Chapter-5

High Accuracy Cubic Spline Finite Difference Approximation for the Solution of One-space Dimensional Non-linear Wave Equations*

5.1. Introduction

We consider the one-space dimensional non-linear wave equation

\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + g\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right), \quad 0 < x < 1, \quad t > 0
\]  

(5.1.1)

with the following initial conditions

\[
u(x, 0) = \phi(x), \quad \frac{\partial u}{\partial t}(x, 0) = \psi(x), \quad 0 \leq x \leq 1
\]  

(5.1.2)

and the boundary conditions

\[
u(0, t) = a_0(t), \quad u(1, t) = b_0(t), \quad t \geq 0.
\]  

(5.1.3)

We assume that the conditions (5.1.2) and (5.1.3) are given with sufficient smoothness to maintain the order of accuracy in the numerical method under consideration.

In this chapter, we derive a new three-level compact finite difference method of \(O(k^2 + k^2 h^2 + h^4)\) based on cubic spline approximation for the numerical solution of the

differential equation (5.1.1) subject to the initial conditions (5.1.2) and the boundary conditions (5.1.3).

In the literature several numerical schemes have been developed for the solution of the non-linear differential equation (5.1.1) [101,107,109,122,125]. First, Bickley [19] and Fyfe [53] have discussed the second order accurate cubic spline method for the solution of linear two point boundary value problems. Jain et al. [65] have derived fourth order cubic spline method for solving the non-linear two point boundary value problems with significant first derivative terms. Recently, Kadalbajoo et al. [73,76], Khan et al. [83], and Kumar et al. [88] have studied on the use of cubic spline technique for solving singular two point boundary value problems.

During the last three decades, there has been much effort to develop stable numerical methods based on cubic spline approximations methods for the solution of time-dependent partial differential equations. In 1973, Papamichael and Whiteman [141] have used a cubic spline technique of lower order accuracy to solve one-dimensional heat conduction equation. Then using the same technique Raggett and Wilson [145] have solved one-dimensional wave equation. Later, Fleck Jr [52] has proposed a cubic spline method for solving the wave equation of non-linear optics. In recent years, Rashidinia et al. [148], and Ding et al. [43] have discussed spline methods for the solution of linear hyperbolic equations with first derivative terms. More recently, Mohanty et al. [126] have developed high accuracy cubic spline method for the solution of one-space dimensional quasi-linear parabolic equations. In this chapter, using nine grid points (Fig. 5.1.1), we discuss a new three level implicit cubic spline finite difference method of accuracy two in time and four in space for the solution of one-space dimensional non-linear wave equation. In this method we require only three evaluation of function $g$. In next section, we propose the finite difference method based on cubic spline approximation. In section 5.3, we derive in detail the said method. In section 5.4, we discuss the application of the proposed method to telegraphic equation and wave equation in polar coordinates, and discuss their stability analysis. Difficulties were experienced in the past for the high order cubic spline solution of wave equation in polar coordinates. The solution usually deteriorates in the vicinity of the singularity. In this section, we modify our technique in such a way that the solution retains its order and accuracy everywhere in the solution region. In section 5.5, we have shown the applicability of the
method to the one space dimensional Telegraph equation and discussed the unconditional stability of the method. In section 5.6, we discuss the higher order approximation at first time level, in order to compute the proposed numerical method of same accuracy, and compare the numerical results of proposed high accuracy cubic spline finite difference method with the corresponding second order accuracy cubic spline method. Concluding remarks are given in section 5.7.

\[
\begin{array}{c|c|c}
(l-1,j+1) & (l, j+1) & (l+1,j+1) \\
\uparrow & k & \downarrow \\
(l,j) & \leftarrow \quad \rightarrow \\
(l-1,j-1) & (l, j-1) & (l+1,j-1)
\end{array}
\]

Fig.5.1.1: (Schematic representation of three-level implicit scheme)

