if $\Lambda_1 \cap \Lambda_2 = \emptyset$ for $\Lambda_1, \Lambda_2 \subseteq V$, then $\mathcal{H}_{\Lambda_1 \cup \Lambda_2} = H_{\Lambda_1} \otimes H_{\Lambda_2}$ and $\mathcal{A}_{\Lambda_1}$ is isomorphic to the $\mathcal{C}^*$-subalgebra $\mathcal{A}_{\Lambda_1} \otimes I_{\Lambda_2}$, where $I_{\Lambda_2}$ is the identity operator on $H_{\Lambda_2}$. Further, if $\Lambda_1 \subseteq \Lambda_2$, one can identify $\mathcal{A}_{\Lambda_1}$ with the subalgebra $\mathcal{A}_{\Lambda_1} \otimes I_{\Lambda_2 \setminus \Lambda_1}$ of $\mathcal{A}_{\Lambda_2}$. Let the identification map be given by $i_{\Lambda_2, \Lambda_1} : A \in \mathcal{A}_{\Lambda_1} \rightarrow A \otimes I_{\Lambda_2 \setminus \Lambda_1} \in \mathcal{A}_{\Lambda_2}$. The collection $\{\mathcal{A}_\Lambda | \Lambda \subseteq V\}$ along with the collection of maps $\{i_{\Lambda_2, \Lambda_1}\}$ has the structure of a directed system of $\mathcal{C}^*$-algebras. Therefore, there exists a $\mathcal{C}^*$-algebra $\mathcal{A}$ with an identity $I$, which is the inductive limit of the collection $\{\mathcal{A}_\Lambda | \Lambda \subseteq V\}$ of $\mathcal{C}^*$-algebras with identity $I_\Lambda$. i.e., there exists a $\mathcal{C}^*$-algebra $\mathcal{A}$ and injective $*$-homomorphisms $i_\Lambda : \mathcal{A}_\Lambda \rightarrow \mathcal{A}$ such that,

$$\Lambda_1 \subseteq \Lambda_2 \rightarrow i_{\Lambda_1}(\mathcal{A}_{\Lambda_1}) \subseteq i_{\Lambda_2}(\mathcal{A}_{\Lambda_2}),$$

$$\bigcup_{\Lambda \subseteq V} i_\Lambda(\mathcal{A}_\Lambda) = \mathcal{A},$$

and

$$i_\Lambda(I_\Lambda) = I; \quad \forall \Lambda \subseteq V.$$ 

$^1$Throughout this chapter and the next, all $\Lambda$'s, $X$'s and $Y$'s which feature as subsets of either $V$ or $Z^*$, with or without subscripts, should be taken to be finite unless stated otherwise.
Also, for
\[ \Lambda_1 \cap \Lambda_2 = \emptyset; \quad [i_{\Lambda_1}(A_{\Lambda_1}), i_{\Lambda_2}(A_{\Lambda_2})] = 0, \]

where \([., .]\) is the commutator. We will hence forth leave out the \(i_{\Lambda_2, \Lambda_1}'s\) and \(i_{\Lambda}'s\) whenever no confusion can arise and regard \(A_{\Lambda}'s\) as subalgebras of \(A\). This object \(A\), along with the net of local \(C^*\)-algebras \(\{A_{\Lambda}\}_{\Lambda \in \mathcal{V}}\) is a quasi-local algebra (The orthogonality relation \(\perp \) between \(\Lambda\)'s is defined by \(\Lambda_1 \perp \Lambda_2 \) if \(\Lambda_1 \cap \Lambda_2 = \emptyset\)). It is worth noting that \(A\) is an uniformly matricial algebra (UHF), and hence a separable \(C^*\)-algebra which is simple [Rob 81]. The local algebra \(A_{\Lambda}\) represents the physical observables associated with the spins located in a finite region \(\Lambda\), whereas the quasi-local algebra \(A\), corresponds to the observables of the infinite spin system.

### 3.2 Interactions

**Definition 3.2.0.8** An interaction \(\Phi\) is a function from the collection \(\mathcal{F}\) of finite subsets \(X\) of \(V\) into the Hermitian (self adjoint) elements in \(A\) such that, for every finite \(X \subseteq V\), \(\Phi(X) \in A_X\).

**Definition 3.2.0.9** An interaction \(\Phi\) is said to be of the nearest neighbour type if, \(\Phi(X) = 0\) whenever \(X\) is not a simplex of the graph \(G\).

Now for a finite \(X\), \(\Phi(X)\) represents the interaction energy of the spins confined to \(X \subseteq V\). Hence, the total interaction energy for a finite \(\Lambda \subseteq V\), consists of the interaction energy of all finite subsystems \(X \subseteq \Lambda\). Thus, we
define this total energy as the Hamiltonian $H(\Lambda)$ associated with $\Lambda \subseteq V$, i.e

$$H(\Lambda) = \sum_{X \subseteq \Lambda} \Phi(X).$$

$H(\Lambda)$ is a Hermitian (self adjoint) element of $A_{\Lambda}$.

### 3.3 Time Evolution

In order to study the evolution of the infinite spin system, we write down the following equation of motion:

$$\frac{dA^\Lambda_t}{dt} = i[H(\Lambda), A^\Lambda_t], \quad A^\Lambda_t \in A_{\Lambda}.$$  

Here, $t \mapsto A^\Lambda_t$ describes the evolution of the observable $A \in A_{\Lambda}$. This equation of motion defines a rule by which the observables associated with a finite $\Lambda \subseteq V$, evolve. With every $A \in A_{\Lambda}$, it associates the observable $\tau_t^\Lambda(A) = A^\Lambda_t = e^{iH(\Lambda)t}Ae^{-iH(\Lambda)t}$, which yields the quantum evolution of the finite spin system. Clearly, $\tau_t^\Lambda(A)$ is an element of $A_{\Lambda}$ and $\tau_t^\Lambda$ is a one-parameter group of *-automorphisms of $A_{\Lambda}$, which defines the time evolution of the finite subsystem confined to $\Lambda \subseteq V$. As the system consists of infinite number of spins, computing the time evolution of a fixed observable $A \in A_{\Lambda_0}$, where $\Lambda_0 \subseteq V$, entails calculating the limit of $\tau_t^\Lambda(A)$ as $\Lambda \to \infty$. Here we adopt the convention that, $\Lambda \to \infty$ indicates $\Lambda$ eventually contains all finite subsets of $V$. It is our endeavour to show that for a certain class of potentials this limit exists for all $A \in A_{\Lambda_0}$. In order to make this notion of convergence precise, we observe that the collection $\mathcal{F}$ of all finite subsets $\Lambda$ of $V$ which is partially ordered by inclusion, is an increasing directed set. Hence, when
we say that a net $S_A$ converges to $S$ in $A$, as $\Lambda \to \infty$ ($\Lambda$ eventually contains all finite subsets of $V$), we mean that for a given $\epsilon > 0$, there exists a finite subset $\Lambda'$ of $V$ such that, $\|S_A - S\| < \epsilon$, whenever $\Lambda \supseteq \Lambda'$. This is equivalent to showing that for a given $\epsilon > 0$, there exists a finite subset $\Lambda'$ of $V$ such that, $\|S_{A_1} - S_{A_2}\| < \epsilon$ whenever $A_1 \supseteq A'$ and $A_2 \supseteq A'$.

Next, note that the time evolution $\tau_t^A (A)$ of a finite system can be expanded in terms of commutators as

$$
\tau_t^A (A) = e^{iH(A)t}Ae^{-iH(A)t} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} [H(A), A]^{(n)},
$$

(3.3.1)

where

$$
[B, A]^{(0)} = A, \quad [B, A]^{(1)} = [B, A] = BA - AB,
$$

and

$$
[B, A]^{(n+1)} = [B, [B, A]^{(n)}].
$$

This formula is easily verified by taking derivatives of the expression in the middle and that of the expression on the right hand-side of (3.3.1).

In order to establish the dynamics of the spin system, we prove the following proposition.

**Proposition 3.3.0.10** Let $\Phi$ be a nearest neighbour type of interaction for the quantum spin system on the infinite graph $G(V, E)$ with valency $\alpha$ such that,

$$
\sup_{v \in V} \left( \sum_{X \ni v} \|\Phi(X)\| \right) < \infty.
$$

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Then, for $A \in \mathcal{A}_{\Lambda_0}$ with $\Lambda_0 \subseteq V$, we have

$$\|[H(A), A]^{(n)}\| \leq \|A\|e^{\|\Lambda_0\|n!}\left(2\left(\sup_{v \in V} \sum_{X \ni v} \|\Phi(X)\|\right)e^{(\alpha+1)}\right)^n. \quad (3.3.2)$$

**Proof** Take $A \in \mathcal{A}_{\Lambda_0}$, where $\Lambda_0 \subseteq V$. One has, $\Phi(X) \in \mathcal{A}_X$ for $X \subseteq V$.

Now the local algebras $\mathcal{A}_{\Lambda_1}, \mathcal{A}_{\Lambda_2}$ commute whenever $\Lambda_1 \cap \Lambda_2 = \emptyset$.

Therefore,

$$\|[H(A), A]^{(n)}\| = \left\| \sum_{X_1 \subseteq \Lambda} \cdots \sum_{X_n \subseteq \Lambda} [\Phi(X_n), \ldots [\Phi(X_1), A]] \right\|$$

$$= \sum_{X_1 \subseteq \Lambda} \cdots \sum_{X_n \subseteq \Lambda} \|[\Phi(X_n), \ldots [\Phi(X_1), A]]\|$$

$$\leq \sum_{X_1 \cap S_0 \neq \emptyset} \cdots \sum_{X_n \cap S_{n-1} \neq \emptyset} \|[\Phi(X_n), \ldots [\Phi(X_1), A]]\|$$

where

$$S_0 = \Lambda_0$$

and

$$S_j = X_j \cup X_{j-1} \cup \ldots \cup X_1 \cup \Lambda_0, \quad \text{for } j \geq 1.$$

Since $\Phi$ is a nearest neighbour type of interaction potential, on applying lemma 3.1.0.7, we notice that if

$$[\Phi(X_j), \ldots [\Phi(X_1), A]] \neq 0,$$

where

$$[\Phi(X_j), \ldots [\Phi(X_1), A]] \in \mathcal{A}_{S_j},$$

then,

$$|X_i| \leq \alpha + 1, \quad \forall i = 1, 2, \ldots, j,$$

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where \( \alpha \) is the valency of the graph \( G(V, E) \). Therefore,

\[
|S_j| \leq |X_j| + |X_{j-1}| + \cdots + |X_1| + |\Lambda_0| \\
\leq j(\alpha + 1) + |\Lambda_0|,
\]

