CHAPTER 4

CONSTRUCTION OF GOLDEN GRAPHS-I

4.1 INTRODUCTION:

In this chapter we have constructed golden graphs. First we have proved logically that $G + P_4$, $G + C_5$ and tree with diameter 6 as golden graphs. And using the graph $G + P_4$, we have constructed golden graphs by taking $k$ copies of $G + P_4$ and each copy of $G + P_4$ is attached to isolated vertex. And also constructed golden graphs using $G + C_5$ and tree with diameter 6 with the same construction. In the end of this chapter, we have constructed golden graphs using $C_5$, $C_5 + K_1$ and $P_4 + K_1$.

4.2 EXISTING RESULTS:

Theorem 4.1[19]: For the spectrum $[\lambda_1, \lambda_2, \ldots, \lambda_n]$ of a graph $G$ the following statement hold:

The numbers $\lambda_1, \ldots, \lambda_n$ are real and $\lambda_1 + \ldots + \lambda_n = 0$.

If $G$ contains no edges, we have $\lambda_1 = \ldots = \lambda_n = 0$.

If $G$ contains at least one edge, we have

$1 \leq r \leq n - 1$ \hspace{1cm} (0.1)

$-r \leq m \leq 1$ \hspace{1cm} (0.2)
In (0.1), the upper bond is attained if and only if $G$ is complete graph, and while the lower bound is reached if and only if the components of $G$ consists of graphs $K_2$ and possibly $K_1$.

In (0.2), the upper bond is attained if and only if the components $G$ are complete graph, and the lower bound if and only if a components of $G$ having the greatest index is a bipartite graph. If $G$ is connected, the lower bound in (0.1) is replaced with $2\cos\frac{\pi}{n+1}$. Then equality holds if and only if $G$ is a path.

**Theorem 4.2[19]**: The characteristic polynomial of the complete product of regular graphs $G_1$ and $G_2$ is given by the relation:

$$\phi(P_{G_1+G_2},x) = \frac{\phi(G_1,x)\phi(G_2,x)}{(x-\eta)(x-3)}[(x-\eta)(x-3)-n_1n_2]$$

**4.3 MAIN RESULTS:**

**Theorem 4.3**: Let $G$ be any graph, then $G+P_4$ is golden graph.

**Proof**: Let $G$ be any graph with order $n$ and size $m$ and $P_4$, a path on 4 vertices.

$G+P_4$ represented by
Characteristic polynomial of $G + P_4$ given by

$$
\phi(G + P_4, x) = (-1)^{4} \phi(G, x) \phi(P_4, x) - (-1)^{n} \phi(P_4, x) \phi(\overline{G}, -1 - x) \\
- (-1)^{n+4} \phi(\overline{G}, -1 - x) \phi(P_4, -1 - x)
$$

$$
= (x^2 + x + 1). (x^2 + x - 1). \phi(G, x) + (-1)^{n} . (x^2 + x + 1). (x^2 + x - 1). \phi(\overline{G}, -1 - x) \\
- (-1)^{n+4} (x^2 + x + 1). (x^2 + x - 1). \phi(\overline{G}, -1 - x)
$$

In the above expression first term is divisible by $(x^2 + x - 1)$ and so second, third term.

$\therefore \phi(G + P_4, x)$ is divisible by $(x^2 + x - 1)$

Hence $G + P_4$ is a golden graph.

**Example 4.1:**
Spectra of the graph [figure 4.2] is -1.8255, -1.6180, -1, -1, -1.4179, 0.6180, 5.2434.

**Theorem 4.4:** Let $G$ be any graph, then $G + C_5$ is golden graph.

**Proof:** Let $G$ be any graph with $n$ vertices and size $m$ and $C_5$ be a cycle on 5 vertices.

$G + C_5$ is represented as
Characteristic polynomial of $G + C_5$ given by

$$\phi(G + C_5, x) = (-1)^5 \phi(G, x).\phi(C_5, -x - 1) + (-1)^n \phi(C_5, x).\phi(G, -1 - x)$$

$$- (-1)^{n+5} \phi(G, -1 - x).\phi(C_5, -1 - x)$$

$$= -(x - 2).\left(x^2 + x - 1\right)^2 \phi(G, x) + (-1)^n (x - 2).\left(x^2 + x - 1\right)^2 \phi(G, -1 - x)$$

$$- (-1)^{n+5} (x - 2).\left(x^2 + x - 1\right)^2 \phi(G, -1 - x)$$

In this expression first term is divisible by $(x^2 + x - 1)$ and so second, third term.