5.2. The finite difference method based on cubic spline approximation

The solution domain \([0,1] \times [t>0]\) is divided into \((N+1) \times J\) mesh with the spatial step size \(h = 1/(N+1)\) in \(x\)-direction and the time step size \(k>0\) in \(t\)-direction respectively, where \(N\) and \(J\) are positive integers. The mesh ratio parameter is given by \(\lambda = (k/h) > 0\). Grid points are defined by \((x_l,t_j) = (lh, jk)\), \(l = 0,1,2,...,N+1\) and \(j = 0,1,2,...,J\). The notations \(u_l^j\) and \(U_l^j\) are used for the discrete approximation and the exact value of \(u(x,t)\) at the grid point \((x_l,t_j)\), respectively.

For the derivation of the cubic spline finite difference method for the solution of differential equation (5.1.1), we follow the ideas given by Jain and Aziz [65]. We use the cubic spline approximations in \(x\)-direction and second order finite difference approximation in \(t\)-direction.
At the grid point \((x_l, t_j)\), we may write the differential equation (5.1.1) as

\[
U_{ml}^j - U_{xl}^j = g(x_l, t_j, U_{xl}^j, U_{sx}^j, U_{sl}^j) \equiv G_l^j, \text{ (say),}
\]

(5.2.1)

Let \(S_j(x)\) be the cubic spline interpolating polynomial of the function \(u(x,t)\) at the grid point \((x_l, t_j)\), and is given by

\[
S_j(x) = \frac{(x_j - x)^3}{6h} M_{j+1}^l + \frac{(x_j - x_{l-1})^3}{6h} M_{j}^l + \left(U_{j-1}^l - \frac{h^2}{6} M_{j-1}^l\right) \left(\frac{x_j - x}{h}\right)
\]

\[
+ \left(U_{j}^l - \frac{h^2}{6} M_{j}^l\right) \left(\frac{x_j - x_{l-1}}{h}\right), \quad x_{l-1} \leq x \leq x_j; \quad l = 1,2,\ldots,N+1; \quad j = 1,2,\ldots,J.
\]

(5.2.2)

which satisfies at \(j\)th-level the following properties

(i) \(S_j(x)\) coincides with a polynomial of degree three on each \([x_{l-1}, x_l]\), \(l = 1,2,\ldots,N+1; \quad j = 1,2,\ldots,J,\)

(ii) \(S_j(x) \in C^2[0,1] \), and

(iii) \(S_j(x_l) = U_{j}^l, \quad l = 0,1,\ldots,N+1; \quad j = 1,2,\ldots,J.\)

The derivatives of cubic spline function \(S_j(x)\) are given by

\[
S_j'(x) = \frac{-(x_j - x)^2}{2h} M_{j+1}^l + \frac{(x_j - x_{l-1})^2}{2h} M_{j}^l + \left(U_{j-1}^l - U_{j}^l\right) \frac{h}{6} M_{j-1}^l - \frac{h}{6} [M_{j}^l - M_{j-1}^l]
\]

(5.2.3a)

\[
S_j''(x) = \frac{(x_j - x)}{h} M_{j+1}^l + \frac{(x_j - x_{l-1})}{h} M_{j}^l
\]

(5.2.3b)

where

\[
M_j^l = S_j''(x_l) = U_{xl}^j = U_{ml}^j - G(x_l, t_j, U_{xl}^j, U_{sx}^j, U_{sl}^j),
\]

(5.2.4)

\[
m_j^l = S_j'(x_l) = U_{sl}^j = \frac{U_{j}^l - U_{j-1}^l}{h} + \frac{h}{6} [M_{j-1}^l + 2M_j^l], \quad x_{l-1} \leq x \leq x_l
\]

(5.2.5)
and replacing ‘h’ by ‘-h’, we get
\[ m'_i = S'_j(x_i) = U_{i,i} = \frac{U_{i+1,i} - U_{i-1,i}}{2h} - \frac{h}{6}[M_{i+1,i} + 2M_{i,i}] \quad \text{x}_i \leq x \leq \text{x}_{i+1} \] (5.2.6)