Thus, we get

\[
\| [H(\Lambda), A]^{(n)} \| \leq 2^n \| A \| \sum_{v_i \in S_0} \cdots \sum_{v_n \in S_{n-1}} \sum \| \Phi(X_i) \| \cdots \| \Phi(X_1) \| \\
\leq 2^n \| A \| \prod_{i=1}^n ((i-1)(\alpha + 1) + |\Lambda_0|) \left( \sup_{v_i \in V} \sum_{X_i \in \mathcal{X}} \| \Phi(X_i) \| \right)^n \\
\leq 2^n \| A \| \prod_{i=1}^n ((i-1)(\alpha + 1) + |\Lambda_0|) \left( \sup_{v \in V} \sum_{X \in \mathcal{X}_v} \| \Phi(X) \| \right)^n \\
\leq 2^n \| A \| (n(\alpha + 1) + |\Lambda_0|)^n \left( \sup_{v \in V} \sum_{X \in \mathcal{X}_v} \| \Phi(X) \| \right)^n.
\]

Now \( a^n \leq n!e^n \), for \( a > 0 \) hence,

\[
\| [H(\Lambda), A]^{(n)} \| \leq \| A \| e^{\| \Lambda_0 \|} 2^n n! \left( \sup_{v \in V} \sum_{X \in \mathcal{X}_v} \| \Phi(X) \| \right)^n e^{n(\alpha + 1)} \\
\leq \| A \| e^{\| \Lambda_0 \|} n! \left( 2 \left( \sup_{v \in V} \sum_{X \in \mathcal{X}_v} \| \Phi(X) \| \right) e^{(\alpha + 1)} \right)^n.
\]

Notice that this estimate is independent of \( \Lambda \) and hence, we have

\[
[H(\Lambda), A]^{(n)} \to \sum_{X_1 \in V} \cdots \sum_{X_n \in V} [\Phi(X_n), [\cdots [\Phi(X_1), A]]]
\]
as \( \Lambda \to \infty \).

\( \triangle \)

**Theorem 3.3.0.11** Let \( \Phi \) be a nearest neighbour type of interaction potential for the quantum spin system on the infinite graph \( G(V, E) \) with valency \( \alpha \) such
that,
\[ \sup_{v \in \mathcal{V}} \left( \sum_{X \notin v} \| \Phi(X) \| \right) < \infty. \]

Then, there exists a strongly continuous, one–parameter group of \( * \)-automorphisms \( \tau_t \) of \( \mathcal{A} \) such that, for all \( A \in \mathcal{A} \) we have

\[ \tau_t(A) = \lim_{\Lambda \to \infty} \tau^\Lambda_t(A), \]

where

\[ \tau^\Lambda_t(A) = e^{i(H(\Lambda)t} A e^{-iH(\Lambda)t}, \]

and the limit is uniform for \( t \) on compact sets.

**Proof** We shall use the fact that

\[ [H(\Lambda), A]^{(n)} \rightarrow \sum_{X_1 \notin V} \ldots \sum_{X_n \notin V} [\Phi(X_1), \ldots [\Phi(X_n), A]], \]

as \( \Lambda \to \infty \) and inequality 3.3.2 to demonstrate that for \( A \in \mathcal{A}_{\Lambda_0} \), the limit of \( \tau^\Lambda_t(A) \) exists as \( \Lambda \to \infty \).

Put

\[ T = \left( 2 \left( \sup_{v \in \mathcal{V}} \sum_{X \notin v} \| \Phi(X) \| \right) e^{(\alpha+1)} \right)^{-1}. \]

It follows from equation 3.3.1 that for \( A \in \mathcal{A}_{\Lambda_0} \),

\[ \| \tau^\Lambda_1(A) - \tau^\Lambda_2(A) \| \]

\[ \leq \| \sum_{n=0}^{N} \frac{i^n}{n!} ([H(\Lambda_1), A]^{(n)} - [H(\Lambda_2), A]^{(n)}) t^n \| + \| \sum_{n=N+1}^{\infty} \frac{i^n}{n!} [H(\Lambda_1), A]^{(n)} t^n \| \]

\[ + \| \sum_{n=N+1}^{\infty} \frac{i^n}{n!} [H(\Lambda_2), A]^{(n)} t^n \|. \]
Let $\epsilon > 0$ be given, and choose $t$ such that $|t| \leq t_1 \leq T$. It follows from inequality 3.3.2 in proposition 3.3.0.10 that, one can find $N \in \mathbb{Z}^+$ such that for $l = 1, 2$

\[
\| \sum_{n=N+1}^{\infty} \frac{i^n}{n!} [H(\Lambda_t), A]^{(n)} t^n \| \leq \sum_{n=N+1}^{\infty} \frac{1}{n!} [H(\Lambda_t), A]^{(n)} ||t^n|| \\
\leq \|A\| e^{|\Lambda|} \sum_{n=N+1}^{\infty} \left( \frac{t_1}{T} \right)^n \\
< \frac{\epsilon}{4}.
\]

Now, using the fact that

\[
[H(\Lambda), A]^{(n)} \rightarrow \sum_{\mathcal{X}_1 \subseteq \mathcal{V}} \ldots \sum_{\mathcal{X}_n \subseteq \mathcal{V}} [\Phi(\mathcal{X}_n), \ldots [\Phi(\mathcal{X}_1), A]],
\]

as $\Lambda \rightarrow \infty$, we can find a finite subset $\Lambda'$ of $\mathcal{V}$ such that,

\[
\| \frac{i^n}{n!} ([H(\Lambda_1), A]^{(n)} - [H(\Lambda_2), A]^{(n)}) \| < \frac{t_1^n \epsilon}{2^N + 2},
\]

for all $n \leq N$ whenever $\Lambda_1 \supseteq \Lambda'$ and $\Lambda_2 \supseteq \Lambda'$. Thus, given $\epsilon > 0$, there exists a finite $\Lambda' \subseteq \mathcal{V}$ such that,

\[
\|\tau_t^{\Lambda_1}(A) - \tau_t^{\Lambda_2}(A)\| < \epsilon,
\]

whenever $\Lambda_1 \supseteq \Lambda'$ and $\Lambda_2 \supseteq \Lambda'$. Hence, it follows that the convergence is uniform in $t$ on any closed subinterval of $(-T, T)$ and in a ball around zero.

Since for $t \in (-T, T)$, the mapping $A \mapsto \tau_t^{\Lambda}(A)$ is a $^*$-automorphism and

\[
\bigcup_{\Lambda \subseteq \mathcal{V}} \mathcal{A}_\Lambda
\]

is dense in $\mathcal{A}$, we conclude that

\[
\lim_{\Lambda \rightarrow \infty} \tau_t^{\Lambda}(A),
\]

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exists for all $A \in \mathcal{A}$ and $t \in (-T, T)$. Therefore,

$$\tau_t(A) = \lim_{\Lambda \to \infty} \tau_t^\Lambda(A)$$

exists for all $A \in \mathcal{A}$ and $t \in (-T, T)$, and thus, defines a $^*$-automorphism of $\mathcal{A}$ for each $t \in (-T, T)$. If we take $t, s$ and $t + s$ in the interval $[-T, T]$ and use the group property of $\tau_t^\Lambda$, then on taking the limit as $\Lambda \to \infty$, we get

$$\tau_s \circ \tau_t(A) = \tau_{t+s}(A).$$

This group property of $\tau_t$ for $|t| < T$ allows us to define $\tau_t$ for all values of $t$. The strong continuity of $\tau_t$ follows from the series expansion.

\[ \Delta \]

### 3.4 Equilibrium State and the KMS Condition

We now focus our attention on the study of equilibrium states of the quantum spin system on an infinite graph. It is known that the equilibrium states of infinite systems are stationary. The analytic properties of these states are going to be the object of our study. In the sequel, we establish the existence of the thermodynamic limit of the local Gibbs states, and derive some interesting properties connected with these states.

As discussed earlier, there is a Hamiltonian $H(\Lambda) \in \mathcal{A}_\Lambda$ associated with each finite $\Lambda \subseteq V$. We are interested in the thermodynamic limit of the local Gibbs states $\rho_\Lambda$. Let us start by defining a local Gibbs state $\rho_\Lambda$ for a finite $\Lambda \subseteq V$ as,

$$\rho_\Lambda(A) = \frac{\text{Tr}(e^{-\beta H(\Lambda)}A)}{\text{Tr}(e^{-\beta H(\Lambda)})},$$
where $A \in \mathcal{A}_\Lambda$. Here $\beta = KT^{-1}$, where $K$ is the Boltzmann’s constant and $T$ the temperature.

**Definition 3.4.0.12** Let $\{\rho_\Lambda\}$ be the collection of the local Gibbs states defined on the local algebras $\mathcal{A}_\Lambda$. If there is a state $\rho$ on $\mathcal{A}$ such that, $\rho$ is the weak$^*$-limit of a net of extensions of $\rho_\Lambda$ to $\mathcal{A}$, then we call $\rho$ the thermodynamic limit of the local Gibbs states. If $\hat{\rho}_\Lambda$ is one such net of extensions, then for arbitrary $A \in \mathcal{A}_\Lambda_0$ and $\Lambda_\alpha \supseteq \Lambda_0$,

$$\lim_{\Lambda_\alpha \to \infty} \rho_{\hat{\Lambda}_\alpha}(A) = \rho(A).$$

Notice that the thermodynamic limit need not be unique, as different weak$^*$-limit points of the extensions of $\rho_\Lambda$ to $\mathcal{A}$ give rise to different thermodynamic limits of $\rho_\Lambda$.

A state obtained as the thermodynamic limit of the local Gibbs states $\{\rho_\Lambda\}$ will be called the equilibrium state of the infinite quantum spin system.

Now, the thermodynamic limit of the local Gibbs states $\{\rho_\Lambda\}$ exists by virtue of the fact that each $\rho_\Lambda$ can be extended to the whole of $\mathcal{A}$, and if $\hat{\rho}_\Lambda$ is one such extension, then the collection $\{\hat{\rho}_\Lambda\}$ being weak$^*$-compact, one can always find an accumulation point $\rho$. Since $\mathcal{A}$ is separable, we can extract a sequence $\hat{\rho}_{\Lambda_n}$ from the net $\hat{\rho}_\Lambda$ such that,

$$\rho(A) = \lim_{n \to \infty} \rho_{\hat{\Lambda}_n}(A),$$

for all $A \in \mathcal{A}_\Lambda$ and all $\Lambda$. Thus, the thermodynamic limit of the local Gibbs states $\rho_\Lambda$ exists. In the discussion that follows, we derive an interesting prop-
erty of the local Gibbs states $\rho_\Lambda$ and establish the Kubo-Martin-Schwinger (KMS) condition for the equilibrium state $\rho$ of the infinite system.