$\therefore \phi(G + C_5, x)$ is divisible by $(x^2 + x - 1)$

Hence $G + C_5$ is a golden graph.

**Example 4.2:**
The eigen values of the graph [as in figure 4.4] is -1.8730, -1.6180, -1.6180, -1, -1, 0.6180, 0.6180, 5.8730.

**Corollary 4.5**: $C_5 + K_1$ is a golden graph.

**Proof**: The proof is obvious.
Spectra of $C_5 + K_1$ [as in figure 4.5] is $-1.6180, -1.6180, -1.4495, 0.6180, 0.6180, 3.4495$.

**Definition 4.6:** Let $G$ be a graph obtained by taking $k$ copies $C_5 + K_1$ and attaching each copy of $C_5 + K_1$ of vertex of degree 5 to an isolated vertex $u$ as shown in the figure 4.6.

**Theorem 4.7:** Let $G$ be graph as shown in the figure 4.6, then $G$ golden graph.

**Proof:** Let $u$ be the vertex of $G$ as shown in the figure 4.6 and $w$ is adjacent to $u$. 
Characteristic polynomial of \( G \) given by

\[
\phi(G, x) = x\phi(G - u, x) - \sum_{w \in \ell} \phi(G - u - w, x)
\]

\[
= x(\phi(C_5 + K_1, x))^k - k[\phi(C_5, x), \phi(C_5 + K_1, x)]
\]

By Corollary 4.5, first term is a polynomial of golden graph, since

\( \phi(C_5 + K_1, x) \) is divisible by \( (x^2 + x - 1) \) and also second term.

Therefore \( \phi(G, x) \) is divisible by \( (x^2 + x - 1) \).

Hence \( G \) is a golden graph.

Example 4.3:
The eigenvalues of the graph [figure 4.7] are -1.9793, -1.6180, -1.6180, -1.6180, -1.4694, -1.1303, 0.0808, 0.6180, 0.6180, 0.6180, 1.1057, 3.4817, 3.9108.

**Corollary 4.8**: $P_4 + K_1$ (Fan Graph) is a golden graph.

**Proof**: The proof is obvious.

**Definition 4.9**: Let $G$ be a graph obtained by taking $k$ copies $P_4 + K_1$ and attaching each copy of $P_4 + K_1$ of vertex of degree 4 to an isolated vertex $u$ as shown in the figure.

**Theorem 4.10**: Let $G$ be a graph as in the figure 4.8, then $G$ is golden graph.
**Proof:** Let \( u \) be the vertex of \( G \) as shown in the figure 4.8 and \( w \) is adjacent to \( u \).

Characteristic polynomial of \( G \) given by

\[
\phi(G, x) = x\phi(G - u, x) - \sum_{v \in V} \phi(G - u - w, x)
\]

\[
= x(\phi(P_4 + K_1, x))^k - k[\phi(P_4, x)\phi(P_4 + K_1, x)^{k-1}]
\]

By Corollary 4.8, first term is divisible by \((x^2 + x - 1)\) as \( P_4 + K_1 \) is golden graph and second term is divisible by \((x^2 + x - 1)\) as \( P_4 \) is the golden graph.

\[\therefore \quad \phi(G, x) \text{ is divisible by } (x^2 + x - 1)\]

Therefore \( G \) is a golden graph.

**Example 4.4:**

![Diagram of a graph](image)

FIGURE 4.9

The spectra of the graph [figure 4.9] is 3.0437, 2.9354, -1.8241, -1.4728, 0.3285, -0.5482, 0.4626, -1.6180, -1.6180, 0.6180, 0.6180

**Note:** The graph shown below \( H_1 \) is golden graph
The spectra of the graph $H_1$ [figure 4.10] is 2.3028, 0.6180, 0, -1.3028, -1.6180

**Definition 4.11**: Let $G$ be a graph obtained by taking $k$ copies of $H_1$ (as in figure 4.10) and attaching each copy of $H_1$ of vertex of degree 2 to an isolated vertex $u$ as shown in the figure 4.11.

**Theorem 4.12**: Let $G$ be the graph as in the figure 4.11, then $G$ is golden graph.
**Proof:** Let $u$ be the vertex of $G$ as shown in the figure 4.11 and $w$ is adjacent to $u$.