Using the continuity of the first derivative at \((x_i,t_j)\), that is \(S'_j(x_i+) = S'_j(x_i-)\), we obtain the following relation for \(l = 1, 2, \ldots, N-1:\)
\[ U_{i+1} - 2U_i + U_{i-1} = \frac{h^2}{6}(M_{i+1,i} + 2M_{i,i} + M_{i-1,i}) \] (5.2.7)

Combining (5.2.5) and (5.2.6), we obtain
\[ m'_i = S'_j(x_i) = U_{i,i} = \frac{U_{i+1,i} - U_{i-1,i}}{2h} - \frac{h}{12}[M_{i+1,i} - M_{i-1,i}] \] (5.2.8)

Further, from (5.2.5), we have
\[ m'_{i+1} = S'_j(x_{i+1}) = U_{i+1,i} = \frac{U_{i+2,i} - U_{i,i}}{h} + \frac{h}{6}[M_{i+1,i} + 2M_{i,i}] \] (5.2.9)

and from (5.2.6), we have
\[ m'_{i-1} = S'_j(x_{i-1}) = U_{i-1,i} = \frac{U_{i,i} - U_{i-2,i}}{h} - \frac{h}{6}[M_{i+1,i} + 2M_{i,i}] \] (5.2.10)

Note that, (5.2.4), (5.2.7), (5.2.8), (5.2.9) and (5.2.10) are important properties of the cubic spline function \(S_j(x)\).

By Taylor series expansion of the function \(U(x,t)\) about the grid point \((x_l, t_j)\), we have
\[ U_{i,1} = U_i + hU_{i,t} + \frac{h^2}{2}U_{i,tt} + \frac{h^3}{6}U_{i,ttt} + O(h^4), \] (5.2.11a)
\[ U_{i,2} = U_i + kU_{i,t} + \frac{k^2}{2}U_{i,tt} + \frac{k^3}{6}U_{i,ttt} + O(k^4). \] (5.2.11b)

Using (5.2.11a) and (5.2.11b), we get the following
\[ U_{i,i} = \frac{1}{2h}(U_{i+1,i} - U_{i-1,i}) + O(h^2) = \frac{1}{2h}(U_{i+1,i} - U_{i-1,i}), \]
\[ U_{i,i} = \frac{1}{h^2}(U_{i+1,i} + 2U_{i,i} + U_{i-1,i}) + O(h^2) = \frac{1}{h^2}(U_{i+1,i} + 2U_{i,i} + U_{i-1,i}). \]
\[ U^i_d = \frac{1}{2k} (U_i^{i+1} - U_i^{i-1}) + O(k^2) = \frac{1}{2k} (U_i^{i+1} - U_i^{i-1}) , \]
\[ U^i_d = \frac{1}{k} (U_i^{i+1} - 2U_i^i + U_i^{i-1}) + O(k^2) = \frac{1}{k} (U_i^{i+1} - 2U_i^i + U_i^{i-1}) , \]
and so on.

Hence, we consider the following approximations:

\[ \overline{U}^i_d = (U_i^{i+1} - U_i^{i-1}) / (2k) , \tag{5.2.12a} \]
\[ \overline{U}^i_{d+1} = (U_i^{i+1} - U_i^{i-1}) / (2k) , \tag{5.2.12b} \]
\[ \overline{U}^i_{d-1} = (U_i^{i+1} - U_i^{i-1}) / (2k) , \tag{5.2.12c} \]
\[ \overline{U}^i_n = (U_i^{i+1} - 2U_i^i + U_i^{i-1}) / (k^2) , \tag{5.2.12d} \]
\[ \overline{U}^i_{n+1} = (U_i^{i+1} - 2U_i^i + U_i^{i-1}) / (k^2) , \tag{5.2.12e} \]
\[ \overline{U}^i_{n-1} = (U_i^{i+1} - 2U_i^i + U_i^{i-1}) / (k^2) , \tag{5.2.12f} \]
\[ \overline{U}^i_s = (U_i^{i+1} - U_i^{i-1}) / (2h) , \tag{5.2.13a} \]
\[ \overline{U}^i_{s+1} = (3U_i^{i+1} - 4U_i^i + U_i^{i-1}) / (2h) , \tag{5.2.13b} \]
\[ \overline{U}^i_{s-1} = (-3U_i^{i+1} + 4U_i^i - U_i^{i-1}) / (2h) . \tag{5.2.13c} \]