**Proposition 3.4.0.13** Let $A, B \in A_\Lambda$ and $\beta > 0$. There exists a complex valued function $F^A_{A,B}$, which is analytic everywhere and uniformly bounded in the strip $0 \leq \Im z \leq \beta$ such that, for real $t$,

$$F^A_{A,B}(t) = \rho_\Lambda(\tau^A_t(B))$$

and

$$F^A_{A,B}(t + i\beta) = \rho_\Lambda(\tau^A_t(B)A).$$

If $\beta < 0$, then there exists a complex valued function $F^A_{A,B}$, which is analytic everywhere and uniformly bounded in the strip $\beta \leq \Im z \leq 0$ such that, for real $t$,

$$F^A_{A,B}(t) = \rho_\Lambda(\tau^A_t(B))$$

and

$$F^A_{A,B}(t + i\beta) = \rho_\Lambda(\tau^A_t(B)A).$$

**Proof** Let $A, B \in A_\Lambda$ and $\beta > 0$. Since $H(\Lambda) \in A_\Lambda$, where $A_\Lambda$ is a matrix algebra, $\tau^A_t(B) = e^{iH(\Lambda)t}B e^{-iH(\Lambda)t}$ makes sense for all complex $t$ and hence, has an analytic extension to the entire complex plane. Therefore, it follows that $\rho_\Lambda(\tau^A_t(B))$ can be extended to an entire function $F^A_{A,B}(z)$ on $\mathbb{C}$. Now, for real $t$, consider

$$\rho_\Lambda(\tau_t^A(B)) = \frac{\text{Tr}(e^{-\beta H(\Lambda)}A e^{iH(\Lambda)(t+i\beta)}B e^{-iH(\Lambda)(t+i\beta)})}{\text{Tr}(e^{-\beta H(\Lambda)})}$$

$$= \frac{\text{Tr}(e^{-\beta H(\Lambda)} e^{iH(\Lambda)t} B e^{-iH(\Lambda)t} A)}{\text{Tr}(e^{-\beta H(\Lambda)})}$$

$$= \rho_\Lambda(\tau_t^A(B)A).$$
The last equality follows from the cyclicity of the trace. Further, $|F_{A,B}^r(z)|$ is bounded in the open strip and $|F_{A,B}^r(z)| \leq \|A\|\|B\|$ on the boundary of the strip. Therefore, it follows from a version of the Phragmen–Lindelöf theorem ([Rob 81], Prop 5.3.5) that, the maximum of the function $|F_{A,B}^r(z)|$ is attained on the boundary. Hence the theorem holds for $\beta > 0$. Similarly, the theorem can be proved for $\beta < 0$. \[\square\]

In order to study the analytic property of the thermodynamic limit $\rho$ of the local Gibbs states $\rho_A$, one needs to prove the following proposition.

**Proposition 3.4.0.14** Let $\{\Lambda_n\}$ be a sequence of finite subsets of $V$ such that, $\lim_{n \to \infty} \rho_{\Lambda_n}(A) = \rho(A)$, $\forall A \in \mathcal{A}_{\Lambda_0}$ and all $\Lambda_0 \subseteq V$. Then, for $A, B \in \mathcal{A}_{\Lambda_0}$, we have

$$\lim_{n \to \infty} \rho_{\Lambda_n}(A_{\tau}^{\Lambda_n}(B)) = \rho(A_{\tau}(B)),$$

where the limit exists for all real $t$ and uniformly for $t$ in a small ball around zero.

**Proof** Let $A, B \in \mathcal{A}_{\Lambda_0}$, where $\Lambda_0 \subseteq V$. Now we have from theorem 3.3.0.11 that,

$$\lim_{n \to \infty} \tau_{t}^{\Lambda_n}(B) = \tau_t(B),$$

for $B \in \mathcal{A}_{\Lambda_0}$, where the limit is uniform in $t$, in some ball around zero. Therefore, given $\epsilon > 0$ and a fixed $t$, there exists $n_0 \in \mathbb{Z}^+$, which can be chosen independent of $t$ in a ball around zero such that, for $n, m > n_0$,

$$\|\tau_{t}^{\Lambda_n}(B) - \tau_{t}^{\Lambda_m}(B)\| < \frac{\epsilon}{4\|A\|} \quad \text{and} \quad \|\tau_{t}^{\Lambda_m}(B) - \tau_{t}(B)\| < \frac{\epsilon}{4\|A\|}.$$
Further, since
\[ \lim_{n \to \infty} \rho_{\Lambda_n}(A \tau_t^{\Lambda_m}(B)) = \rho(A \tau_t^{\Lambda_m}(B)), \]
we have for given \(m, n > n_0\)
\[ |\rho_{\Lambda_n}(A \tau_t^{\Lambda_m}(B)) - \rho(A \tau_t^{\Lambda_m}(B))| < \frac{\epsilon}{2}. \]
These estimates allow us to arrive at the following inequalities:
\[
|\rho(A \tau_t(B)) - \rho_{\Lambda_n}(A \tau_t^{\Lambda_n}(B))| \leq |\rho(A \tau_t(B)) - \rho(A \tau_t^{\Lambda_m}(B))| \\
+ |\rho(A \tau_t^{\Lambda_m}(B)) - \rho_{\Lambda_n}(A \tau_t^{\Lambda_m}(B))| \\
+ |\rho_{\Lambda_n}(A \tau_t^{\Lambda_m}(B)) - \rho_{\Lambda_n}(A \tau_t^{\Lambda_n}(B))| \\
\leq \|\tau_t^{\Lambda_n}(B) - \tau_t^{\Lambda_m}(B)\|\|A\| \\
+ \|\tau_t^{\Lambda_m}(B) - \tau_t(B)\|\|A\| + \frac{\epsilon}{2} \\
< \epsilon.
\]
This proves the proposition for real \(t\), and \(t\) in a small ball around zero. \(\triangle\)

It is evident that the time evolution bears some relation with the equilibrium state of the infinite system. One such relation is the Kubo–Martin–Schwinger (KMS) condition. This condition may be formulated as follows for the equilibrium state \(\rho\).

**Theorem 3.4.0.15** Let \(\rho\) be the equilibrium state of the quantum spin system on the infinite graph \(G(V, E)\) and \(A, B \in \mathcal{A}\). Then, for \(\beta > 0\), there exists a function \(F_{A,B}\), which is analytic in the open strip \(0 < \Re z < \beta\), continuous and uniformly bounded in the closed strip \(0 \leq \Re z \leq \beta\) such that,
\[ F_{A,B}(t) = \rho(A \tau_t(B)) \quad \text{and} \quad F_{A,B}(t + i\beta) = \rho(\tau_t(B)A). \]
If \( \beta < 0 \), then there exists a function \( F_{A,B} \), which is analytic in the open strip \( \beta < \Im z < 0 \), continuous and uniformly bounded in the closed strip \( \beta \leq \Im z \leq 0 \) such that,

\[
F_{A,B}(t) = \rho(A\tau_t(B)) \quad \text{and} \quad F_{A,B}(t + i\beta) = \rho(\tau_t(B)A).
\]

**Proof** We shall prove the theorem for the case \( \beta > 0 \). Let \( \{\Lambda_n\} \) be a sequence of finite subsets of \( \mathbb{Z}^r \) such that, \( \lim_{n \to \infty} \rho_{\Lambda_n}(B) = \rho(B) \), for all \( B \in \mathcal{A}_{\Lambda_0} \) and all \( \Lambda_0 \subseteq V \). It follows from proposition 3.4.0.13, that, for \( \beta > 0 \) and \( A, B \in \mathcal{A}_{\Lambda_0} \), there exists a sequence of entire functions \( F_{A,B}^{\Lambda_n}(z) \), which is uniformly bounded in the closed strip \( 0 \leq \Im z \leq \beta \) such that, for real \( t \),

\[
F_{A,B}^{\Lambda_n}(t) = \rho_{\Lambda_n}(A\tau_t^{\Lambda_n}(B)) \quad \text{and} \quad F_{A,B}^{\Lambda_n}(t + i\beta) = \rho_{\Lambda_n}(\tau_t^{\Lambda_n}(B)A).
\]

Therefore, it follows from proposition 3.4.0.14, that this sequence converges pointwise on the real axis and in a neighbourhood of zero. Hence, as a consequence of Vitali's theorem, see [Tit 91], the sequence \( F_{A,B}^{\Lambda_n} \) of analytic functions converges uniformly on every compact subset of the strip to a function \( F_{A,B} \), which is analytic in the open strip \( 0 < \Im z < \beta \), continuous and uniformly bounded in the closed strip \( 0 \leq \Im z \leq \beta \) such that,

\[
F_{A,B}(t) = \rho(A\tau_t(B)) \quad \text{and} \quad F_{A,B}(t + i\beta) = \rho(\tau_t(B)A)
\]

The general case can be handled by approximating arbitrary \( A \in \mathcal{A} \) by local elements and using a version of the Phragmen–Lindelöf theorem ([Rob 81], Prop 5.3.5).
For $\beta < 0$, the theorem can be proved along the same lines by considering
the closed strip $\beta \leq \Im z \leq 0.$

\textbf{Corollary 3.4.0.16} The equilibrium state $\rho$ of the infinite spin system is
invariant under time evolution given by the automorphism group $\tau_t$.

\textbf{Proof} Take $B = I$ in proposition (3.4.0.14).

Thus, having established the existence of an equilibrium state of the spin
system on an infinite graph, we set our sights on proving the maximum en-
tropy principle for the infinite spin system. In view of this, we attempted to
establish the existence of thermodynamic quantities such as mean entropy
and the free energy of this infinite system. To this end we constructed a
nested sequence $\{G_n(V_n, E_n)\}$ of finite subgraphs of the infinite connected
graph $G(V, E)$, with set of vertices $V_n$ and collection of edges $E_n$. Each of
these subgraphs $G_n(V_n, E_n)$ is constructed from the preceding subgraph by
simply adding those vertices of the graph $G(V, E)$, which are connected to it
by an edge. The choice of the initial subgraph can be arbitrary. The inves-
tigation concerning the existence of mean entropy and the free energy of the
infinite system in the state $\rho$, entails computing the limit of the entropy per
site $S_{\rho}(V_n) = -\log(\text{Tr}(e^{-\beta H(V_n)}))$, as $n \to \infty$. Here
$|V_n|$ denotes the cardinality of the set of vertices of the subgraph $G_n(V_n, E_n)$.