Characteristic polynomial of $G$ given by

$$
\phi(G, x) = x \phi(G - u, x) - \sum_{w \in V} \phi(G - u - w, x)
$$

$$
= x (\phi(H_1, x))^k - k[\phi(P_4, x) \phi(H_1, x)^{k-1}]\n$$

First term is characteristic polynomial of golden graph $H_1$ [figure 4.10], since $\phi(H_1, x)$ is divisible by $(x^2 + x - 1)$ and second term is divisible by $(x^2 + x - 1)$, as $\phi(P_4, x)$ is characteristic polynomial of $P_4$ which is golden graph.

Therefore $\phi(G, x)$ is divisible by $(x^2 + x - 1)$

Therefore $G$ is a golden graph

**Example 4.5:**

![Figure 4.11](image-url)
The spectra of the graph [figure 4.11] is -1.8976, -1.6180, -1.6180, -1.3028, -0.4891, 0, 0.6180, 0.6180, 0.8493, 2.3028, 2.5374.

**Definition 4.13:** Let $H$ be any graph with vertices $u_1, u_2, \ldots, u_n$. Let $G$ be the graph obtained by taking $k$ copies of $H$ and attaching the vertex $u_i$ of each copy of $H$ and one copy of $H_1$ of degree 3 to an isolated vertex $u$ as shown in the figure 4.12.

**Theorem 4.14:** Let $G$ be the graph as in the figure 4.12, then $G$ is golden graph.
**Proof:** Let $u$ be the vertex of $G$ as shown in the figure 4.12 and $u_1$ is adjacent to $u$.

Characteristic polynomial of $G$ given by

$$
\phi(G,x) = x\phi(G-u,x) - \sum_{u=u} \phi(G-u-u_1,x)
$$

$$
= x(\phi(H,x))^k \phi(H_1,x) - \phi(P_4,x)\phi(H,x)^k - \phi(H_1,x).k[\phi(H,x)^{k-1}\phi(H_2,x)]
$$

Where $H_1$ is the graph as in figure [4.10], and $H_2$ is the graph obtained by deleting vertex $u_1$ adjacent to $u$.

By Figure 4.10, first term is divisible by $(x^2 + x - 1)$ as $H_1$ is golden graph and second term is divisible by $(x^2 + x - 1)$ as $P_4$ is the golden graph and also third term is divisible by $(x^2 + x - 1)$ as $H_1$ is golden graph.

$\therefore$ $\phi(G,x)$ is divisible by $(x^2 + x - 1)$

Hence $G$ is a golden graph.
Example 4.6:

![Graph Figure 4.13]

The spectra of the graph [figure 4.13] are -1.8608, -1.6180, -1.1701, -1, -0.2541, 0.6180, 0.6889, 2.1149, 2.4812.

Definition 4.15: Let $G$ be the graph obtained by attaching $k$ copies of $C_5$ to an isolated vertex $u$ as shown in figure 4.14.
Theorem 4.16: Let $G$ be a graph as shown in the figure 4.14, then $G$ is golden graph.

Proof: Let $u$ be the vertex of $G$ as shown in the figure 4.11 and $w$ is adjacent to $u$.

Characteristic polynomial of $G$ given by

$$\phi(G,x) = x\phi(G-u,x) - \sum_{w} \phi(G-u-w,x)$$

$$= x(\phi(C_5,x))^k - k[\phi(P_4,x)\phi(C_5,x)^{k-1}]$$

First term is divisible by $(x^2 + x - 1)$ as $C_5$ is golden graph and second term is divisible by $(x^2 + x - 1)$ as $P_4$ is the golden graph.

$\therefore \phi(G,x)$ is divisible by $(x^2 + x - 1)$
Hence $G$ is a golden graph.

**Example 4.7:**

![Graph Diagram](image)

The spectra of the graph [figure 4.15] is -2.0980, -1.6180, -1.6180, -1.6180, -0.3565, 0.6180, 0.6180, 0.6180, 1.1899, -2, 2.2562

**Definition 4.17:** Let $H$ be any graph with vertices $u_1,u_2,\ldots,u_n$. Let $G$ be the graph obtained by taking $k$ copies of $H$ and one $C_5$ and attaching the vertex $u_i$ of each copy of $H$ and one vertex of $C_5$ to an isolated vertex $u$ as shown in figure 4.16.
**Theorem 4.18:** Let \( G \) be a graph as in the figure 4.16, then \( G \) is golden graph.