Further

\[ \overline{G}^i_d = G(x_i, t_j, U_i^i, \overline{U}^i_d, \overline{U}^i_d) , \tag{5.2.14a} \]
\[ \overline{G}^i_{d+1} = G(x_i, t_j, U_i^i, \overline{U}^i_{d+1}, \overline{U}^i_{d+1}) , \tag{5.2.14b} \]
\[ \overline{G}^i_{d-1} = G(x_i, t_j, U_i^i, \overline{U}^i_{d-1}, \overline{U}^i_{d-1}) . \tag{5.2.14c} \]

Since the derivative values of \( S_j(x) \) defined by (5.2.4), (5.2.8), (5.2.9) and (5.2.10) are not known at each grid point \((x_i, t_j)\), we use the following approximations for the derivatives of \( S_j(x) \).

\[ \overline{M}^i_d = \overline{U}^i_d - \overline{G}^i_d , \tag{5.2.15a} \]
\[ \overline{M}^i_{d+1} = \overline{U}^i_{d+1} - \overline{G}^i_{d+1} , \tag{5.2.15b} \]
\[ \overline{M}^i_{d-1} = \overline{U}^i_{d-1} - \overline{G}^i_{d-1} , \tag{5.2.15c} \]
\[ \hat{m}_i = \frac{U_{i+1} - U_i}{2h} - \frac{h}{12} \left[ M_{i+1} - M_i \right] \],

\[ \hat{m}_{i+1} = \frac{U_{i+1} - U_i}{h} + \frac{h}{6} \left[ M_{i+1} + 2M_i \right] \],

\[ \hat{m}_{i-1} = \frac{U_{i} - U_{i-1}}{h} - \frac{h}{6} \left[ M_i + 2M_{i+1} \right] \].

Now we define the following approximations:

\[ \hat{G}_i = G(x_i, t_j, U_i', \hat{m}_i, \overline{U}_i') \],

\[ \hat{G}_{i+1} = G(x_{i+1}, t_j, \hat{m}_{i+1}, \overline{U}_{i+1}') \],

\[ \hat{G}_{i-1} = G(x_{i-1}, t_j, \hat{m}_{i-1}, \overline{U}_{i-1}') \],

in which we use the cubic spline function \( U_i' = S_j(x_i) \), approximation of its first order space derivative defined by (5.2.16a)-(5.2.16c) in \( x \)-direction and central difference approximations of time derivative defined by (5.2.12a)-(5.2.12c) in \( t \)-direction.

Then, a cubic spline finite difference method of Numerov type with accuracy of \( O(k^2 + k^2 h^2 + h^4) \) for the solution of differential equation (5.1.1) may be written as

\[ 6\lambda^2 \left[ U_{i+1} - 2U_i + U_{i-1} \right] = \frac{k^2}{2} \left[ \overline{U}_{i+1} + \overline{U}_{i-1} + 10\overline{U}_i \right] \]

\[ -\frac{k^2}{2} \left[ \hat{G}_{i+1} + \hat{G}_{i-1} + 10\hat{G}_i \right] + \hat{T}_i \]

where \( \hat{T}_i = O(k^4 + k^4 h^2 + k^2 h^4) \).

