OPTIMUM STEP-STRESS PARTIALLY ACCELERATED LIFE TEST PLANS FOR THE TRUNCATED LOGISTIC DISTRIBUTION WITH TYPE-I AND TYPE-II CENSORING

5.1 INTRODUCTION

In step-stress PALT, a test item is first run at normal use condition and, if it does not fail, then it is run at accelerated condition until failure occurs or the observation is censored.

This Chapter focuses on optimum design of step-stress PALT for the truncated logistic distribution with type-I and type-II censoring. Truncated distributions arise when sample selection is not possible in some sub-region of the sample space. The failure rate of the truncated logistic distribution truncated at point zero, is increasing and is more realistically bounded below and above by a non-zero finite quantity (see 1.4.2.8).

The work done in the literature on the design of step-stress PALT prior to this has been reviewed in 1.10.6.

Notations

\[ q_1 \quad \text{Proportion of units failed at use condition} \]
\( n_{q_1} \) or \( n_u \)  
Number of units failed at use condition

\( q_r \)  
Proportion of units failed before censoring

\( r \)  
Number of units failed before censoring at which the test is terminated under type-II censoring (case I: number of failures before censoring pre-specified)

\( n_{q_r} \)  
Number of units failed before censoring at which the test is terminated under type-II censoring (case II: proportion of units failing before censoring pre-specified)

\( q_r - q_1 \)  
Proportion of units failed at accelerated condition

\( n(q_r - q_1) \) or \( n_a \)  
Number of units failed at accelerated condition

\( 1 - q_r \)  
Proportion of censored units

\( n(1 - q_r) \) or \( n_c \)  
Number of censored units

\( n \)  
Total number of test items in a step-stress PALT  
\( (n = n_{q_r} + n(1 - q_r) \) or \( n = n_u + n_a + n_c \))

\( \beta \)  
Acceleration factor, \( \beta > 1 \)

\( T \)  
Lifetime of an item at use condition

\( Y \)  
Total lifetime of an item under type-I censoring and type-II censoring (case I: number of failures before censoring pre-specified)

\( Y_i \)  
Observed value of the total lifetime \( Y_i \) of item \( i, i = 1, 2, \ldots, n \) under type-I censoring
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For type-I censoring:
\[ \delta_{1i} = \begin{cases} 1, & y_i \leq \tau, \quad i = 1, 2, \ldots, n_u \\ 0, & \text{otherwise} \end{cases} \]

\[ \delta_{2i} = \begin{cases} 1, & \tau < y_i \leq \eta, \quad i = n_u + 1, \ldots, n_u + n_a \\ 0, & \text{otherwise} \end{cases} \]

For type-II censoring (case I: number of failures pre-specified)

\[ \delta_{1i} = \begin{cases} 1, & y_i \leq \tau, \quad i = 1, 2, \ldots, n_u \\ 0, & \text{otherwise} \end{cases} \]

\[ \delta_{2i} = \begin{cases} 1, & \tau < y_i \leq y_{(r)}, \quad i = n_u + 1, \ldots, r \\ 0, & \text{otherwise} \end{cases} \]

For type-II censoring (case II: proportion of units failing before censoring pre-specified)

\[ \delta_{1i} = \begin{cases} 1, & u_{(i)} \leq u_{(nq_l)}, \quad i = 1, 2, \ldots, nq_l \\ 0, & \text{otherwise} \end{cases} \]

\[ \delta_{2i} = \begin{cases} 1, & u_{(nq_l)} < u_{(i)} \leq u_{(nq_r)}, \quad i = (nq_l + 1), \ldots, nq_r \\ 0, & \text{otherwise} \end{cases} \]

\[ \bar{\delta}_{li} = 1 - \delta_{li} \]

\[ \bar{\delta}_{2i} = 1 - \delta_{2i} \]

\[ A = \frac{1}{1 + \exp \left( -\frac{\mu}{\sigma} \right)} \]

\[ P_u \quad \text{Probability of an item fails at use condition:} \]
\[ P_u = \frac{1}{A} \left( \frac{1}{1 + \exp \left( -\frac{\tau - \mu}{\sigma} \right)} - \frac{1}{1 + \exp \left( \frac{\mu}{\sigma} \right)} \right), \]

under type-I censoring and type-II censoring (case I: number of failures pre-specified)

\[ P_u = \frac{1}{A} \left( \frac{1}{1 + \exp \left( -\frac{\mu - \mu_{(nq)}}{\sigma} \right)} - \frac{1}{1 + \exp \left( \frac{\mu}{\sigma} \right)} \right), \]

under type-II censoring (case II: proportion of units failing before censoring pre-specified)

\[ P_a = \text{Probability of an item fails at accelerated condition:} \]

\[ P_a = \frac{1}{A} \left( \frac{1}{1 + \exp \left( -\frac{(\eta - \tau)\beta + \tau - \mu}{\sigma} \right)} - \frac{1}{1 + \exp \left( -\frac{\tau - \mu}{\sigma} \right)} \right), \]

under type-I censoring,

\[ P_a = \frac{1}{A} \left( \frac{1}{1 + \exp \left( -\frac{(y(r) - \tau)\beta + \tau - \mu}{\sigma} \right)} - \frac{1}{1 + \exp \left( -\frac{\tau - \mu}{\sigma} \right)} \right), \]

under type-II censoring (case I: number of failures pre-specified), and

\[ P_a = \frac{1}{A} \left( \frac{1}{1 + \exp \left( -\frac{u_{(nq)} + (u_{(nq)} - u_{(nq)})\beta - \mu}{\sigma} \right)} - \frac{1}{1 + \exp \left( -\frac{u_{(nq)} - \mu}{\sigma} \right)} \right), \]

under type-II censoring (case II: proportion of units failing before censoring pre-specified).
pre-specified).

\[ P_c = 1 - (P_u + P_n) \]

\[ P_c = \frac{1}{A} \left( \frac{1}{1 + \exp \left( -\frac{1}{\sigma} (\eta - \tau) + \tau - \mu \right)} - \frac{1}{1 + \exp \left( \frac{\mu}{\sigma} \right)} \right), \]

under type-I censoring,

\[ P_c = \frac{1}{A} \left( \frac{1}{1 + \exp \left( -\frac{1}{\sigma} (y_r - \tau) + \tau - \mu \right)} - \frac{1}{1 + \exp \left( \frac{\mu}{\sigma} \right)} \right), \]

under type-II censoring (case I: number of failures pre-specified), and

\[ P_c = \frac{1}{A} \left( \frac{1}{1 + \exp \left( -\frac{1}{\sigma} u_{(nq)} + (u_{(nq+1)} - u_{(nq)}) \beta - \mu \right)} - \frac{1}{1 + \exp \left( \frac{\mu}{\sigma} \right)} \right) \]

under type-II censoring (case II: proportion of units failing before censoring pre-specified).