**Proof:** Let \( u \) be the vertex of \( G \) as shown in the figure 4.16 and \( u_i \) is adjacent to \( u \). Characteristic polynomial of \( G \) given by

\[
\phi(G, x) = x \phi(G - u, x) - \sum_{v \in V} \phi(G - u - u_i, x)
\]

\[
= x \phi(C_5, x)(\phi(H, x))^k - \phi(P_4, x)\phi(H, x)^k - k[\phi(C_5, x)\phi(H, x)\phi(H, x)^{k-1}]
\]
First term is divisible by \((x^2 + x - 1)\) as \(C_5\) is golden graph and second term is divisible by \((x^2 + x - 1)\) as \(P_4\) is the golden graph and third term is divisible by \((x^2 + x - 1)\) as \(C_5\) is golden graph.

\[\therefore \phi(G,x) \text{ is divisible by } (x^2 + x - 1)\]

Hence \(G\) is a golden graph.

**Example 4.8:**

![FIGURE 4.17]
The spectra of the graph [figure 4.17] is -1.4812, 0.3111, 2.1701, -1.6180, 0.6180, -1.6180, -1, 0.6180.

**Definition 4.19:** Let $G = T$ be a tree of diameter 6 on $n$ vertices as shown in figure 4.18.

![Figure 4.18](image)

**Theorem 4.20:** Let $G = T$ be a tree as shown in the figure 4.18, then $G$ is golden graph.

**Proof:** Let $G$ be a tree with diameter 6 as shown in figure[].

Characteristic polynomial of $G$ is given by

$$
\phi(G, x) = x \phi(G - u, x) - \sum_{w \in V} \phi(G - u - w, x)
$$

$$
= x^{n-8} \left( \phi(P_4, x) \right)^2 - 2 \left[ x^{n-8} \phi(K_2, x) \phi(P_4, x) \right] - (n-8) \left[ \phi(P_4, x)^2 \right]
$$

First term in the expression is divisible by $(x^2 + x - 1)$ and second term and also third term as $P_4$ is the golden graph.

$$
\therefore \phi(G, x) \text{ is divisible by } x^2 + x - 1
$$

Hence $G$ is a golden graph.
Example 4.9:

The spectra of the graph [figure 4.19] is $2.2059, \ -2.2059, \ -1.6180, \ -1.3376, \ 1.6180, \ 1.3376, \ -0.6180, \ -0.5870, \ 0.6180, \ 0.5870, \ 0, \ 0$.

Definition 4.21: Let $H$ be any graph with vertices $u_1, u_2, \ldots, u_n$. Let $G$ be the graph obtained by taking $k$ copies of $H$ and one tree $T$ (as shown in the figure 4.19) and attaching the vertex $u_i$ of each copy of $H$ and central vertex of tree $T$ to an isolated vertex $u$ as shown in the figure.
Theorem 4.22: Let $G$ be a graph as shown in the figure 4.20, then $G$ is a golden graph.

Proof: Characteristic polynomial of $G$ is given by 

$$
\phi(G, x) = x\phi(G - u, x) - \sum_{u_i \sim u} \phi(G - u - u_i, x)
$$

$$
= x\phi(T, x)[\phi(H, x)]^k - [\phi(T, x)]\phi(H, x) - k[\phi(H_1, x)\phi(H, x)^{k-1}\phi(T, x)]
$$
Where $T_i$ is the graph obtained from the graph $T$ by deleting the vertex $u_i$ adjacent to $u$ and $H_i$ is the graph obtained from the graph $H$ by deleting the vertex $u_i$ adjacent to $u$. 

In this equation, first term is divisible by $(x^2 + x - 1)$ as it contains polynomial of tree which is golden graph (by theorem 4.20) and second term consists polynomial of $T_i$ which divisible by $(x^2 + x - 1)$ and also third term. Therefore $\phi(G,x)$ is divisible by $(x^2 + x - 1)$

Hence $G$ is golden graph.

Example 4.10:
FIGURE 4.21

The spectra of the graph [figure 4.21] is 0, 1, 1, 1.9440, -1.6614, -1.6180, 1, 0.6180, -0.1413, -0.1413, 1.6180, 1.6180, -1, -0.6180, -0.6180, 0, -1, 1.