5.3 Derivation of the method

At the grid point \((x_l, t_j)\), we may write the differential equation (5.1.1) as

\[ U_{n1}' - U_{xx} = g(x_l, t_j, U_l', U_{x1}', U_{n1}') \equiv G_l' \text{ (say)} \]
and at the same point \((x_i, t_j)\), let us denote
\[
U_{ab} = \left( \frac{\partial U_{ab}}{\partial x^a \partial U^b} \right)_i, \quad \alpha'_i = \left( \frac{\partial G_{ij}}{\partial U_i} \right)_j, \quad \beta'_i = \left( \frac{\partial G_{ij}}{\partial U_i} \right)_j.
\] (5.3.2)

By Taylor series expansion, we have
\[
6 \lambda^2 \left[ U_{jt}^{t+1} - 2U_{jt}^i + U_{jt}^{t-1} \right] = \frac{k^2}{2} \left[ \overline{U}_{n1}^{t+1} + \overline{U}_{n1}^{t-1} + 10\overline{U}_{n1}^i \right] \]
\[
- \frac{k^2}{2} \left[ G_{i+1}^{t+1} + G_{i-1}^{t+1} + 10G_i^j \right] + O(k^4 + k^4h^2 + k^2h^4).
\] (5.3.3)

Simplifying the approximations (5.2.12a)-(5.2.13c), we get
\[
\overline{U}_{i1}^j = U_{i1}^j + O(k^2), \quad (5.3.4a)
\]
\[
\overline{U}_{i1}^j = U_{i1}^j + O(k^2 + k^2h + k^2h^2), \quad (5.3.4b)
\]
\[
\overline{U}_{i1}^j = U_{i1}^j + O(k^2 - k^2h + k^2h^2), \quad (5.3.4c)
\]
\[
\overline{U}_{n1}^i = U_{n1}^i + O(k^2), \quad (5.3.4d)
\]
\[
\overline{U}_{n1}^i = U_{n1}^i + O(k^2 + k^2h + k^2h^2), \quad (5.3.4e)
\]
\[
\overline{U}_{n1}^i = U_{n1}^i + O(k^2 - k^2h + k^2h^2), \quad (5.3.4f)
\]
\[
\overline{U}_{i1}^j = U_{i1}^j + \frac{k^2}{6} U_{30} + O(h^4), \quad (5.3.5a)
\]
\[
\overline{U}_{i1}^j = U_{i1}^j - \frac{k^2}{2} U_{30} + O(h^4 + h^4), \quad (5.3.5b)
\]
\[
\overline{U}_{i1}^j = U_{i1}^j - \frac{k^2}{2} U_{30} + O(-h^4 + h^4). \quad (5.3.5c)
\]

With the help of the approximations (5.3.2), (5.3.4a) - (5.3.4c) and (5.3.5a) - (5.3.5c), from (5.2.14a)-(5.2.14c), we obtain
\[
\overline{G_i}^j = g \left( x_i, t_j, U_{ij}^i, U_{ij}^j, U_{i1}^j + \frac{k^2}{6} U_{30} + O(h^4), U_{i1}^j + O(k^2) \right)
\]
\[
= g \left( x_i, t_j, U_{ij}^i, U_{ij}^j, U_{i1}^j \right) + \frac{k^2}{6} U_{30} \alpha'_i + O(k^2 + h^4)
\]
\[
\frac{\alpha_{j+1}^i}{\alpha_{j+1}^i} = \frac{\alpha_j^i}{\alpha_j^i} \pm O(\epsilon), \\
\frac{\beta_{j+1}^i}{\beta_{j+1}^i} = \frac{\beta_j^i}{\beta_j^i} \pm O(\epsilon).
\]

where we have used the approximations:

Now using the approximations (5.3.4d)-(5.3.4f) and (5.3.6a)-(5.3.6c), and simplifying (5.2.16a)-(5.2.16c), we get

\[
\hat{m}_j^i = \frac{U_{j+1}^i - U_{j}^i}{2h} - \frac{h}{12} \left[ \overline{M}_{j+1}^i - \overline{M}_{j}^i \right]
\]

\[
\hat{m}_j^i = \frac{U_{j+1}^i - U_{j}^i}{2h} - \frac{h}{12} \left[ \overline{U}_{j+1}^i - \overline{U}_{j}^i \right] + \frac{h}{12} \left[ \overline{G}_{j+1}^i - \overline{G}_{j}^i \right]
\]