Ordered failure times under type-II censoring (case I: number of failures pre-specified)

\[ y(1) \leq \ldots \leq y(n_u) \leq \tau \leq y(n_u+1) \leq \ldots \leq y(r) \]

Ordered failure times under type-II censoring (case II: proportion of units failing before censoring pre-specified)

\[ u(1) < \ldots < u_{(nq)} < u_{(nq+1)} < \ldots < u_{(nq_r)} \]
5.1.1 Truncated Logistic Distribution

The cdf, pdf, reliability function, and hazard function, respectively, of the truncated logistic distribution are

\[ F(t) = \frac{1}{A} \left( \frac{1}{1 + \exp\left( -\frac{t - \mu}{\sigma} \right)} - \frac{1}{1 + \exp\left( \frac{\mu}{\sigma} \right)} \right), \quad t > 0, \quad (5.1) \]

(Mood, Graybill and Boes (1974))

\[ f(t) = \frac{\exp\left( -\frac{t - \mu}{\sigma} \right)}{A \sigma \left( 1 + \exp\left( -\frac{t - \mu}{\sigma} \right) \right)^2}, \quad t > 0, \quad (5.2) \]

\[ R(t) = \frac{\exp\left( -\frac{t - \mu}{\sigma} \right)}{A \left( 1 + \exp\left( -\frac{t - \mu}{\sigma} \right) \right)}, \quad t > 0, \quad (5.3) \]

and

\[ h(t) = \frac{1}{\sigma \left( 1 + \exp\left( -\frac{t - \mu}{\sigma} \right) \right)}, \quad t > 0. \quad (5.4) \]

The hazard function in (5.4) is an increasing function of t, and is bounded by

\[ \frac{1}{\sigma \left( 1 + \exp\left( \frac{\mu}{\sigma} \right) \right)}, \quad \text{and} \quad \frac{1}{\sigma}. \]

Thus, the total lifetime \( Y \) of an item under type-I censoring and type-II censoring (case I: number of failures pre-specified) is

\[ Y = \begin{cases} T, & T \leq \tau \\ \tau + \beta^{-1}(T - \tau), & \text{otherwise} \end{cases}, \quad (5.5) \]
the cdf and pdf of total lifetime $Y$ of an item under both type-I censoring and type-II censoring (case I: number of failures pre-specified), respectively, are

$$F(y) = \begin{cases} \frac{1}{A} \left( \frac{1}{1 + \exp \left( \frac{y - \mu}{\sigma} \right)} - \frac{1}{1 + \exp \left( \frac{\mu}{\sigma} \right)} \right) , & 0 < y \leq \tau \\ \frac{1}{A} \left( \frac{1}{1 + \exp \left( -\tau + (y - \tau)^{\beta - \mu} \right)} - \frac{1}{1 + \exp \left( \frac{\mu}{\sigma} \right)} \right) , & \tau < y \end{cases}$$

(5.6)

$$f(y) = \begin{cases} \frac{\exp \left( \frac{y - \mu}{\sigma} \right)}{A\sigma \left( 1 + \exp \left( \frac{y - \mu}{\sigma} \right) \right)^2} , & 0 < y \leq \tau \\ \frac{\beta \exp \left( \frac{-\tau + (y - \tau)^{\beta - \mu}}{\sigma} \right)}{A\sigma \left( 1 + \exp \left( \frac{-\tau + (y - \tau)^{\beta - \mu}}{\sigma} \right) \right)^2} , & y > \tau \end{cases}$$

(5.7)

Also, the total lifetime $U$ of an item under type-II censoring (case II: proportion of units failing before censoring pre-specified) is

$$U = \begin{cases} T , & T \leq u_{(nq_1)} \\ u_{(nq_1)} + \beta^{-1}(T - u_{(nq_1)}) , & \text{otherwise} \end{cases}$$

(5.8)

the cdf and pdf of total lifetime $U$ of an item under type-II censoring (case II: proportion of units failing before censoring pre-specified), respectively, are
F(u) = \begin{cases} 
\frac{1}{A} \left( \frac{1}{1 + \exp\left\{-\frac{u - \mu}{\sigma}\right\}} - \frac{1}{1 + \exp\left\{\frac{\mu}{\sigma}\right\}} \right), & 0 < u \leq u_{(nq)}_1 \\
\frac{1}{A} \left( \frac{1}{1 + \exp\left\{-\frac{u_{(nq)}_1 + \beta(u - u_{(nq)}_1) - \mu}{\sigma}\right\}} - \frac{1}{1 + \exp\left\{\frac{\mu}{\sigma}\right\}} \right), & u_{(nq)}_1 < u,
\end{cases} 
(5.9)

and

f(u) = \begin{cases} 
\frac{\exp\left\{-\frac{u - \mu}{\sigma}\right\}}{A\sigma\left(1 + \exp\left\{-\frac{u - \mu}{\sigma}\right\}\right)^2}, & 0 < u \leq u_{(nq)}_1 \\
\frac{\beta\exp\left\{-\frac{u_{(nq)}_1 + (u - u_{(nq)}_1)\beta - \mu}{\sigma}\right\}}{A\sigma\left(1 + \exp\left\{-\frac{u_{(nq)}_1 + (u - u_{(nq)}_1)\beta - \mu}{\sigma}\right\}\right)^2}, & u > u_{(nq)}_1.
\end{cases} 
(5.10)

5.2 OPTIMUM TIME-CENSORED STEP-STRESS PALT MODEL

In the present section, we (Srivastava and Mittal (2010)) have formulated an optimum time-censored step-stress PALT plan for the truncated logistic distribution.

Assumptions

1) The lifetime of an item tested at both use, and at accelerated condition follows the truncated logistic distribution.
2) The lifetimes of test items are independent and identically distributed random variables.

5.2.1 Test Procedure

1) All 'n' items are first tested at use condition.
2) If any item out of 'n' items does not fail at use condition by pre-specified time $\tau$, then it is put on accelerated condition, and run until censoring time $\eta$.
3) The test is continued until:
   a) all test items fail, or
   b) a prescribed censoring time $\eta$;
      whichever occurs earlier, and the test conditions remain the same.

5.2.2 Log-Likelihood Function

The likelihood function $L$ for $n$ independent observations is

$$
L = L(\beta, \mu, \sigma) = \prod_{i=1}^{n} L_i \left( \beta, \mu, \sigma; (y_i, \delta_{i1}, \delta_{i2}) \right)
$$

$$
= \prod_{i=1}^{n} \left( f_i(y_i) \delta_{i1} \left( f_2(y_i) \delta_{i2} \left( R(\eta) \right) \right) \right)^{\delta_{i1}} \left( \exp \left( -\frac{y_i - \mu}{\sigma} \right) \right)^{\delta_{i1}} \left( \beta \exp \left( -\frac{(y_i - \tau)\beta + \tau - \mu}{\sigma} \right) \right)^{\delta_{i2}} \left( \exp \left( -\frac{(\eta - \tau)\beta + \tau - \mu}{\sigma} \right) \right)^{\delta_{i2}}.
$$

(5.11)
It is usually easier to maximize the natural logarithm of the likelihood function rather than the likelihood function itself. Thus, the log-likelihood of (5.11) is

\[
\ln L = \ln L(\mu, \sigma) = n_a \ln \beta - (n_a + n_u) \ln \sigma + n \ln \left(1 + \exp\left(-\frac{\mu}{\sigma}\right)\right)
\]

\[
-\sum_{i=1}^{n} 2\delta_{1i} \ln \left(1 + \exp\left(-\frac{y_i - \mu}{\sigma}\right)\right) - \sum_{i=1}^{n} \delta_{2i} \left(\frac{(y_i - \tau)\beta + \tau - \mu}{\sigma}\right)
\]

\[
-\sum_{i=1}^{n} 2\delta_{2i} \ln \left(1 + \exp\left(-\frac{(y_i - \tau)\beta + \tau - \mu}{\sigma}\right)\right) - \sum_{i=1}^{n} \delta_{li} \delta_{2i} \left(\frac{(\eta - \tau)\beta + \tau - \mu}{\sigma}\right)
\]

\[
-\sum_{i=1}^{n} \delta_{li} \xi_{2i} \ln \left(1 + \exp\left(-\frac{(\eta - \tau)\beta + \tau - \mu}{\sigma}\right)\right) - \sum_{i=1}^{n} \delta_{li} \left(\frac{y_i - \mu}{\sigma}\right).
\]

(5.12)

### 5.2.3 Parameter Estimation

The first and second order partial derivatives of (5.12) with respect to \(\beta, \mu,\) and \(\sigma\) for ‘n’ observations are given in the Appendix A.13.