The entropy $S_{\rho}(V_n) = -\text{Tr}(\rho_{V_n} \log(\rho_{V_n}))$, where $\rho_{V_n}$ is the density matrix
corresponding to the restriction of the state $\rho$ to the local algebra $A_{V_n}$, as-
associated with the subgraph $G_n(V_n, E_n)$. But, all attempts at proving the
existence of these limits failed, primarily because of the absence of spatial

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homogeneity. For, unlike in the case of a quantum spin system on a lattice, the local entropy is not translation invariant. Besides, the absence of translation invariance also hindered the investigation pertaining to the existence of free energy of the spin system on the infinite graph. Despite the fact that the local entropy satisfies the strong subadditivity property, none of the results pertaining to the existence of the limit of objects such as $\frac{f(x)}{x}$ as $x \to \infty$, where $f$ is a real valued subadditive function defined on $\mathbb{R} (\mathbb{R}^+, \mathbb{Z}^+)$, could be applied in this case. Such results are known to have a role to play, in demonstrating the existence of mean entropy for quantum spin systems on a lattice with deterministic interaction potentials [Rue 69]. Thus, the question of existence of these quantities remains unresolved. Therefore, one conjectures that the maximum entropy principle may not hold for a quantum spin system on an infinite connected graph with deterministic interaction potential of the nearest neighbour type. However, in the case of some random models of a spin glass, subadditivity along with the appropriate ergodic theorem have been employed to establish the existence of some thermodynamic quantities under fairly stringent conditions on the random interaction potential. For, in the study of equilibrium spin glass theory through random models on a lattice, van Hemmen et al [Hem 83, Ent 83] have shown that the thermodynamic limit of the local free energy $F(A)$ exists. In fact, it has been established that the free energy of the infinite system exists as a non-random limit of $\frac{F(A)}{|A|}$, with probability one. Thus, one is obliged to conclude that the attempt at understanding the behaviour of a quantum spin glass through
quantum spin systems on an infinite graph, has not proved to be very useful. Therefore, one is obliged to take recourse to the more traditional approach.

As mentioned earlier, in the more traditional line of thinking, quantum spin glasses have been studied as systems of quantum spins interacting through random interactions. These models are essentially Ising-type models with random coupling. Extensive investigations on the existence of the thermodynamic limit have been made e.g. van Hemmen et al [Hem 83, Ent 83], and the equilibrium statistical mechanics of such systems has been studied. Although quantum spin glasses admit a natural dynamics, no attempt has been made to study the dynamics of a quantum spin glass. Hence, we study the dynamics of a quantum spin glass, as a quantum spin system on an infinite lattice with random interactions. We establish the existence of a family of one-parameter groups of \( \star \)-automorphisms \( \{ \tau_t(\omega) \} \), of the quasi-local algebra \( \mathcal{A} \) associated with the infinite system. Here \( \omega \) lives in a probability space \((\Omega, S, P)\), where \( \Omega \) is a set, \( S \) a sigma algebra and \( P \) a complete probability measure. The strong measurability of \( (t, \omega) \mapsto \tau_t(\omega)(A) \), for all \( A \in \mathcal{A} \) is established. Some interesting algebraic properties of the automorphism groups \( \tau_t(\omega) \) as well as those of their generators \( \delta(\omega) \) have been derived.

### 3.5 Description of the Random Model

Consider a quantum spin system with spins located at the vertices of an infinite lattice \( \mathbb{Z}^d \). The interaction between spins of course taken to be random. A quasi-local UHF algebra similar to the one in 3.1, constructed over the
finite subsets of $Z^r$, is associated with this spin system. One can order the collection of all finite subsets of $Z^r$ by inclusion. With each point in $Z^r$, one associates a two dimensional Hilbert space $\mathcal{H}_x$. Then with each finite $\Lambda \subseteq Z^r$, we associate the tensor product space

$$\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x,$$

where $\Lambda \subseteq Z^r$. We define a local $C^*$-algebra for each finite $\Lambda \subseteq Z^r$ by $\mathcal{A}_\Lambda = \mathcal{L}(\mathcal{H}_\Lambda)$, where $\mathcal{L}(\mathcal{H}_\Lambda)$ denotes the space of all bounded linear operators on $\mathcal{H}_\Lambda$. Now if $\Lambda_1 \cap \Lambda_2 = \emptyset$ for $\Lambda_1, \Lambda_2 \subseteq Z^r$, then $\mathcal{H}_{\Lambda_1 \cup \Lambda_2} = \mathcal{H}_{\Lambda_1} \otimes \mathcal{H}_{\Lambda_2}$ and $\mathcal{A}_{\Lambda_1}$ is isomorphic to the $C^*$-subalgebra $\mathcal{A}_{\Lambda_1} \otimes I_{\Lambda_2}$, where $I_{\Lambda_2}$ is the identity operator on $\mathcal{H}_{\Lambda_2}$. Further, if $\Lambda_1 \subseteq \Lambda_2$, one can identify $\mathcal{A}_{\Lambda_1}$ with the subalgebra $\mathcal{A}_{\Lambda_1} \otimes I_{\Lambda_2 \setminus \Lambda_1}$ of $\mathcal{A}_{\Lambda_2}$. Let the identification map be given by $i_{\Lambda_2, \Lambda_1} : A \in \mathcal{A}_{\Lambda_1} \to A \otimes I_{\Lambda_2 \setminus \Lambda_1} \in \mathcal{A}_{\Lambda_2}$. The collection $\{\mathcal{A}_\Lambda | \Lambda \subseteq Z^r\}$, with the collection of maps $\{i_{\Lambda_2, \Lambda_1}\}$ has the structure of a directed system of $C^*$-algebras. Therefore, there exists a $C^*$-algebra $\mathcal{A}$ with an identity, which is the inductive limit of the collection $\{\mathcal{A}_\Lambda | \Lambda \subseteq Z^r\}$ of $C^*$-algebras with identity $I_{\Lambda}$. i.e., there exists a $C^*$-algebra $\mathcal{A}$ and injective *-homomorphisms $i_\Lambda : \mathcal{A}_\Lambda \to \mathcal{A}$ such that,

$$\Lambda_1 \subseteq \Lambda_2 \Rightarrow i_{\Lambda_1}(\mathcal{A}_{\Lambda_1}) \subseteq i_{\Lambda_2}(\mathcal{A}_{\Lambda_2}),$$

$$\bigcup_{\Lambda \subseteq Z^r} i_\Lambda(\mathcal{A}_\Lambda) = \mathcal{A}$$

and

$$i_{\Lambda}(I_{\Lambda}) = I \quad \forall \Lambda \subseteq Z^r.$$
Also, for
\[ \Lambda_1 \cap \Lambda_2 = \emptyset; \quad [i_{\Lambda_1}(\mathcal{A}_{\Lambda_1}), i_{\Lambda_2}(\mathcal{A}_{\Lambda_2})] = 0, \]
where \([., .]\) is the commutator.

We will hence forth leave out the \(i_{\Lambda_2, \Lambda_1}\)'s and \(i_{\Lambda}\)'s whenever no confusion can arise and regard \(\mathcal{A}_{\Lambda}\)'s as subalgebras of \(\mathcal{A}\). This object \(\mathcal{A}\) along with the net of local \(C^*\)-algebras \(\{\mathcal{A}_{\Lambda}\}_{\Lambda \subseteq \mathbb{Z}^\nu}\) is a quasi-local algebra (The orthogonality relation \(\perp\) between \(\Lambda\)'s is defined by \(\Lambda_1 \perp \Lambda_2\) if \(\Lambda_1 \cap \Lambda_2 = \emptyset\)). It is easily seen that the quasi-local UHF algebra \(\mathcal{A}\), is a separable \(C^*\)-algebra with no non-trivial closed ideals. Hence, it is a simple \(C^*\)-algebra [Rob 81].

The local algebras \(\mathcal{A}_{\Lambda}\) represent the physical observables associated with the spins located in \(\Lambda\), whereas the quasi-local algebra \(\mathcal{A}\) corresponds to the observables associated with the infinite spin system.

Having described the kinematical structure of the quantum spin system on the lattice \(\mathbb{Z}^\nu\), we now turn our attention to the action of the symmetry group associated with the lattice \(\mathbb{Z}^\nu\), on the observable algebra \(\mathcal{A}\). To this end, for each \(x \in \mathbb{Z}^\nu\), choose an unitary mapping \(V(x) : \mathcal{H}_x \rightarrow \mathcal{H}_x\), where \(\mathcal{H}_x\) is the underlying Hilbert space at \(x\). Now for \(x_1, x_2 \in \mathbb{Z}^\nu\), define \(V(x_2, x_1) : \mathcal{H}_{x_1} \rightarrow \mathcal{H}_{x_2}\), by \(V(x_2, x_1) = V(x_2)V(x_1)^{-1}\). It is clear that for \(x_1, x_2, x_3 \in \mathbb{Z}^\nu\), \(V(x_3, x_1) = V(x_3, x_2)V(x_2, x_1)\). Furthermore, for each \(a \in \mathbb{Z}^\nu\), define \(V_x(a) : \mathcal{H}_x \rightarrow \mathcal{H}_{x+a}\) as \(V_x(a) = V(x + a, x)\). Thus, if for each \(\Lambda \subseteq \mathbb{Z}^\nu\), one were to define \(V_\Lambda(a) : \mathcal{H}_\Lambda \rightarrow \mathcal{H}_{\Lambda+a}\) as
\[
V_\Lambda(a) = \bigotimes_{x \in \Lambda} V_x(a),
\]
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then $V_{\Lambda}(a)$ is an isomorphism and one has $^2 V_{\Lambda}(a)^{-1} = V_{\Lambda+a}(-a)$. We can now introduce an action $\alpha$ of $\mathbb{Z}^\nu$ as $^*$-automorphisms of $\mathcal{A}$ as follows. For each $a \in \mathbb{Z}^\nu$, define

$$\alpha_a(A) = V_{\Lambda}(a)AV_{\Lambda}(a)^{-1}; \quad \forall A \in \mathcal{A}_{\Lambda}.$$ 

Thus, $\alpha$ is consistently defined on the union of local $C^*$-algebras $\bigcup \mathcal{A}_{\Lambda}$; as an isometric $^*$-isomorphism and hence, can be extended by continuity to an automorphism of $\mathcal{A}$, as

$$\alpha_a(\mathcal{A}_{\Lambda}) = \mathcal{A}_{\Lambda+a}.$$ 

Therefore, it follows from the quasi-local structure of $\mathcal{A}$ that

$$\lim_{a \to \infty} \|[\alpha_a(A), B]\| = 0, \quad \forall A, B \in \mathcal{A}$$

i.e., $\mathcal{A}$ is asymptotically abelian.

### 3.6 Random Interactions

**Definition 3.6.0.17** An interaction $\Psi$ of the quantum spin system on the infinite lattice $\mathbb{Z}^\nu$, is a mapping from the collection of finite subsets $X$ of $\mathbb{Z}^\nu$ into the Hermitian (self adjoint) elements of $\mathcal{A}$ such that, for every finite $X \subseteq \mathbb{Z}^\nu$, $\Psi(X) \in \mathcal{A}_X$.

Before we introduce random interactions, one has to define the notion of measurability of Banach space valued functions on a measure space $(\Omega, \mathcal{S}, m)$, where $\Omega$ is a set, $\mathcal{S}$ a sigma algebra and $m$ a sigma-finite measure on $\Omega$.