\[
\hat{m}_j^i = U_{j+1}^i - U_{j}^i - \frac{h^2}{6} U_{30} - \frac{h^2}{12} U_{12} + \frac{h}{12} \left[ G_{j+1}^i - G_{j}^i \right] + O(k^2 + k^2 h^2 + h^4)
\]

\[
\hat{m}_j^i = U_{j+1}^i - U_{j}^i - \frac{h^2}{6} U_{30} - \frac{h^2}{12} U_{12} + \frac{h}{12} \left[ G_{j+1}^i - G_{j}^i \right] + O(k^2 + k^2 h^2 + h^4)
\]

\[
\hat{m}_j^i = m_j^i + O(k^2 + k^2 h^2 + h^4)
\]
\[ \tilde{m}_{i+1} = \frac{U_{i+1}^j - U_i^j}{h} + \frac{h}{6} \left[ \overline{M}_i^j + 2 \overline{M}_{i+1}^j \right] \]
\[ = \frac{U_{i+1}^j - U_i^j}{h} + \frac{h}{6} \left[ \overline{G}_i^j + 2 \overline{G}_{i+1}^j \right] \]
\[ = U_i^j + \frac{h}{2} U_{20} + \frac{h^2}{6} U_{30} + \frac{h^3}{24} U_{40} + \frac{h}{6} \left[ \overline{U}_i^j + 2 \overline{U}_{i+1}^j \right] \]
\[ - \frac{h}{6} \left[ G_i^j + 2 G_{i+1}^j \right] + O(k^2 + k^2 h^2 + h^4) \]
\[ = U_i^j + \frac{h}{2} U_{20} + \frac{h^2}{6} U_{30} + \frac{h^3}{24} U_{40} + \frac{h}{6} \left[ 3 \overline{U}_i^j + 2 h \overline{U}_{i+1}^j \right] \]
\[ - \frac{h}{6} \left[ 3G_i^j + 2 hG_{i+1}^j \right] + O(k^2 + h^3 + k^2 h^2 + h^4) \]
\[ = U_i^j + \frac{h}{2} U_{20} + \frac{h^2}{6} U_{30} + \frac{h^3}{24} U_{40} + \frac{h}{6} \left[ 3 \overline{U}_i^j + 2 h \overline{U}_{i+1}^j \right] \]
\[ - \frac{h}{6} \left[ 3 \left( \overline{U}_i^j - \overline{U}_{i+1}^j \right) + 2 h \left( \overline{U}_i^j - \overline{U}_{i+1}^j \right) \right] + O(k^2 + h^3 + k^2 h^2 + h^4) \]
\[ = U_i^j + \frac{h}{2} U_{20} + \frac{h^2}{6} U_{30} + \frac{h^3}{24} U_{40} + \frac{h}{6} \left[ 3 \overline{U}_i^j + 2 h \overline{U}_{i+1}^j \right] + O(k^2 + h^3 + k^2 h^2 + h^4) \]
\[ = U_i^j + h U_{20} + \frac{h^2}{2} U_{30} + O(k^2 + h^3 + k^2 h^2 + h^4) \]

But,
\[ m_{i+1}^j = U_{i+1}^j = U_i^j + h U_{20} + \frac{h^2}{2} U_{30} + O(h^3) \]
\[ \therefore \quad \tilde{m}_{i+1} = m_{i+1}^j + O(k^2 + k^2 h^2 + k^4) . \quad (5.3.7b) \]

Similarly,
\[ \tilde{m}_{i-1} = m_{i-1}^j + O(k^2 + k^2 h^2 + k^4) . \quad (5.3.7c) \]