### 5.2.4 Fisher Information Matrix

It is the 3×3 symmetric matrix of the expectations of the negative of the second order partial derivatives of the log-likelihood function with respect to \(\beta, \mu,\) and \(\sigma\). The Fisher information matrix for \(n\) independent observations is
\[
F = \begin{bmatrix}
E\left[ -\frac{\partial^2 \ln L}{\partial \beta^2} \right] & E\left[ -\frac{\partial^2 \ln L}{\partial \beta \partial \mu} \right] & E\left[ -\frac{\partial^2 \ln L}{\partial \beta \partial \sigma} \right] \\
E\left[ -\frac{\partial^2 \ln L}{\partial \mu \partial \beta} \right] & E\left[ -\frac{\partial^2 \ln L}{\partial \mu^2} \right] & E\left[ -\frac{\partial^2 \ln L}{\partial \mu \partial \sigma} \right] \\
E\left[ -\frac{\partial^2 \ln L}{\partial \sigma \partial \beta} \right] & E\left[ -\frac{\partial^2 \ln L}{\partial \sigma \partial \mu} \right] & E\left[ -\frac{\partial^2 \ln L}{\partial \sigma^2} \right]
\end{bmatrix}
\] 
(5.13)

The calculation of the elements of the Fisher information matrix is shown in the Appendix A.13.

### 5.2.5 Optimal Test Plan

The optimal value of \( \tau \) is found by minimizing the asymptotic variance of the acceleration factor.

The NMinimize option of Mathematica 6.0 has been used to formulate the optimal plan.

### 5.3 OPTIMUM FAILURE-CENSORED STEP-STRESS PALT MODEL

This section provides theory for the optimum failure-censored step-stress PALT plan for the truncated logistic distribution using two different methods. In the first method, it has been assumed that number of failures is known. This method has been discussed in 5.3.1. In the second method, it has been assumed that proportion of units which will fail before censoring, is known. The later assumption is more realistic than the former one. The second method has been discussed in 5.3.2.
5.3.1 Type-II Censoring (Case I: Number of Failures Pre-Specified)

In the present section, we (Srivastava and Mittal (2010)) have formulated an optimum failure-censored step-stress PALT plan for the truncated logistic distribution.

Assumptions

1) The lifetime of an item tested at both use, and at accelerated condition follows the truncated logistic distribution.
2) The lifetimes of test items are independent and identically distributed random variables.
3) The number of failures is pre-specified.

We have estimated failure time of the last unit using

\[ E[X_{(r)}] = F^{-1}\left(\frac{r}{n+1}\right), \]  

(David and Nagaraja pg. 80 (2003))

where ‘r’ is total number of failures before censoring.

5.3.1.1 Test Procedure

1) All 'n' items are first tested at use condition.
2) If any item out of ‘n’ items do not fail at use condition by stress changing time ‘t’, then it is put on accelerated condition, and run until the predetermined number of failures ‘r’ is reached.

5.3.1.2 Log-Likelihood Function

The likelihood function L for n independent observations is
\begin{align}
L &= L(\beta, \mu, \sigma) = \prod_{i=1}^{n} L_i(\beta, \mu, \sigma; (y_i, \delta_{1i}, \delta_{2i})) \\
&= \prod_{i=1}^{n} \left( f_1(y_i) \right)^{\delta_{1i}} \left( f_2(y_i) \right)^{\delta_{2i}} \left( R(y_{(r)}) \right)^{\bar{\delta}_{1i} \bar{\delta}_{2i}} \\
&= \prod_{i=1}^{n} \left( \frac{\exp \left\{ -\frac{y_i - \mu}{\sigma} \right\}}{A \sigma \left( 1 + \exp \left\{ -\frac{y_i - \mu}{\sigma} \right\} \right)^2} \right)^{\delta_{1i}} \left( \frac{\beta \exp \left\{ -\frac{(y_i - \tau)\beta + \tau - \mu}{\sigma} \right\}}{A \sigma \left( 1 + \exp \left\{ -\frac{(y_i - \tau)\beta + \tau - \mu}{\sigma} \right\} \right)^2} \right)^{\delta_{2i}} \\
&\times \left( \frac{\exp \left\{ -\frac{(y_{(r)} - \tau)\beta + \tau - \mu}{\sigma} \right\}}{A \left( 1 + \exp \left\{ -\frac{(y_{(r)} - \tau)\beta + \tau - \mu}{\sigma} \right\} \right)} \right)^{\bar{\delta}_{1i} \bar{\delta}_{2i}}. \\
&\quad \text{(5.14)}
\end{align}

It is usually easier to maximize the natural logarithm of the likelihood function rather than the likelihood function itself.

Thus, the log-likelihood of (5.14) is

\begin{align}
\ln L &= \ln L(\beta, \mu, \sigma) = n_1 \ln \beta - r \ln \sigma + n \ln \left( 1 + \exp \left\{ -\frac{\mu}{\sigma} \right\} \right) - \sum_{i=1}^{n} \delta_{1i} \left( \frac{y_i - \mu}{\sigma} \right) \\
&\quad - \sum_{i=1}^{n} 2\delta_{1i} \ln \left( 1 + \exp \left\{ -\frac{y_i - \mu}{\sigma} \right\} \right) - \sum_{i=1}^{n} \delta_{2i} \left( \frac{(y_i - \tau)\beta + \tau - \mu}{\sigma} \right) \\
&\quad - \sum_{i=1}^{n} 2\delta_{2i} \ln \left( 1 + \exp \left\{ -\frac{(y_i - \tau)\beta + \tau - \mu}{\sigma} \right\} \right) - \sum_{i=1}^{n} \delta_{1i} \bar{\delta}_{2i} \left( \frac{(y_{(r)} - \tau)\beta + \tau - \mu}{\sigma} \right) \\
&\quad - \sum_{i=1}^{n} \bar{\delta}_{1i} \bar{\delta}_{2i} \ln \left( 1 + \exp \left\{ -\frac{(y_{(r)} - \tau)\beta + \tau - \mu}{\sigma} \right\} \right). \\
&\quad \text{(5.15)}
\end{align}
5.3.1.3 Parameter Estimation

The first and second order partial derivatives of (5.15) with respect to $\beta$, $\mu$, and $\sigma$ for all ‘n’ observations are given in the Appendix A.14.