---

$^2$Here $V_{\Lambda}(a)^{-1}$ denotes the inverse of $V_{\Lambda}(a)$. 
Definition 3.6.0.18 Let $(\Omega, \mathcal{S}, m)$ be a measure space. A function $f : \Omega \to B$ where $B$ is a Banach space, is said to be weakly measurable if, for every $\phi \in B^*$, the map $\omega \mapsto \phi(f(\omega))$ is $\mathcal{S}$-measurable. $f$ is said to be strongly measurable if, there exists a sequence of countably valued functions strongly convergent to $f$ almost everywhere on $\Omega$ [Hil 57].

In case $m$ is a finite measure, then we may replace “countably valued” in the above definition by “simple”. It can be shown that the notions of strong and weak measurability are equivalent if $B$ is separable.

Definition 3.6.0.19 Let $(\Omega, \mathcal{S}, P)$ be a probability space and $J$ some index set. If $T_j$ is a measure preserving automorphism of $\Omega$, for each $j \in J$, then the action of $T_j$'s is said to be ergodic if, for $A \in \mathcal{S}$, $P(A) = 0$ or $1$ whenever $T_jA = A$, for all $j \in J$.

From now on, let $(\Omega, \mathcal{S}, P)$ be a complete probability space, where $\Omega$ is a complete separable metric space. $\mathcal{S}$ is the sigma algebra of subsets of $\Omega$, containing the Borel sigma algebra $\mathcal{B}$ generated by open sets in $\Omega$. $P$ is the completion of a probability measure defined on $\mathcal{B}$.

Definition 3.6.0.20 Let $\mathcal{F}$ be the collection of all finite subsets of $\mathbb{Z}^\nu$. A random interaction is a map $\Phi : \mathcal{F} \times \Omega \to A$ such that, for each $\omega \in \Omega$, $\Phi(\cdot, \omega)$ is an interaction of the quantum spin system on $\mathbb{Z}^\nu$ and $\omega \mapsto \Phi(X, \omega)$ is strongly measurable for every $X \in \mathcal{F}$.

Now, for finite $A \subseteq \mathbb{Z}^\nu$, the Hamiltonian associated with the spins confined
to the region $\Lambda$ is given by a Hermitian (self adjoint) element

$$H(\Lambda, \omega) = \sum_{X \subseteq \Lambda} \Phi(X, \omega),$$

for $\omega \in \Omega$. Clearly, $H(\Lambda, \omega)$ is strongly measurable since each $\Phi(X, \omega)$ is strongly measurable on $\Omega$.

In order to construct the dynamics of the quantum spin system with random interactions, we have to restrict the class of random interactions $\Phi$. To this end, we introduce a measure preserving group of automorphisms $\{T_a\}_{a \in \mathbb{Z}'}$ with an ergodic action on the probability space $\Omega$, and thereby restrict the class of interactions to those $\Phi$ which satisfy the following condition:

$$\Phi(X + a, T_{-a}\omega) = a_{\omega}(\Phi(X, \omega)).$$

From now on, we shall consider only those random interactions $\Phi$ which satisfy the above condition. Therefore, $H(\Lambda + a, T_{-a}\omega) = a_{\omega}(H(\Lambda, \omega))$.

**Definition 3.6.0.21** Let $\Phi$ be a random interaction. The interaction $\Phi(., \omega)$ is said to have a finite range if, the set

$$\Delta_\omega = \{x \in \mathbb{Z}'| \exists X \ni x; \text{ such that } 0 \in X, \text{ and } \Phi(X, T_0\omega) \neq 0, \text{ for some } a \in \mathbb{Z}'\}$$

is a finite subset of $\mathbb{Z}'$. We may call $\Delta_\omega$ the range of $\Phi(., \omega)$.

**Remark** Clearly, for such $\Phi(., \omega)$'s, whenever $^3 X - X \not\subseteq \Delta_\omega$, $\Phi(X, \omega) = 0$.

For, if $X - X \not\subseteq \Delta_\omega$, then there exists $x, y \in X$ such that, $x - y \not\in \Delta_\omega$.

But, $x - y \in X - y$, therefore $X - y \not\subseteq \Delta_\omega$. Now, since $0 \in X - y$, it

---

$^3$For $X \subseteq \mathbb{Z}'$, $X - X = \{x - y|x, y \in X\}$. 65
follows from the above definition that $\Phi(X - y, T_a \omega) = 0$ for all $a \in \mathbb{Z}^\nu$. In particular, on putting $a = y$, we get $\Phi(X - y, T_y \omega) = 0$. Therefore, $\Phi(X, \omega) = \alpha_y(\Phi(X - y, T_y \omega)) = 0$.

**Definition 3.6.0.22** The random interaction $\Phi$ is said to be a finite range random interaction if, $\Phi(., \omega)$ has a finite range $\Delta_\omega$ for almost every $\omega \in \Omega$, and $\omega \mapsto |\Delta_\omega|$ is a measurable function of $\omega$. Here $|$ denotes the cardinality of a set.

It is clear from the above remark that if $\Phi$ is a finite range random interaction, then for almost every $\omega \in \Omega$, $\Phi(X, \omega) = 0$, whenever $|X| > |\Delta_\omega|$.

We use the ergodicity of the measure preserving group of automorphisms to establish the following fact.

**Lemma 3.6.0.23** Let $\Phi$ be a finite range random interaction. Since the action of the measure preserving group of automorphisms $\{T_a\}$ is ergodic, the function $\omega \mapsto |\Delta_\omega|$ is almost surely constant.

**Proof** We show that $\Delta_\omega = \Delta_{T_b \omega}$, for all $b \in \mathbb{Z}^\nu$. Fix $b \in \mathbb{Z}^\nu$. Let $x \in \Delta_\omega$. Then there exists a finite $X \ni x$ such that, $0 \in X$ and $\Phi(X, T_a \omega) \neq 0$ for some $a \in \mathbb{Z}^\nu$. i.e. there exists $X \ni x$ such that, $0 \in X$ and $\Phi(X, T_{a-b} (T_b \omega)) \neq 0$, for some $a \in \mathbb{Z}^\nu$. Therefore, $x \in \Delta_{T_b \omega}$. Conversely, let $x \in \Delta_{T_b \omega}$. Then there exists $X \ni x$ such that, $0 \in X$ and $\Phi(X, T_a (T_b \omega)) \neq 0$, for some $a \in \mathbb{Z}^\nu$. This implies that there exists $X \ni x$ such that, $0 \in X$ and $\Phi(X, T_{a+b} \omega) \neq 0$, for some $a \in \mathbb{Z}^\nu$. Hence, $x \in \Delta_\omega$. Thus, $\Delta_\omega = \Delta_{T_b \omega}$. Since $b$ is arbitrary, this holds for all $b \in \mathbb{Z}^\nu$. Now, since $\Phi(., \omega)$ has a finite range $\Delta_\omega$ for almost
every \( \omega \in \Omega \), we have \(|\Delta_\omega| = |\Delta_{T_a}\omega|\), for almost every \( \omega \in \Omega \). Therefore, it is readily concluded that the measurable function \( \omega \mapsto |\Delta_\omega| \) is invariant almost everywhere with respect to the measure preserving group of automorphisms \( \{T_a\} \). Since the action of the group is ergodic, the lemma follows.

**Lemma 3.6.0.24** Let \( \Phi \) be a finite range random interaction of the quantum spin system on an infinite lattice \( \mathbb{Z}^\nu \), satisfying

\[
\sup_{a \in \mathbb{Z}^\nu} \left( \sum_{X \ni 0} \|\Phi(X, T_a \omega)\| \right) < \infty
\]

almost everywhere, then the function

\[
\omega \mapsto \sup_{a \in \mathbb{Z}^\nu} \left( \sum_{X \ni 0} \|\Phi(X, T_a \omega)\| \right)
\]

is almost surely constant.

**Proof** The function \( \omega \mapsto \Phi(X, \omega) \) is strongly measurable for all finite \( X \subseteq \mathbb{Z}^\nu \). Therefore, it follows easily that \( \omega \mapsto \|\Phi(X, \omega)\| \) is a numerically valued measurable function on \( \Omega \). Since for \( a \in \mathbb{Z}^\nu \), \( T_a \) is a measure preserving automorphism of \( \Omega \), clearly, \( \omega \mapsto \|\Phi(X, T_a \omega)\| \) is a measurable function of \( \omega \). Next, let \( X_1, X_2, \ldots \) be the countable collection of all finite subsets of \( \mathbb{Z}^\nu \) containing 0. Since \( \Phi(., \omega) \) has a finite range for almost every \( \omega \in \Omega \), the sum of non-negative terms

\[
\sum_{X \ni 0} \|\Phi(X, T_a \omega)\|,
\]

is finite almost everywhere. Therefore, the series \( \sum_{n=1}^{\infty} \|\Phi(X_n, T_a \omega)\| \) converges to \( \sum_{X \ni 0} \|\Phi(X, T_a \omega)\| \), almost everywhere. i.e.,

\[
\sum_{n=1}^{\infty} \|\Phi(X_n, T_a \omega)\| = \sum_{X \ni 0} \|\Phi(X, T_a \omega)\|,
\]

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almost everywhere. Now, each of the terms of the series is a measurable function of \( \omega \). Hence, the measurability of \( \omega \mapsto \sum_{X \ni 0} \| \Phi(X, T_a \omega) \| \) follows, for \( a \in \mathbb{Z}^\nu \). Thus,

\[
\omega \mapsto \sup_{a \in \mathbb{Z}^\nu} \left( \sum_{X \ni 0} \| \Phi(X, T_a \omega) \| \right)
\]

is a measurable function of \( \omega \). Also, for almost every \( \omega \in \Omega \),

\[
\sup_{a \in \mathbb{Z}^\nu} \left( \sum_{X \ni 0} \| \Phi(X, T_a (T_b \omega)) \| \right) = \sup_{a \in \mathbb{Z}^\nu} \left( \sum_{X \ni 0} \| \Phi(X, T_a \omega) \| \right),
\]

for all \( b \in \mathbb{Z}^\nu \). Thus,

\[
\omega \mapsto \sup_{a \in \mathbb{Z}^\nu} \left( \sum_{X \ni 0} \| \Phi(X, T_a \omega) \| \right)
\]

is a measurable function which is invariant under the action of the measure preserving group of automorphisms almost everywhere. Since the action of the automorphism group is ergodic, it follows that the function

\[
\omega \mapsto \sup_{a \in \mathbb{Z}^\nu} \left( \sum_{X \ni 0} \| \Phi(X, T_a \omega) \| \right)
\]

is almost surely constant.

\[\Delta\]