Now, with the help of the approximations (5.3.7a)-(5.3.7c) and (5.3.4a)-(5.3.4c), from (5.2.17a)-(5.2.17c), we obtain

\[ \tilde{G}_i^j = g(x_i, t_j, U_i^j, m_i^j + O(k^2 + k^2 h^2 + h^4), U_{i+1}^j + O(k^2)) \]
\[ = g(x_i, t_j, U_i^j, m_i^j, U_{i+1}^j) + O(k^2 + k^2 h^2 + h^4) \]
\[ = G_i^j + O(k^2 + k^2 h^2 + h^4) . \quad (5.3.8a) \]
\[ \hat{G}_{i+1} = g(x_{i+1}, t_{j+1}, U_{i+1}^j, m_{i+1}^j) + O(k^2 + k^2 h^2 + h^4), U_{i+1}^j + O(k^2), \]
\[ = G_{i+1}^j + O(k^2 + k^2 h^2 + h^4) \]  
\[ \hat{G}_{i-1} = g(x_{i-1}, t_{j-1}, U_{i-1}^j, m_{i-1}^j) + O(k^2 + k^2 h^2 + h^4), U_{i+1}^j + O(k^2), \]
\[ = G_{i-1}^j + O(k^2 + k^2 h^2 + h^4) \]  

where we have used the approximations:
\[ \alpha_{i \pm 1}^j = \alpha_i^j \pm O(h), \]
\[ \beta_{i \pm 1}^j = \beta_i^j \pm O(h). \]

Then, at each grid point \((x_i, t_j)\), a cubic spline finite difference method of Numerov type with accuracy of \(O(k^2 + k^2 h^2 + h^4)\) for the solution of differential equation (5.1.1) may be written as
\[ 6\lambda^2 \left[ U_{i+1}^j - 2U_i^j + U_{i-1}^j \right] = \frac{k^2}{2} \left[ \bar{U}_{i+1}^j + \bar{U}_{i-1}^j + 10\bar{U}_i^j \right] 
- \frac{k^2}{2} \left[ \hat{G}_{i+1}^j + \hat{G}_{i-1}^j + 10\hat{G}_i^j \right] + \hat{T}_i^j, \]  
(5.3.9)

where \(\hat{G}_{i+1}^j, \hat{G}_{i-1}^j\) and \(\hat{G}_i^j\) are given by (5.3.8a)-(5.3.8c).

Using the approximations (5.3.8a)-(5.3.8c) in (5.3.9) and using (5.3.3), we obtain the local truncation error
\[ -\frac{k^2}{2} \left[ G_{i+1}^j + O(k^2 + k^2 h^2 + h^4) \right] 
+ G_{i-1}^j + O(k^2 + k^2 h^2 + h^4) 
+ 10(G_i^j + O(k^2 + k^2 h^2 + h^4)) \]
\[ + \hat{T}_i^j = -\frac{k^2}{2} \left[ G_{i+1}^j + G_{i-1}^j + 10G_i^j \right] + O(k^4 + k^4 h^2 + k^2 h^4). \]

We obtain the local truncation error \(\hat{T}_i^j = O(k^4 + k^4 h^2 + k^2 h^4)\).

Note that, the initial and Dirichlet boundary conditions are given by (5.1.2) and (5.1.3), respectively. Incorporating the initial and boundary conditions, we can write the
method (5.2.18) in a tri-diagonal matrix form. If the differential equation (5.1.1) is linear, we can solve the linear system using Gauss-elimination (tri-diagonal solver) method; in the non-linear case, we can use Newton-Raphson iterative method to solve the non-linear system (Kelly [82], Hageman and Young [58]).

5.4. Application to wave equation with singular coefficients and stability analysis

Consider the hyperbolic equation with singular coefficient

\[ \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{\gamma}{r} \frac{\partial u}{\partial r} + f(r,t), \quad 0 < r < 1, \quad t > 0. \]  

(5.4.1)

For \( \gamma = 1 \) and 2, above equation represents the one-space dimensional wave equation in cylindrical and spherical co-ordinates, respectively. The initial and the Dirichlet boundary conditions are prescribed by

\[ u(r,0) = \phi(r), \quad u_r(r,0) = \psi(r), \quad 0 \leq r \leq 1 \]  

(5.4.2)

\[ u(0,t) = q_0(t), \quad u(1,t) = q_1(t), \quad t \geq 0 \]  

(5.4.3)

Assume that \( f(r,t) \in C^2(0,1) \times [t > 0] \) and the conditions (5.4.2) and (5.4.3) are given with sufficient smoothness to maintain the order of accuracy in the numerical method under consideration.