5.3.1.4 Fisher Information Matrix

It is the $3 \times 3$ symmetric matrix of the expectations of the negative of the second order partial derivatives of the log-likelihood function with respect to $\beta$, $\mu$, and $\sigma$. The Fisher information matrix for $n$ independent observations is

$$
F = \begin{bmatrix}
E\left[-\frac{\partial^2 \ln L}{\partial \beta^2}\right] & E\left[-\frac{\partial^2 \ln L}{\partial \beta \partial \mu}\right] & E\left[-\frac{\partial^2 \ln L}{\partial \beta \partial \sigma}\right] \\
E\left[-\frac{\partial^2 \ln L}{\partial \mu \partial \beta}\right] & E\left[-\frac{\partial^2 \ln L}{\partial \mu^2}\right] & E\left[-\frac{\partial^2 \ln L}{\partial \mu \partial \sigma}\right] \\
E\left[-\frac{\partial^2 \ln L}{\partial \sigma \partial \beta}\right] & E\left[-\frac{\partial^2 \ln L}{\partial \sigma \partial \mu}\right] & E\left[-\frac{\partial^2 \ln L}{\partial \sigma^2}\right]
\end{bmatrix}.
$$

(5.16)

The calculation of the elements of the Fisher information matrix is shown in the Appendix A.14.

5.3.1.5 Optimal Test Plan

The optimal value of ‘$\tau$’ is found by minimizing the asymptotic variance of the acceleration factor.

The NMinimize option of Mathematica 6.0 has been used to formulate the optimal plan.
5.3.2 For Type-II Censoring (Case II: Proportion of Units Failing Before Censoring Pre-Specified)

Here, we (Srivastava and Mittal (2013b)) have designed optimal step-stress PALT using type-II censoring and truncated logistic distribution.

Assumptions

1) The lifetime of an item tested at both use, and at accelerated condition follows the truncated logistic distribution.
2) The lifetimes of test items are independent and identically distributed random variables.
3) The proportion of items failing before censoring is pre-specified.

5.3.2.1 Test Procedure

1) All ‘n’ items should be first run simultaneously at use condition.
2) When ‘nq_l’ items have failed at use condition, the surviving ‘n(1−q_l)’ items are put to test at accelerated condition.
3) The test is terminated when n(q_r−q_l) units have failed at accelerated condition.

5.3.2.2 Log-Likelihood Function

The likelihood function L for n independent observations is
\[ L = L(\beta, \mu, \sigma) = \prod_{i=1}^{n} L_i(\beta, \mu, \sigma; (u_{(i)}, \delta_{l_i}, \delta_{2_i})) \]

\[ = \prod_{i=1}^{n} \left( f_1(u_{(i)})^\delta_{l_i} (f_2(u_{(i)}))^{\delta_{2_i}} (R(u_{(nq,i)}))^\delta_{0_1} \delta_{0_2} \right) \]

\[ = \prod_{i=1}^{n} \left( \exp \left\{ -\frac{u_{(i)} - \mu}{\sigma} \right\} \right)^{\delta_{l_i}} \]

\[ \frac{A\sigma \left( 1 + \exp \left\{ -\frac{u_{(i)} - \mu}{\sigma} \right\} \right)^2}{\beta \exp \left\{ -\frac{u_{(nq,i)} + (u_{(i)} - u_{(nq,i)})\beta - \mu}{\sigma} \right\}} \]

\[ \left( \frac{\exp \left\{ -\frac{u_{(nq,i)} + (u_{(nq,i)} - u_{(nq,i)})\beta - \mu}{\sigma} \right\}}{A \left( 1 + \exp \left\{ -\frac{u_{(nq,i)} + (u_{(nq,i)} - u_{(nq,i)})\beta - \mu}{\sigma} \right\} \right)^2} \right)^{\delta_{2_i}} \]

\[ \left( \frac{\exp \left\{ -\frac{u_{(nq,i)} + (u_{(nq,i)} - u_{(nq,i)})\beta - \mu}{\sigma} \right\}}{A \left( 1 + \exp \left\{ -\frac{u_{(nq,i)} + (u_{(nq,i)} - u_{(nq,i)})\beta - \mu}{\sigma} \right\} \right)^2} \right)^{\delta_{0_1} \delta_{0_2}} \]

\[ . \quad (5.17) \]

It is usually easier to maximize the natural logarithm of the likelihood function rather than the likelihood function itself.

Thus, the log-likelihood of (5.17) is
\[
\ln L = \ln L(\beta, \mu, \sigma) = (nq_i - nq_l) \ln \beta - nq_i \ln \sigma + n \ln \left(1 + \exp\left(-\frac{\mu}{\sigma}\right)\right)
\]

\[
- \sum_{i=1}^{n} 2\delta_{i1} \ln \left(1 + \exp\left(-\frac{u_{(i)} - \mu}{\sigma}\right)\right) - \sum_{i=1}^{n} \delta_{2i} \left(\frac{u_{(nq_i)} + (u_{(i)} - u_{(nq_l)})\beta - \mu}{\sigma}\right)
\]

\[
- \sum_{i=1}^{n} 2\delta_{2i} \ln \left(1 + \exp\left(-\frac{u_{(nq_i)} + (u_{(i)} - u_{(nq_l)})\beta - \mu}{\sigma}\right)\right) - \sum_{i=1}^{n} \delta_{1i} \left(\frac{u_{(i)} - \mu}{\sigma}\right)
\]

\[
- \sum_{i=1}^{n} \delta_{1i} \delta_{2i} \left(\frac{u_{(nq_i)} + (u_{(nq_l)} - u_{(nq_i)})\beta - \mu}{\sigma}\right)
\]

\[
- \sum_{i=1}^{n} \delta_{1i} \delta_{2i} \ln \left(1 + \exp\left(-\frac{u_{(nq_i)} + (u_{(nq_l)} - u_{(nq_i)})\beta - \mu}{\sigma}\right)\right).
\]

**5.3.2.3 Parameter Estimation**

The first and second order partial derivatives of (5.18) with respect to \(\beta, \mu,\) and \(\sigma\) ‘n’ observations are given in the Appendix A.15.

**5.3.2.4 Fisher Information Matrix**

It is the 3×3 symmetric matrix of the expectations of the negative of the second order partial derivatives of the log-likelihood function with respect to \(\beta, \mu,\) and \(\sigma\).

The Fisher information matrix for \(n\) independent observations is

\[
F = \begin{bmatrix}
E\left[ -\frac{\partial^2 \ln L}{\partial \beta^2} \right] & E\left[ -\frac{\partial^2 \ln L}{\partial \beta \partial \mu} \right] & E\left[ -\frac{\partial^2 \ln L}{\partial \beta \partial \sigma} \right] \\
E\left[ -\frac{\partial^2 \ln L}{\partial \mu \partial \beta} \right] & E\left[ -\frac{\partial^2 \ln L}{\partial \mu^2} \right] & E\left[ -\frac{\partial^2 \ln L}{\partial \mu \partial \sigma} \right] \\
E\left[ -\frac{\partial^2 \ln L}{\partial \sigma \partial \beta} \right] & E\left[ -\frac{\partial^2 \ln L}{\partial \sigma \partial \mu} \right] & E\left[ -\frac{\partial^2 \ln L}{\partial \sigma^2} \right]
\end{bmatrix}
\]
The calculation of the elements of the Fisher information matrix is shown in the Appendix A.15.