### 3.7 Random Evolution

For a finite spin system confined to a region \( \Lambda \subseteq \mathbb{Z}^\nu \), and for \( \omega \in \Omega \), the equation of motion is given by

\[
\frac{dA_t^\Lambda(\omega)}{dt} = i[H(\Lambda, \omega), A_t^\Lambda(\omega)], \quad A_t^\Lambda(\omega) \in \mathcal{A}_\Lambda.
\]

Here \( t \mapsto A_t^\Lambda(\omega) \) describes the evolution of the observable \( A \in \mathcal{A}_\Lambda \). This yields the time evolution given by

\[
\tau_t^\Lambda(\omega)(A) = A_t^\Lambda(\omega) = e^{iH(\Lambda, \omega)t} A e^{-iH(\Lambda, \omega)t},
\]

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for $\omega \in \Omega$ and for all $A \in \mathcal{A}_\Lambda$. Clearly, $\tau_t^\Lambda(\omega)(A)$ is an element of $\mathcal{A}_\Lambda$. In fact, for all $\omega \in \Omega$, $\tau_t^\Lambda(\omega)$ is a one-parameter group of $^*$-automorphisms of $\mathcal{A}_\Lambda$. Since the spin system consists of infinite number of spins, the construction of the time evolution of a fixed observable $A \in \mathcal{A}_{\Lambda_0}$, where $\Lambda_0 \subseteq \mathbb{Z}^\nu$ involves taking the limit of $\tau_t^\Lambda(\omega)(A)$ as $\Lambda \to \infty$. Here, we adopt the convention that $\Lambda \to \infty$ indicates, $\Lambda$ eventually contains all finite subsets of $\mathbb{Z}^\nu$. This notion of convergence has been made precise in subsection 3.3, in chapter 3. We shall show that for a certain class of random interaction potentials, this limit exists for almost every $\omega \in \Omega$ and for all $A \in \mathcal{A}_\Lambda$, where $\Lambda \subseteq \mathbb{Z}^\nu$.

**Definition 3.7.0.25** Let $S$ be an operator on the Banach space $X$. An element $x \in X$ is defined to be analytic for $S$ if $x \in D(S^n)$, for all $n = 1, 2, \ldots$, and if the series

$$\sum_{n=0}^{\infty} \frac{(it)^n}{n!} \|S^n x\|$$

has a positive radius of convergence.

**Definition 3.7.0.26** Let $t \mapsto \tau_t$ be a strongly continuous group of automorphisms of a $C^*$-algebra $\mathcal{A}$. An element $A \in \mathcal{A}$ is called analytic for $\tau_t$, if there exists a strip $I_\lambda = \{z | |\Im z| < \lambda\}$ in $\mathbb{C}$, a function $f : I_\lambda \to \mathcal{A}$ such that,

1. $f(t) = \tau_t(A), \forall t \in \mathbb{R},$

2. $z \mapsto f(z)$ is strongly analytic.

An element $A \in \mathcal{A}$, is said to be entire analytic for $\tau_t$ if, there exists a function, $f : \mathbb{C} \to \mathcal{A}$, which is strongly analytic in the entire complex plane and $f(t) = \tau_t(A), \forall t \in \mathbb{R}.$

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In order to construct a family of one-parameter groups of $\ast$-automorphisms which determine the evolution of the spin system, we shall invoke the theory of derivations of $C^*$-algebras which usually arise as generators of automorphism groups. To this end, we have the following proposition.

**Proposition 3.7.0.27** Let $\Phi$ be an interaction of a quantum spin system satisfying

$$P_\Phi(x) = \sum_{x \in X} \|\Phi(X)\| < \infty,$$

for all $x \in L$, where $L$ is a countable set. It follows that there exists a derivation $\delta$ of the quantum spin algebra $A$ such that the domain of $\delta$,

$$D(\delta) = \bigcup_{\Lambda \subseteq \mathbb{Z}^n} A_\Lambda,$$

and for $A \in A_\Lambda$,

$$\delta(A) = i \sum_{X : \Lambda \neq \emptyset} [\Phi(X), A].$$

The derivation $\delta$ is norm-closable and its closure $\overline{\delta}$ is the generator of a strongly continuous one-parameter group of $\ast$-automorphisms $\tau$ of $A$ if, and only if, one of the following conditions is satisfied: either $\overline{\delta}$ possesses a dense set of analytic elements or $(I + \alpha \delta)(D(\delta)) = A, \alpha \in \mathbb{R} \setminus \{0\}$. Finally, if $\delta$ generates the group $\tau$ and if $\tau^A_t(A) = e^{iH(A)t} Ae^{-iH(A)t}$, then

$$\lim_{\Lambda \to \infty} \|\tau_t(A) - \tau^A_t(A)\| = 0$$

for all $A \in A$, uniformly, for $t$ in compacts.

**Proof** See [Rob 81], vol 2, prop 6.2.3, pg 248.
Theorem 3.7.0.28 Let $\Phi$ be a finite range random interaction of the quantum spin system on a lattice $\mathbb{Z}^n$, satisfying

$$\sup_{\omega \in \mathbb{Z}^n} \left( \sum_{X \in \Lambda} \| \Phi(X, T_{\omega}) \| \right) < \infty$$

almost everywhere. Then, for almost every $\omega \in \Omega$, there exists a strongly continuous, one-parameter group of $*$-automorphisms $\tau_t(\omega)$ of $\mathcal{A}$ such that,

$$\lim_{\Lambda \to \infty} \tau^\Lambda_t(\omega)(A) = \tau_t(\omega)(A), \quad \forall A \in \mathcal{A}$$

and uniformly, for $t$ in compacts, where $\tau^\Lambda_t(\omega)(A) = e^{iH(\lambda, \omega)t} A e^{-iH(\lambda, \omega)t}$.

$\tau_t(\omega)$ is called the evolution group of the spin system whenever the limit exists.

Proof Now, whenever $\Phi(\cdot, \omega)$ has a finite range $\Delta_{\omega}$ for $\omega \in \Omega$, we have for $x \in \mathbb{Z}^n$,

$$P_\Phi(\omega)(x) = \sum_{X \ni x} \| \Phi(X, \omega) \|$$

$$= \sum_{X - x \ni 0} \| \alpha_x(\Phi(X - x, T_{\omega}) \|$$

$$= \sum_{X - x \ni 0} \| \Phi(X - x, T_{\omega}) \|$$

$$\leq \sum_{Y \ni 0} \| \Phi(Y, T_{\omega}) \| < \infty,$$

On appealing to the above proposition, there exists a derivation $\delta(\omega)$ of $\mathcal{A}$ such that, the domain of $\delta(\omega)$,

$$D(\delta(\omega)) = \bigcup_{\Lambda \subseteq \mathbb{Z}^n} \mathcal{A}_\Lambda$$

and for $A \in \mathcal{A}_\Lambda$,

$$\delta(\omega)(A) = i \sum_{X \cap \Lambda \neq \emptyset} [\Phi(X, \omega), A].$$

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Next, we shall show that $D(\delta(\omega))$ is a dense set of analytic elements for $\delta(\omega)$ and hence establish that the derivation $\delta(\omega)$ is norm-closable by the above proposition.

Take $A \in \mathcal{A}_{\Lambda_0}$, where $\Lambda_0 \subseteq \mathbb{Z}^\nu$. One has $\Phi(X, \omega) \in \mathcal{A}_X$, for finite $X \subseteq \mathbb{Z}^\nu$.

Now the local algebras $\mathcal{A}_{\Lambda_1}$, $\mathcal{A}_{\Lambda_2}$ commute whenever $\Lambda_1 \cap \Lambda_2 = \emptyset$.

Therefore, we have

$$
\|(\delta(\omega))^n(A)\| = \|\sum_{X_1 \cap S_0 \neq \emptyset} \cdots \sum_{X_n \cap S_{n-1} \neq \emptyset} [\Phi(X_n, \omega), \ldots [\Phi(X_1, \omega), A]]\| \\
\leq \sum_{X_1 \cap S_0 \neq \emptyset} \cdots \sum_{X_n \cap S_{n-1} \neq \emptyset} \|[\Phi(X_n, \omega), \ldots [\Phi(X_1, \omega), A]]\|,
$$

where $S_0 = \Lambda_0$ and

$$S_j = \Lambda_0 \cup \bigcup_{i=1}^j X_i, \quad \text{for } j \geq 1.$$

Since $\Phi(., \omega)$ has a finite range $\Delta_\omega$, it follows that $\Phi(X, \omega) = 0$, whenever $|X| > |\Delta_\omega|$.

Therefore, if

$$[\Phi(X_j, \omega), \ldots [\Phi(X_1, \omega), A]] \neq 0,$$

where

$$[\Phi(X_j, \omega), \ldots [\Phi(X_1, \omega), A]] \in \mathcal{A}_{S_j},$$

then

$$|X_i| \leq |\Delta_\omega|, \quad \forall i = 1, 2, \ldots, j.$$

Therefore,

$$|S_j| \leq \sum_{i=1}^j |X_i| + |\Lambda_0|$$

$$\leq j|\Delta_\omega| + |\Lambda_0|$$

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Thus, whenever
\[
\sup_{a \in \mathbb{Z}^r} \left( \sum_{X \in \mathcal{X}} \|\Phi(X, T_a \omega)\| \right) < \infty,
\]
we get
\[
\|((\delta(\omega))^n(A))\| \leq 2^n \|A\| \sum_{i=1}^n \prod_{i=1}^n (|\Delta_w| + |\Lambda_0|) \left( \sup_{x_i \in X_{n-1}} \left( \sum_{X \in \mathcal{X}} \|\Phi(x_i, T_i \omega)\| \right) \right)
\]
\[
\leq 2^n \|A\| \left( \sum_{i=1}^n \prod_{i=1}^n (|\Delta_w| + |\Lambda_0|) \right)^{n} \left( \sup_{a \in \mathbb{Z}^r} \left( \sum_{X \in \mathcal{X}} \|\Phi(X, T_a \omega)\| \right) \right)^n.
\]

Now, \(a^n \leq n!\) for \(a > 0\) hence,
\[
\|((\delta(\omega))^n(A))\| \leq \|A\| e^{n!} 2^n n! \left( \sup_{x \in \mathbb{Z}^r} \left( \sum_{X \in \mathcal{X}} \|\Phi(X, T_a \omega)\| \right) \right) e^n |\Delta_w|.
\]