5.3.2.5 **Optimal Test Plan**

The optimal value of ‘$q_1$’ is found by using the D-optimality criterion which consists in minimizing the reciprocal of the determinant of the Fisher information matrix.

The NMinimize option of *Mathematica 6.0* has been used to formulate the optimal plan.

5.4 **CONFIDENCE INTERVAL**

The MLEs $\hat{\beta}$, $\hat{\mu}$, and $\hat{\sigma}$ are approximately normally distributed in large samples, so $(\hat{\mu}, \hat{\sigma}, \hat{\beta}) \sim N((\mu, \sigma, \beta), F^{-1})$.

The two-sided $100(1 - \alpha')\%$ approximate confidence interval for the parameter $\mu$ is given by

$$\hat{\mu} \pm z_{\alpha'/2} \sqrt{\text{var}(\hat{\mu})},$$

where $Z_{\alpha'/2}$ is the $(1 - \alpha'/2)\text{th}$ quantile of the standard normal distribution, and $\sqrt{\text{var}(\hat{\mu})}$ is obtained by taking the square root of the first diagonal element of $F^{-1}$.

Similarly, the two-sided $100(1 - \alpha')\%$ approximate confidence intervals for the parameters $\sigma$, and acceleration factor $\beta$, can be obtained.

The main disadvantage of the approximate $100(1 - \alpha')\%$ confidence interval is that it may yield negative lower bounds though the parameter takes only positive values. Since parameters $\sigma$ and $\beta$ take positive values, therefore their respective
approximate two-sided $100(1 - \alpha')\%$ confidence intervals as suggested by Meeker and Escobar (1998) are:

$\left( \hat{\sigma} \left[ -\frac{z_{\alpha'/2}}{\sqrt{\text{var}(\hat{\sigma})}} / \hat{\sigma} \right], \hat{\sigma} \left[ \frac{z_{\alpha'/2}}{\sqrt{\text{var}(\hat{\sigma})}} / \hat{\sigma} \right] \right)$, and

$\left( \hat{\beta} \left[ -\frac{z_{\alpha'/2}}{\sqrt{\text{var}(\hat{\beta})}} / \hat{\beta} \right], \hat{\beta} \left[ \frac{z_{\alpha'/2}}{\sqrt{\text{var}(\hat{\beta})}} / \hat{\beta} \right] \right)$.

### 5.5 ILLUSTRATIVE EXAMPLES

#### 5.5.1 Numerical Example: Type-I Censoring

The following hypothetical data set has been considered.

$n = 35, \beta = 1.1, \mu = 0.5, \sigma = 3, \eta = 6.8$

**5.5.1.1 Optimal Plan**

For $n = 35, \beta = 1.1, \mu = 0.5, \sigma = 3,$ and $\eta = 6.8$, the optimal value of $\tau'$ is obtained by using D-optimality criterion. It is obtained as $\tau^* = 4.41263$ using NMinimize option of *Mathematica 6*.

**5.5.1.2 Simulated Data**

The data in Table 5.1 gives 35 simulated observations based on data $n = 35, \beta = 1.1, \mu = 0.5, \sigma = 3, \eta = 6.8,$ and $\tau^* = 4.41263$. Thus, the total number of units tested at use condition is $n_u = 17$, and at accelerated condition it is $n_a = 9$. 
Table 5.1: Simulated failure times ($n = 35, \beta = 1.1, \mu = 0.5, \sigma = 3, \eta = 6.8, \tau^* = 4.41263, \text{ and } n_c = 9$)

<table>
<thead>
<tr>
<th>Use Condition</th>
<th>Failure Times</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step-Stress</td>
<td>0.941951, 2.176240, 2.207960, 0.244373, 2.128270, 0.403601, 1.763830, 1.385200, 2.158640, 3.006270, 1.154010, 1.010620, 4.407310, 3.481320, 3.896940, 3.004100, 0.555658</td>
</tr>
<tr>
<td>Accelerated Condition</td>
<td>5.593110, 6.753510, 5.324340, 6.525530, 6.051710, 4.764450, 5.809210, 4.949270, 5.633080</td>
</tr>
</tbody>
</table>

5.5.1.3 MLEs of the Design Parameters

The MLEs of the model parameters $\mu$, $\sigma$, and the acceleration factor $\beta$ obtained using simulated data given in 5.5.1.2 are $\hat{\mu} = -0.957196$, $\hat{\sigma} = 4.41033$, and $\hat{\beta} = 1.46292$.

These estimates are obtained using the NMaximize option of Mathematica 6.0.

5.5.1.4 Confidence Intervals

$100(1 - \alpha')\%$ confidence intervals for the parameters are obtained using the inverse of the observed Fisher information matrix $\hat{F}^{-1}$, and it is given by

$$\hat{F}^{-1} = \begin{pmatrix} 0.898411 & 10.0328 & 3.06744 \\ 10.0328 & 181.509 & 49.9538 \\ 3.06744 & 49.9538 & 14.5182 \end{pmatrix}.$$
The observed value of $F^{-1}$, that is, $\hat{F}^{-1}$ is determined by substituting the estimated parameters $\hat{\beta}$, $\hat{\mu}$, and $\hat{\sigma}$ for the true parameters in the asymptotic variance-covariance matrix. The square root of the diagonal elements of $\hat{F}^{-1}$ gives the standard error of an estimator. Thus, the 95% approximate confidence intervals for $\beta$, $\mu$, and $\sigma$, respectively, are

1.000000 < $\beta$ ≤ 5.20877,

$-27.36330 \leq \mu \leq 25.4489$, and

0.811086 ≤ $\sigma$ ≤ 23.9814.

Because, the range of parameter ‘$\beta$’ is greater than one. Therefore, lower limit of its confidence interval cannot be less than one. Thus, we replace the lower limit by one whenever the lower limit comes out to be less than one.

### 5.5.1.5 Graphical Goodness of Fit

Figure 5.1 shows the associated step-stress truncated logistic probability plot at normal use and at accelerated condition with type-I censoring. The failure times at normal use and at accelerated condition given in Table 5.1 are both arranged in increasing order, and are ranked from $i = 1, 2, \ldots, k, \ldots, n$. The plot is obtained by taking these ranked failure times on X-axis, and their corresponding ordinate values, viz,

$$
\mu - \sigma \ln \left( \frac{A_i}{n+1} + \frac{1}{1+\exp\{\mu/\sigma\}} \right)^{-1} - 1
$$
at normal use, and

$$
\tau^* - \frac{1}{\beta} \left( \tau^* - \mu + \sigma \ln \left( \frac{A_i}{n+1} + \frac{1}{1+\exp\{\mu/\sigma\}} \right)^{-1} - 1 \right)
$$
at accelerated condition,
These ordinate values are obtained using (5.6), and (5.7), respectively.

![Truncated logistic distribution probability plot](image)

**Figure 5.1: Truncated logistic distribution probability plot for the simulated data under type-I censoring**

The plotted points tend to follow straight line, which is substantiated by fitting straight line to these points resulting in high value of coefficient of determination $r^2 = 0.990$.
Therefore, for the truncated logistic distribution for the simulated data under type-I censoring appears to describe the data adequately.
5.5.1.6 Coverage Probabilities

Coverage probabilities have been obtained to see if the normal approximation holds even for small as well as moderate n values. The simulation study shows that the approximate confidence intervals possess the coverage probabilities closer to the nominal levels for the parameters $\beta$, and $\sigma$. However, this is not so for the parameter $\mu$. This shows that normal approximation can be used even for small, and moderate samples for $\hat{\beta}$, and $\hat{\sigma}$ but not for $\hat{\mu}$. To compute confidence intervals for $\mu$ when n is small, other methods like bootstrap percentile method may be used (Effron (1982)). Table 5.2 gives the estimated coverage probabilities of the two-sided 95 % approximate confidence intervals for $\hat{\mu}$, $\hat{\sigma}$, and $\hat{\beta}$, for selected values of n, and given values of $\beta=1.1$, $\mu=0.5$, $\sigma=3$, and $\eta=6.8$. These results are based on 1000 pseudorandom type-I censored samples of size n from a step-stress truncated logistic model. The simulation results show that even for small values of n the normal approximation for the pdfs of $\hat{\sigma}$, and $\hat{\beta}$ perform well.
Table 5.2: Estimated coverage probabilities (in %) of the approximate 95% confidence interval based on 1000 simulations of type-I censored samples of size n with (β = 1.1, μ = 0.5, σ = 3, and η = 6.8)

<table>
<thead>
<tr>
<th>n</th>
<th>β</th>
<th>μ</th>
<th>σ</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>84.8</td>
<td>63.5</td>
<td>91.3</td>
</tr>
<tr>
<td>15</td>
<td>91.0</td>
<td>55.6</td>
<td>82.9</td>
</tr>
<tr>
<td>20</td>
<td>96.0</td>
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<td>97.3</td>
</tr>
<tr>
<td>25</td>
<td>92.8</td>
<td>67.7</td>
<td>91.6</td>
</tr>
<tr>
<td>30</td>
<td>96.6</td>
<td>70.7</td>
<td>97.1</td>
</tr>
<tr>
<td>35</td>
<td>95.6</td>
<td>73.2</td>
<td>99.2</td>
</tr>
</tbody>
</table>

5.5.2 Numerical Example: Type-II Censoring (Case I: Number of Failures Pre-Specified)

The following hypothetical data set has been considered.

n = 35, β = 1.1, μ = 0.5, σ = 3, y(r) = 7

5.5.2.1 Optimal Plan

For the data set n = 35, β = 1.1, μ = 0.5, σ = 3,

and y(r) = 7, the optimal value of ‘τ’ is obtained using D-optimality criterion. It is obtained as τ∗ = 4.49458, using NMinimize option of Mathematica 6.0.
5.5.2.2 Simulated Data

The data in Table 5.3 gives 35 simulated observations based on data \( n = 35, \beta = 1.1, \mu = 0.5, \sigma = 3, y_r = 7, \) and \( \tau^* = 4.49458. \) Thus, the total number of units tested at use condition is \( n_u = 20, \) and at accelerated condition it is \( n_a = 8. \)

Table 5.3: Simulated failure times \( (n = 35, \beta = 1.1, \mu = 0.5, \sigma = 3, y_r = 7, \tau^* = 4.49458, \) and \( n_c = 7) \)

<table>
<thead>
<tr>
<th>Step-Stress</th>
<th>Failure Times</th>
</tr>
</thead>
<tbody>
<tr>
<td>Use Condition</td>
<td>1.897660, 4.290860, 2.933900, 0.521004, 0.326628, 4.355820, 1.680090, 1.857180, 2.703160, 3.812230, 0.879353, 1.582120, 4.063950, 3.582280, 0.711965, 1.669980, 0.104429, 2.914920, 3.148060, 1.137920</td>
</tr>
<tr>
<td>Accelerated Condition</td>
<td>5.654130, 6.945540, 5.767420, 4.560850, 5.697860, 5.122740, 5.980470, 5.892920</td>
</tr>
</tbody>
</table>

5.5.2.3 MLEs of the Design Parameters

The MLEs of the model parameters \( \mu, \sigma, \) and the acceleration factor \( \beta \) obtained using simulated data given in 5.5.2.2 are \( \hat{\mu} = 1.57225, \hat{\sigma} = 2.97842, \) and \( \hat{\beta} = 1.10622. \)

These estimates are obtained using the NMaximize option of Mathematica 6.0.
5.5.2.4 Confidence Intervals

100(1−α′)% confidence intervals for the parameters are obtained using the inverse of the observed Fisher information matrix \( \hat{F}^{-1} \), and it is given by

\[
\hat{F}^{-1} = \begin{bmatrix}
0.378404 & 1.57076 & 0.806715 \\
1.570760 & 14.5333 & 5.858750 \\
0.806715 & 5.85875 & 2.757020
\end{bmatrix}.
\]

The observed value of \( F^{-1} \), that is, \( \hat{F}^{-1} \) is determined by substituting the estimated parameters \( \hat{\beta}, \hat{\mu}, \) and \( \hat{\sigma} \) for the true parameters in the asymptotic variance-covariance matrix. The square root of the diagonal elements of \( \hat{F}^{-1} \) gives the standard error of an estimator. Thus, the 95% approximate confidence intervals for \( \beta, \mu, \) and \( \sigma \), respectively, are

\[
1.000000 < \beta \leq 3.28992,
\]

\[
-5.899770 \leq \mu \leq 9.04428,
\]

and

\[
0.998718 \leq \sigma \leq 8.88235.
\]

Since the range of parameter ‘\( \beta \)’ is greater than one, therefore the lower limit of its confidence interval cannot be less than one, and so is replaced by one whenever it comes out to be so.

5.5.2.5 Graphical Goodness of Fit

Figure 5.2 shows the associated step-stress truncated logistic probability plot at normal use and at accelerated condition with type-II censoring. This Figure is being added to verify the results obtained, through graphical approach also.

The failure times at normal use and at accelerated condition given in Table 5.3 are both arranged in increasing order, and are ranked from \( i = 1, 2, \ldots, k, \ldots, n \). The plot
is obtained by taking these ranked failure times on X-axis, and their corresponding ordinate values, viz,

\[ \mu - \sigma \ln \left( \frac{A_i}{n+1} + \frac{1}{1 + \exp \{ \mu / \sigma \}} \right)^{-1} - 1 \]

at normal use, and

\[ \tau^* - \frac{1}{\beta} \left( \tau^* - \mu + \sigma \ln \left( \frac{A_i}{n+1} + \frac{1}{1 + \exp \{ \mu / \sigma \}} \right)^{-1} - 1 \right) \]

at accelerated condition,

where \( \tau^* = 4.49458 \) on Y-axis Grosh (1989).

These ordinate values are obtained using (5.6), and (5.7), respectively.

Figure 5.2: Truncated logistic distribution probability plot for the simulated data under type-II censoring (case I: number of failures pre-specified)
The plotted points tend to follow straight line, which is substantiated by fitting straight line to these points resulting in high value of coefficient of determination \( r^2 = 0.982 \).

Therefore, the truncated logistic distribution for the simulated data under type-II censoring when number of failures is pre-specified appears to describe the data adequately.

5.5.2.6 Coverage Probabilities

Coverage probabilities have been obtained to see if the normal approximation holds even for small as well as moderate \( n \) values.