This establishes that \(A\) is an analytic element for \(\delta(\omega)\), with radius of analyticity
\[
r_w \geq \left( 2 \left( \sup_{x \in \mathbb{Z}^r} \left( \sum_{X \in \mathcal{X}} \|\Phi(X, T_a \omega)\| \right) \right) e^n |\Delta_w| \right)^{-1},
\]
where the radius of analyticity \(r_w\) is independent of \(A\). i.e.,
\[
\sum_{n=0}^{\infty} \frac{|t|^n}{n!} \|((\delta(\omega))^n(A))\| < \infty
\]
for
\[
|t| < \left( 2 \left( \sup_{x \in \mathbb{Z}^r} \left( \sum_{X \in \mathcal{X}} \|\Phi(X, T_a \omega)\| \right) \right) e^n |\Delta_w| \right)^{-1}.
\]
Therefore, it follows from the above proposition that, \( \delta(\omega) \) is norm-closable and the norm-closure \( \bar{\delta}(\omega) \) is the generator of an automorphism group \( \tau_t(\omega) \) of \( \mathcal{A} \) such that,

\[
\tau_t^\Lambda(\omega)(A) \to \tau_t(\omega)(A), \quad \forall A \in \mathcal{A}.
\]

The convergence of course being uniform in \( t \). We also have

\[
\delta^\Lambda(\omega)(A) \to \bar{\delta}(\omega)(A), \quad \forall A \in \bigcup_{\Lambda \subseteq \mathbb{Z}^\nu} \mathcal{A}_\Lambda,
\]

where,

\[
\delta^\Lambda(\omega)(A) = i[H(\Lambda, \omega), A], \quad \forall A \in \mathcal{A}.
\]

Since the local elements

\[
A \in \bigcup_{\Lambda \subseteq \mathbb{Z}^\nu} \mathcal{A}_\Lambda,
\]

are analytic for \( \bar{\delta}(\omega) \), the convergence of \( \tau_t^\Lambda(\omega)(A) \) as \( \Lambda \to \infty \), is uniform in a ball around zero. It is also worth noting that,

\[
D = \bigcup_{\Lambda \subseteq \mathbb{Z}^\nu} \mathcal{A}_\Lambda,
\]

is a core for \( \bar{\delta}(\omega) \). Thus, whenever \( \Phi(_, \omega) \) has a finite range and

\[
\sup_{\alpha \in \mathbb{Z}^\nu} \left( \sum_{X \geq 0} \| \Phi(X, T_\alpha \omega) \| \right) < \infty,
\]

there exists a strongly continuous, one-parameter group of automorphisms \( \tau_t(\omega) \) of \( \mathcal{A} \) such that,

\[
\tau_t(\omega)(A) = \lim_{\Lambda \to \infty} \tau_t^\Lambda(\omega)(A), \quad \forall A \in \mathcal{A}.
\]

\( \tau_t(\omega) \) is called the evolution group associated with the infinite spin system.

Since \( \Phi \) is a finite range random interaction, \( \Phi(_, \omega) \) has a finite range \( \Delta_\omega \) for
almost every \( \omega \in \Omega \). Moreover,

\[
\sup_{a \in \mathcal{Z}} \left( \sum_{X \in \mathcal{Z}} \| \Phi(X, T_a \omega) \| \right) < \infty
\]

almost everywhere. Therefore, we conclude that for almost every \( \omega \in \Omega \), there exists a strongly continuous, one-parameter group of \(*\)-automorphisms \( \tau_t(\omega) \) of \( \mathcal{A} \) such that,

\[
\lim_{\Lambda \to \infty} \tau^\Lambda_t(\omega)(A) = \tau_t(\omega)(A), \quad \forall A \in \mathcal{A}.
\]

The convergence being uniform in \( t \) on compact subsets. Thus for almost every \( \omega \in \Omega \), \( \lim_{\Lambda \to \infty} \tau^\Lambda_t(\omega)(A) \) exists for all \( A \in \mathcal{A} \) and determines an evolution group \( \tau_t(\omega) \) associated with the spin system. Besides, for almost every \( \omega \in \Omega \),

\[
\delta^\Lambda(\omega)(A) \to \delta(\omega)(A), \quad \forall A \in \bigcup_{\Lambda \subseteq \mathcal{Z}} \mathcal{A},
\]

where

\[
D = \bigcup_{\Lambda \subseteq \mathcal{Z}} \mathcal{A}
\]

is a core for \( \delta(\omega) \).

**Remark 1** Note that the radius of analyticity \( r_\omega \) of \( A \in D \), for \( \delta(\omega) \) is such that,

\[
r_\omega \geq \left( 2 \left( \sup_{a \in \mathcal{Z}} \left( \sum_{X \in \mathcal{Z}} \| \Phi(X, T_a \omega) \| \right) e^{\| A \|} \right) \right)^{-1},
\]

where in view of lemmas 3.6.0.23 and 3.6.0.24,

\[
\omega \mapsto \left( 2 \left( \sup_{a \in \mathcal{Z}} \left( \sum_{X \in \mathcal{Z}} \| \Phi(X, T_a \omega) \| \right) e^{\| A \|} \right) \right)^{-1}
\]

is almost surely constant.
Remark 2 Now for $b \in \mathbb{Z}^{r}$, $\Phi(\omega)$ has a finite range if, and only if, $\Phi(T_b\omega)$ has a finite range. Also,

$$\sup_{a \in \mathbb{Z}^{r}} \left( \sum_{X \ni 0} \| \Phi(X, T_a\omega) \| \right) = \sup_{a \in \mathbb{Z}^{r}} \left( \sum_{X \ni 0} \| \Phi(X, T_a(T_b\omega)) \| \right).$$

Therefore, $\delta(\omega)$ is norm-closable if, and only if, $\delta(T_b\omega)$ is norm-closable and $\delta^\Lambda(\omega)(A)$ converges to $\delta(\omega)(A)$, if and only if, $\delta^\Lambda(T_b\omega)(A)$ converges to $\delta(T_b\omega)(A)$, for all $A \in D$. Also, note that $D$ is a core for $\delta(\omega)$ if, and only if, it is a core for $\delta(T_b\omega)$. Hence,

$$\tau_t(\omega)(A) = \lim_{\Lambda \to \infty} \tau_t^\Lambda(\omega)(A)$$

defines a strongly continuous group of automorphisms of $\mathcal{A}$ if, and only if,

$$\tau_t(T_b\omega)(A) = \lim_{\Lambda \to \infty} \tau_t^\Lambda(T_b\omega)(A)$$

defines a strongly continuous group of automorphisms of $\mathcal{A}$. i.e. $\tau_t(\omega)$ is an evolution group if, and only if, $\tau_t(T_b\omega)$ is an evolution group.

Let $\mathcal{E}$ denote the sigma algebra of all Lebesgue measurable subsets of $\mathbb{R}$, with Lebesgue measure $\mu$. Let $\mu \times P$ be the Caratheodory extension of the product measure defined on the smallest $\sigma$-algebra $\mathcal{E} \times \mathcal{S}$, containing all measurable rectangles in $\mathbb{R} \times \Omega$. Since $\mu \times P$ is obtained using the Caratheodory extension process, it is complete. Moreover, both $\mu$ and $P$ being $\sigma$-finite, so is $\mu \times P$. Therefore, we have a measurable structure on the product space given by the triple $(\mathbb{R} \times \Omega, \mathcal{E} \times \mathcal{S}, \mu \times P)$, where $\mathcal{E} \times \mathcal{S}$ denotes the smallest sigma algebra containing $\mathcal{E} \times \mathcal{S}$, on which $\mu \times P$ is complete.
Proposition 3.7.0.29 Let $\Phi$ satisfy the assumptions of theorem 3.7.0.28 and $\tau_t(\omega)$ be the strongly continuous, one-parameter group of automorphisms of $\mathcal{A}$, which determine the evolution of the spin system. Then, $\omega \mapsto \tau_t(\omega)(A)$ is strongly, jointly measurable in both $t$ and $\omega$, for all $A \in \mathcal{A}$.

Proof It is sufficient to prove the strong measurability of the map $\omega \mapsto \tau_t(\omega)(A)$, for $A \in \mathcal{A}_{\Lambda_0}$ and all $\Lambda_0 \subseteq \mathbb{Z}^\nu$. Measurability in the case of an arbitrary $A \in \mathcal{A}$ can be established by approximating $A$ in the norm by local elements. Let $A \in \mathcal{A}_{\Lambda_0}$ where $\Lambda_0 \subseteq \mathbb{Z}^\nu$. It follows from theorem 3.7.0.28 that, $\tau_t(\omega)(A) = \lim_{t \to \infty} \tau_t^{\Lambda}(\omega)(A)$, for almost every $\omega \in \Omega$. Now, let $\{\Lambda_n\}$ be a sequence\(^3\) of finite subsets increasing to $\mathbb{Z}^\nu$. i.e.,

$$\Lambda_1 \subset \Lambda_2 \subset \Lambda_3 \subset \cdots, \quad \text{and} \quad \bigcup_{n=1}^{\infty} \Lambda_n = \mathbb{Z}^\nu.$$ 

Then for almost every $\omega \in \Omega$,

$$\tau_t(\omega)(A) = \lim_{n \to \infty} \tau_t^{\Lambda_n}(\omega)(A),$$

where $\tau_t^{\Lambda_n}(\omega)$ can be expressed in terms of commutators as

$$\tau_t^{\Lambda_n}(\omega)(A) = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} [H(\Lambda_n, \omega), A]^{(k)}.$$

Therefore, one has

$$\tau_t(\omega)(A) = \lim_{n \to \infty} \tau_t^{\Lambda_n}(\omega)(A),$$

for almost every $(t, \omega) \in \mathbb{R} \times \Omega$. Since $\omega \mapsto H(\Lambda, \omega)$ is strongly measurable, and strong measurability is preserved under products of functions, it follows

\(^3\) $\Lambda_n$'s can be taken to be cubic regions symmetric about the origin, with faces perpendicular to the co-ordinate axes and edges of length $2n$.
that \( \omega \mapsto [H(\Lambda, \omega), A]^{(k)} \) is strongly measurable for all \( k \in \mathbb{Z}^+ \). Besides, for \( t \in \mathbb{R} \), \( t^k \) is the limit almost everywhere, of numerically valued simple functions, for all \( k \in \mathbb{Z}^+ \). Therefore, the product of \( t^k \) with \([H(\Lambda, \omega), A]^{(k)}\) is the limit almost everywhere of countably valued functions on \( \mathbb{R} \times \Omega \). Hence, each of the terms of the series is a strongly, jointly measurable function of \( t \) and \( \omega \) with respect to the product measure \( \mu \times P \). We know from [Hil 57] (Theorem 3.5.4, Page 74) that, strong measurability is preserved rather, well under taking limits. Therefore, the above series is strongly, jointly measurable in \( t \) and \( \omega \). Since

\[
\tau_t(\omega)(A) = \lim_{n \to \infty} \tau_t^{\Lambda_n}(\omega)(A)
\]

for almost every \((t, \omega)\) in \( \mathbb{R} \times \Omega \), the strong, joint measurability of \((t, \omega) \mapsto \tau_t(\omega)(A)\) follows readily. Hence the proposition follows.

It is seen in the case of quantum spin systems on a lattice \( \mathbb{Z}^\nu \) with translation invariant interactions, that whenever the dynamics exists, the evolution group of \(*\)-automorphisms of the quasi-local algebra, commutes with the symmetry group of automorphisms associated with the lattice \( \mathbb{Z}^\nu \). Here we prove a variant of this property. Before we set about establishing this result, the following fact is worth noting.

**Lemma 3.7.0.30** Let \( \tau_t^\Lambda(\omega) \) be the strongly continuous, one-parameter group of local automorphisms associated with a finite \( \Lambda \subseteq \mathbb{Z}^\nu \), where

\[
\tau_t^\Lambda(\omega)(A) = e^{iH(\Lambda, \omega)t} A e^{-iH(\Lambda, \omega)t}.
\]
Then for all $a \in \mathbb{Z}^*$, we have

$$\alpha_a(\tau^\Lambda_t(\omega)(A)) = \tau^{\Lambda+a}(T_{-a}\omega)(\alpha_a(A)); \forall A \in \mathcal{A}_\Lambda.$$ 

Proof We have

$$\alpha_a(\tau^\Lambda_t(\omega)(A)) = \alpha_a(e^{iH(\Lambda,\omega)t}Ae^{-iH(\Lambda,\omega)t}) = \alpha_a(e^{iH(\Lambda,\omega)t})\alpha_a(A)\alpha_a(e^{-iH(\Lambda,\omega)t}).$$

Therefore, it follows from function calculus for $H(\Lambda, \omega)$ and the identity

$$H(\Lambda + a, T_{-a}\omega) = \alpha_a(H(\Lambda, \omega)) \quad (3.7.4)$$

that,

$$\alpha_a(\tau^\Lambda_t(\omega)(A)) = e^{iH(\Lambda + a, T_{-a}\omega)t}\alpha_a(A)e^{-iH(\Lambda + a, T_{-a}\omega)t}.$$ 

Hence, the lemma follows from this equality.

We will have the occasion to use the above lemma in the proof of the following proposition.

Proposition 3.7.0.31 Let $\tau_t(\omega)$ be the evolution group of the spin system on an infinite lattice $\mathbb{Z}^*$. Then for all $a \in \mathbb{Z}^*$, we have

$$\tau_t(T_{-a}\omega)(\alpha_a(A)) = \alpha_a(\tau_t(\omega)(A)), \quad \forall A \in \mathcal{A}.$$

Proof It is sufficient to prove the above identity for $A \in \mathcal{A}_{\Lambda_0}$ and all $\Lambda_0 \subseteq \mathbb{Z}^*$. The general case follows easily from the fact that an arbitrary $A \in \mathcal{A}$ can be approximated in the norm by local elements. It follows from
theorem 3.7.0.28, and lemma 3.7.0.30 established prior to this proposition that, for $A \in \mathcal{A}_{\Lambda_0}$, where $\Lambda_0 \subseteq \mathbb{Z}^\nu$, and all $a \in \mathbb{Z}^\nu$,

$$\alpha_a(\tau_t(\omega)(A)) = \alpha_a(\lim_{\Lambda \to \infty} (\tau_t^{\Lambda}(\omega))(A)))$$

$$= \lim_{\Lambda \to \infty} \tau_t^{\Lambda+a}(T_{-a}\omega)(\alpha_a(A))$$

$$= \lim_{\Lambda' \to \infty} (\tau_t^{\Lambda'}(T_{-a}\omega)\alpha_a(A))$$

$$= \tau_t(T_{-a}\omega)(\alpha_a(A)),$$

where $\Lambda' = \Lambda + a$. Thus we have established the identity for all local elements. Therefore, this identity can be extended to the whole of $\mathcal{A}$ using the fact that the local elements are norm-dense in $\mathcal{A}$.

**Remark** If $\tau_t(\omega)$ is the evolution group of the spin system on the infinite lattice $\mathbb{Z}^\nu$, then it follows from proposition 3.7.0.31 that, if $A$ is entire-analytic with respect to $\tau_t(\omega)$, then $\alpha_a(A)$ is entire-analytic with respect to $\tau_t(T_{-a}\omega)$, for all $a \in \mathbb{Z}^\nu$.

In the discussion that follows, we establish some interesting algebraic properties of the generators $\bar{S}(\omega)$ of the evolution groups $\tau_t(\omega)$. To this end, we have the following theorem.

**Theorem 3.7.0.32** Let $U_n$ be a sequence of $C_0$-semigroups of contractions on the Banach space $X$, with generators $S_n$ and define the graph $G_\alpha$ by

$$G_\alpha = \lim_{n \to \infty} G(I - \alpha S_n).$$

The following conditions are equivalent:

1. there exists a $C_0$-semigroup $U$ such that,

$$\lim_{n \to \infty} \|(U_{n,t} - U_t)A\| = 0,$$
for all \( A \in X, t \in \mathbb{R}_+ \); uniformly for \( t \) in any finite interval of \( \mathbb{R}_+ \);

2. the sets \( D(G_\alpha) \) and \( R(G_\alpha) \) are norm-dense in \( X \) for some \( \alpha > 0 \).

If these conditions are satisfied, then \( G_\alpha \) is the graph of \( I - \alpha S \), where \( S \) is the generator of \( U \).

**Proof** Refer to theorem 3.1.28 in [Rob 87]. \( \triangle \)

**Remark** One of the situations in which the above theorem can be applied is the following: Let \( S_n \) and \( S \) be the generators of \( C_0 \)-contraction semigroups and suppose there exists a core \( D \) of \( S \) such that,

\[
D \subseteq \bigcup_m \left( \bigcap_{n \geq m} D(S_n) \right)
\]

and

\[
\lim_{n \to \infty} \| (S_n - S) A \| = 0,
\]

for all \( A \in D \). It then follows that \( S \) is the graph limit of the \( S_n \)'s.

This theorem yields the following proposition.

**Proposition 3.7.0.33** Let \( \tau_t(\omega) \) be the evolution group of the spin system and \( D(\delta(\omega)) \) be the domain of the generator of the automorphism group \( \tau_t(\omega) \). Then for all \( a \in \mathbb{Z}^\nu \), we have \( \alpha_a(D(\delta(\omega))) = D(\delta(T_{-a(\omega)}) \) and \( \alpha_a(\delta(\omega))(A) = \delta(T_{-a(\omega)})(\alpha_a(A)) \), for all \( A \in D(\delta(\omega)) \).

**Proof** It is seen from the proof of theorem 3.7.0.28 that,

\[
D = \bigcup_{\Lambda \subseteq \mathbb{Z}^\nu} A_\Lambda,
\]

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is a core for $\delta(\omega)$ and

$$\delta^\Lambda(\omega)(B) \to \delta(\omega)(B); \quad \forall B \in D,$$

where $\delta^\Lambda(\omega)$ is the generator of the local automorphism group $\tau^\Lambda(\omega)$. We have $\tau^\Lambda(\omega)(B) = e^{iH(\Lambda,\omega)t}Be^{-iH(\Lambda,\omega)t}$ and $\delta^\Lambda(\omega)(B) = i[H(\Lambda,\omega), B]$, for all $B \in A$. Let $\{\Lambda_n\}$ be a sequence of finite subsets increasing to $\mathbb{Z}^\nu$, then we have

$$\delta^{\Lambda_n}(\omega)(B) \to \delta(\omega)(B); \quad \forall B \in D.$$

Therefore, we conclude from the remark made after the statement of the above theorem that, $\delta(\omega)$ is the graph limit of $\delta^{\Lambda_n}(\omega)$. Hence, for $A \in D(\delta(\omega))$, there exists a sequence $\{A_n\}$, where $A_n \in D(\delta^{\Lambda_n}(\omega))$ such that, $A_n \to A$ and $\delta^{\Lambda_n}(\omega)(A_n) \to \delta(\omega)(A)$. This implies that $\alpha_a(A_n) \to \alpha_a(A)$ and $\alpha_a(\delta^{\Lambda_n}(\omega)(A_n)) \to \alpha_a(\delta(\omega)(A))$. Now, it follows from the identity 3.7.4 in lemma 3.7.0.30 that,

$$\alpha_a(\delta^{\Lambda_n}(\omega)(A_n)) = \delta^{\Lambda_n+\alpha}(T_{-\omega})(\alpha_a(A_n)).$$

Hence, we have $\alpha_a(A_n) \to \alpha_a(A)$ and $\delta^{\Lambda_n+\alpha}(T_{-\omega})(\alpha_a(A_n)) \to \alpha_a(\delta(\omega)(A))$. Clearly, from remark 2 at the end of theorem 3.7.0.28, $\delta^\Lambda(T_{-\omega})(B)$ converges to $\delta(T_{-\omega})(B)$, for all $B \in D$, and $D$ is a core for $\delta(T_{-\omega})$, where $\delta^\Lambda(T_{-\omega})$ is the generator of the local automorphism group $\tau^\Lambda(T_{-\omega})$. We have

$$\tau^\Lambda(T_{-\omega})(B) = e^{iH(\Lambda, T_{-\omega})t}Be^{-iH(\Lambda, T_{-\omega})t} \text{ and } \delta^\Lambda(T_{-\omega})(B) = i[H(\Lambda, T_{-\omega}), B],$$

for all $B \in A$. Since $\{\Lambda_{n+a}\}$ is a sequence of finite subsets increasing to $\mathbb{Z}^\nu$, it follows that $\delta^{\Lambda_n+\alpha}(T_{-\omega})(B)$ converges to $\delta(T_{-\omega})(B)$, for all $B \in D$. Hence, the remark following theorem 3.7.0.28 implies that $\delta(T_{-\omega})$ is the graph limit of $\delta^{\Lambda_n}(\omega)$.
of $\delta^{\Lambda_n+a}(T_{-a}\omega)$. Therefore, as $\alpha_a(A_n) \to \alpha_a(A)$ and $\delta^{\Lambda_n+a}(T_{-a}\omega)(\alpha_a(A_n)) \to \alpha_a(\delta(\omega)(A))$, where $\alpha_a(A_n) \in D(\delta^{\Lambda_n+a}(T_{-a}\omega))$, one concludes that $\alpha_a(A) \in D(\delta(T_{-a}\omega))$ and $\alpha_a(\delta(\omega))(A) = \delta(T_{-a}\omega)(\alpha_a(A))$. Conversely, it can be shown that if $A \in D(\delta(T_{-a}\omega))$ then $\alpha_{-a}(A) \in D(\delta(\omega)))$. This completes the proof of the proposition.

In the next chapter, we aim to study the Arveson spectrum of the strongly continuous, one-parameter group of automorphisms $\tau_t(\omega)$, which determines the evolution of the spin system. We report an interesting ergodic property of the Arveson spectrum of the evolution group $\tau_t(\omega)$. 

$\Delta$