The simulation study shows that the approximate confidence intervals possess the coverage probabilities closer to the nominal levels for the parameters \( \beta \), and \( \sigma \). However, this is not so for the parameter \( \mu \). This shows that normal approximation can be used even for small, and moderate samples for \( \hat{\beta} \), and \( \hat{\sigma} \) but not for \( \hat{\mu} \). To compute confidence intervals for \( \mu \) when \( n \) is small, other methods like bootstrap percentile method may be used (Effron (1982)).

Table 5.4 gives the estimated coverage probabilities of the two-sided 95 % approximate confidence intervals for \( \hat{\mu}, \hat{\sigma} \), and \( \hat{\beta} \), for selected values of \( n \), \( \beta = 1.1, \mu = 0.5, \sigma = 3 \), and \( y_{(r)} = 7 \). These results are based on 1000 pseudorandom type-II censored samples of size \( n \) from a step-stress truncated logistic model. The simulation results show that even for small values of \( n \) the normal approximation for the pdfs of \( \hat{\sigma} \), and \( \hat{\beta} \) perform well.
Table 5.4: Estimated coverage probabilities (in %) of the approximate 95% confidence interval based on 1000 simulations of type-II censored samples case I of size n with $(\beta = 1.1, \mu = 0.5, \sigma = 3, \text{ and } y_{(r)} = 7)$

<table>
<thead>
<tr>
<th>n</th>
<th>$\beta$</th>
<th>$\mu$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>85.1</td>
<td>61.8</td>
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<td>88.2</td>
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<td>89.4</td>
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</table>

5.5.3 Numerical Example: Type-II Censoring (Case II: Degree of Censoring Pre-Specified)

The following hypothetical data set has been considered.

$n = 36, \beta = 3.5, \mu = 3, \sigma = 2, q_r = 0.8.$

5.5.3.1 Optimal Plan

Assuming $n = 36, \beta = 3.5, \mu = 3, \sigma = 2, \text{ and } q_r = 0.8,$ the optimal value of ‘$q_l$’ is obtained by D-optimality criterion using NMinimize option of *Mathematica 6.0*. It is obtained as $q_l^* = 0.537308.$

Table 5.5 gives the optimal value of ‘$q_l$’ for various sets of parametric values.
Table 5.5: Optimum failure-censored step-stress PALT for n = 36

<table>
<thead>
<tr>
<th>$q_r$</th>
<th>$\beta$</th>
<th>$\mu$</th>
<th>$\sigma$</th>
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<td>3.0</td>
<td>1.0</td>
<td>0.476486</td>
</tr>
</tbody>
</table>
5.5.3.2 Simulated Data

The data in Table 5.6 gives 36 simulated observations based on data $n = 36$, $\beta = 3.5$, $\mu = 3$, $\sigma = 2$, $q_r = 0.8$, and $q^*_l = 0.537308$. Thus, the total number of units tested at use condition is $nq_l = 19$, and at accelerated condition it is $nq_r - nq_l = 10$. 

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
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<td>0.537065</td>
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<td>3.0</td>
<td>2.5</td>
<td>0.562226</td>
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</tbody>
</table>
Table 5.6: Simulated failure times \( (n = 36, \beta = 3.5, \mu = 3, \sigma = 2, q_r = 0.8, q_l^* = 0.537308 \text{ and } n_c = 7) \)

<table>
<thead>
<tr>
<th>Step-Stress</th>
<th>Failure Times</th>
</tr>
</thead>
<tbody>
<tr>
<td>Use Condition</td>
<td>0.825647, 1.274270, 3.522210, 1.689260, 3.634120, 2.678840, 0.509556, 0.876301, 0.928494, 3.127680, 2.878130, 0.384488, 1.666690, 1.443160, 3.433290, 1.752420, 3.139340, 1.500500, 3.893300</td>
</tr>
<tr>
<td>Accelerated Condition</td>
<td>3.929780, 4.483510, 3.982710, 4.721040, 5.092100, 4.265770, 4.561690, 4.841640, 4.271310, 4.049440</td>
</tr>
</tbody>
</table>

5.5.3.3 MLEs of the Design Parameters

The MLEs of the model parameters \( \mu, \sigma, \) and the acceleration factor \( \beta \) obtained using simulated data given 5.5.3.2 are \( \hat{\mu} = 2.32716, \hat{\sigma} = 2.37193, \) and \( \hat{\beta} = 2.28489. \)

These estimates are obtained using the NMaximize option of Mathematica 6.0.

5.5.3.4 Confidence Intervals

100(1 – \( \alpha' \))% confidence intervals for the parameters are obtained using the inverse of the observed Fisher information matrix \( \hat{F}^{-1} \), and it is given by

\[
\hat{F}^{-1} = \begin{pmatrix}
1.34435 & -1.34934 & 1.09024 \\
-1.34934 & 3.93506 & -2.04478 \\
1.09024 & -2.04478 & 1.42118
\end{pmatrix}.
\]
The observed value of $F^{-1}$, that is, $\hat{F}^{-1}$ is determined by substituting the estimated parameters $\hat{\beta}$, $\hat{\mu}$, and $\hat{\sigma}$ for the true parameters in the asymptotic variance-covariance matrix. The square root of the diagonal elements of $\hat{F}^{-1}$ gives the standard error of an estimator. Thus, the 95% approximate confidence intervals for $\beta$, $\mu$, and $\sigma$, respectively, are:

$1.000000 < \beta \leq 6.17751,$

$-1.560890 \leq \mu \leq 6.21521,$

$0.885686 \leq \sigma \leq 6.35219.$

Since the range of parameter ‘$\beta$’ is greater than one, therefore the lower limit of its confidence interval cannot be less than one, and so is replaced by one whenever it comes out to be so.

5.6 SENSITIVITY ANALYSIS

5.6.1 Sensitivity Analysis: Type-I Censoring and Type-II Censoring
(Case I: Number of Failures Pre-Specified)

To use an optimum test plan, one must have information about acceleration factor, $\beta$, and parameters of the model, $\mu$, and $\sigma$, which are usually unknown. Therefore they have to be approximated from experience, similar data, or preliminary tests. Incorrect choice of these gives a non-optimal test plan. With the true values of parameters, we have studied the effects of incorrect pre-estimates of $\beta$, $\mu$, and $\sigma$ in terms of the relative increase of asymptotic variance for truncated logistic distribution under type-I censoring and type-II censoring (case I: number of failures pre-specified) that are presented in Table 5.7 and Table 5.8, respectively.
RV = \left| \frac{Asvar^* - Asvar^0}{Asvar^*} \right| \times 100,

where Asvar^* is the asymptotic variance for the plan obtained with the correctly specified values, and Asvar^0 is the asymptotic variance for the plan obtained with mis-specified values. We have found that if they are not too far from the true values the increase is small, as shown in Table 5.7 and Table 5.8. The result of Sensitivity analysis shows that the proposed plan is robust.
Table 5.7: Sensitivity analysis for step-stress PALT plan with type-I censoring

\((n = 35, \eta = 6.8, \tau^* = 4.41263, \text{ and } \text{AsVar}^* = 0.388484)\)

<table>
<thead>
<tr>
<th>Percentage Change</th>
<th>(\beta)</th>
<th>(\mu)</th>
<th>(\sigma)</th>
<th>RV</th>
<th>Incorrect (\tau^*)</th>
<th>Asvar(^0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1%</td>
<td>1.089</td>
<td>0.495</td>
<td>2.97</td>
<td>2.38054</td>
<td>4.38924</td>
<td>0.379236</td>
</tr>
<tr>
<td>+1%</td>
<td>1.111</td>
<td>0.505</td>
<td>3.03</td>
<td>2.42030</td>
<td>4.43563</td>
<td>0.397887</td>
</tr>
<tr>
<td>-2%</td>
<td>1.078</td>
<td>0.49</td>
<td>2.94</td>
<td>4.72143</td>
<td>4.36546</td>
<td>0.370142</td>
</tr>
<tr>
<td>+2%</td>
<td>1.122</td>
<td>0.510</td>
<td>3.06</td>
<td>4.88102</td>
<td>4.45826</td>
<td>0.407446</td>
</tr>
<tr>
<td>-3%</td>
<td>1.067</td>
<td>0.485</td>
<td>2.91</td>
<td>7.02320</td>
<td>4.34128</td>
<td>0.361200</td>
</tr>
<tr>
<td>+3%</td>
<td>1.133</td>
<td>0.515</td>
<td>3.09</td>
<td>7.38524</td>
<td>4.48052</td>
<td>0.417164</td>
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<tr>
<td>-4%</td>
<td>1.056</td>
<td>0.480</td>
<td>2.88</td>
<td>9.28661</td>
<td>4.31670</td>
<td>0.352407</td>
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<tr>
<td>+4%</td>
<td>1.144</td>
<td>0.520</td>
<td>3.12</td>
<td>9.92525</td>
<td>4.50242</td>
<td>0.427042</td>
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<tr>
<td>-5%</td>
<td>1.045</td>
<td>0.475</td>
<td>2.85</td>
<td>11.5117</td>
<td>4.29172</td>
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<tr>
<td>+5%</td>
<td>1.155</td>
<td>0.525</td>
<td>3.15</td>
<td>12.5097</td>
<td>4.52395</td>
<td>0.437082</td>
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</table>
Table 5.8: Sensitivity analysis for step-stress PALT plan with type-II censoring (case I) \( (n = 35, Y_{(r)} = 7, \tau^* = 4.49458, \text{ and } \text{AsVar}^* = 0.381725) \)

<table>
<thead>
<tr>
<th>Percentage Change</th>
<th>( \beta )</th>
<th>( \mu )</th>
<th>( \sigma )</th>
<th>RV</th>
<th>Incorrect ( \tau^* )</th>
<th>Asvar$^0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1%</td>
<td>1.089</td>
<td>0.495</td>
<td>2.97</td>
<td>2.36217</td>
<td>4.46998</td>
<td>0.372708</td>
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<tr>
<td>+1%</td>
<td>1.111</td>
<td>0.505</td>
<td>3.03</td>
<td>2.40094</td>
<td>4.51878</td>
<td>0.390890</td>
</tr>
<tr>
<td>-2%</td>
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<td>0.490</td>
<td>2.94</td>
<td>4.68531</td>
<td>4.44498</td>
<td>0.363840</td>
</tr>
<tr>
<td>+2%</td>
<td>1.122</td>
<td>0.510</td>
<td>3.06</td>
<td>4.84171</td>
<td>4.54260</td>
<td>0.400207</td>
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<tr>
<td>-3%</td>
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<td>0.485</td>
<td>2.91</td>
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<td>4.41958</td>
<td>0.355118</td>
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<tr>
<td>+3%</td>
<td>1.133</td>
<td>0.515</td>
<td>3.09</td>
<td>7.32229</td>
<td>4.56603</td>
<td>0.409676</td>
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<tr>
<td>-4%</td>
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<td>0.480</td>
<td>2.88</td>
<td>9.21737</td>
<td>4.39376</td>
<td>0.346540</td>
</tr>
<tr>
<td>+4%</td>
<td>1.144</td>
<td>0.520</td>
<td>3.12</td>
<td>9.84347</td>
<td>4.58909</td>
<td>0.419300</td>
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<tr>
<td>-5%</td>
<td>1.045</td>
<td>0.475</td>
<td>2.85</td>
<td>11.4268</td>
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<td>+5%</td>
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<td>12.4055</td>
<td>4.61178</td>
<td>0.429080</td>
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5.6.2 Sensitivity Analysis: Type-II Censoring (Case II: Proportion of Items Failing Before Failure Pre-Specified)

To use an optimum test plan, one needs estimates of the design parameters \( \beta, \mu, \) and \( \sigma \). These estimates sometimes may significantly affect the values of the resulting decision variables; therefore, their incorrect choice may give a poor
estimate of the quantile at a design constant stress. Hence, it is important to conduct sensitivity analysis to evaluate the robustness of the resulting PALT plan. The percentage deviations (PDs) of the optimal settings are measured by

\[
PD = \left( \frac{Z^{**} - Z^*}{Z^*} \right) \times 100,
\]

where \( Z^* \) is the optimal asymptotic variance obtained with the given design parameters, and \( Z^{**} \) is the one obtained when a parameter is mis-specified. Table 5.9 shows the optimal test plan for various deviations from the design parameters. The results show that the optimal setting of \( Z \) is robust to the deviations of those baseline parameters.

Table 5.9: Sensitivity analysis for step-stress PALT plan with type-II censoring (case II) (\( n = 36, q_f = 0.8, \) and \( q_l^* = 0.537308 \))

<table>
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<th>Parameter</th>
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<th>PD</th>
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<td>( \beta )</td>
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<td>0.0159323</td>
</tr>
<tr>
<td>( \beta )</td>
<td>-5%</td>
<td>0.537228</td>
<td>0.0149052</td>
</tr>
<tr>
<td>( \mu )</td>
<td>+5%</td>
<td>0.534043</td>
<td>0.607694</td>
</tr>
<tr>
<td>( \mu )</td>
<td>-5%</td>
<td>0.540533</td>
<td>0.600216</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>+5%</td>
<td>0.540506</td>
<td>0.595246</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>-5%</td>
<td>0.534825</td>
<td>0.462128</td>
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</table>
5.7 COMPARATIVE STUDY FOR TYPE-II CENSORING (CASE II: PROPORTION OF ITEMS FAILING BEFORE CENSORING PRE-SPECIFIED)

In this section, the proposed step-stress PALT model have been compared with the ones designed by Abdel-Ghaly, Attia, and Abdel-Ghani (2002) in terms of log-likelihood functions using the hypothetical failure time data set under step-stress PALT with type-II censoring (case II) given in Table 5.6.

Table 5.10: Comparative study of step-stress PALT models with type-II censoring (case II)

<table>
<thead>
<tr>
<th>PALT Model</th>
<th>Log-Likelihood Function</th>
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<tr>
<td>Proposed Model</td>
<td>−62.3714</td>
</tr>
<tr>
<td>Abdel-Ghaly, Attia, and Abdel-Ghani (2002) Model</td>
<td>−63.1796</td>
</tr>
</tbody>
</table>

Table 5.10 shows that the proposed model performs better than the other failure – censored step-stress PALT models existing in the literature for the given data set.

5.8 CONCLUDING REMARKS

In this chapter, we have formulated optimum step-stress PALT for the truncated logistic distribution with type-I and type-II censoring schemes. The truncated logistic life distribution truncated at point zero, has the failure rate that is increasing, and is more realistically bounded below, and above by a non zero finite
quantity. The procedures developed have been explained using numerical examples.