Putting \( \frac{\gamma}{r} = D(r) \) in (5.4.1) and applying the cubic spline method (5.2.18) to (5.4.1), we obtain

\[ 6 \lambda^2 \delta_r^2 u_i^j = \frac{k^2}{2} \left[ -u_{i+1,j}^j + u_{i-1,j}^j + 10u_{i,j}^j \right] - \frac{k^2}{2} \left[ \hat{G}_{i+1}^j + \hat{G}_{i-1}^j + 10\hat{G}_i^j \right] \]  

(5.4.4)

where

\[ G_i^j = g(r_i,t_j, u_i^j, u_t^j, u_{tt}^j) = D(r_i)u_t^j + f(r_i,t_j). \]

Now,

\[ \hat{G}_i^j = g(r_i,t_j, \tilde{u}_i^j, \tilde{u}_t^j, \tilde{u}_{tt}^j) = D(r_i)\tilde{u}_t^j + f(r_i,t_j) \]

\[ = D_t\tilde{u}_t^j + f_t^j = \frac{1}{2h} D_t(2\mu_\delta^j)u_t^j + f_t^j \]
\[ \bar{G}_{i+1} = g(r_{i+1}, t, \bar{\nu}_{i+1}, \bar{\nu}_{i+1}^2, \bar{\nu}_{i+1}^3) = D_{i+1} \bar{\nu}_{i+1} + f_{i+1} = \frac{1}{2h} D_{i+1}(2\mu, \delta_r, 2\delta_r^2)u_i^j + f_{i+1}^j \]

\[ \bar{G}_{i-1} = g(r_{i-1}, t, \bar{\nu}_{i-1}, \bar{\nu}_{i-1}^2, \bar{\nu}_{i-1}^3) = D_{i-1} \bar{\nu}_{i-1} + f_{i-1} = \frac{1}{2h} D_{i-1}(2\mu, \delta_r, 2\delta_r^2)u_i^j + f_{i-1}^j \]

Then,

\[ \bar{G}_{i+1} - \bar{G}_{i-1} = \frac{1}{2h} D_{i+1}(2\mu, \delta_r, 2\delta_r^2)u_i^j + f_{i+1}^j - \frac{1}{2h} D_{i-1}(2\mu, \delta_r, 2\delta_r^2)u_i^j - f_{i-1}^j \]

\[ = \frac{1}{2h} D_{i+1}(2\mu, \delta_r, 2\delta_r^2)u_i^j - \frac{1}{2h} D_{i-1}(2\mu, \delta_r, 2\delta_r^2)u_i^j + f_{i+1}^j - f_{i-1}^j, \]

\[ \bar{G}_{i} + 2\bar{G}_{i+1} = \left[ \frac{1}{2h} D_{i}(2\mu, \delta_r)u_i^j + f_{i+1}^j \right] + 2 \left[ \frac{1}{2h} D_{i+1}(2\mu, \delta_r, 2\delta_r^2)u_i^j + f_{i+1}^j \right] \]

\[ = \frac{1}{2h} D_{i}(2\mu, \delta_r) + 2D_{i+1}(2\mu, \delta_r, 2\delta_r^2)u_i^j + f_{i+1}^j + 2f_{i+1}^j, \]

\[ \bar{G}_{i} + 2\bar{G}_{i-1} = \left[ \frac{1}{2h} D_{i}(2\mu, \delta_r)u_i^j + f_{i+1}^j \right] + 2 \left[ \frac{1}{2h} D_{i-1}(2\mu, \delta_r, 2\delta_r^2)u_i^j + f_{i-1}^j \right] \]

\[ = \frac{1}{2h} D_{i}(2\mu, \delta_r) + 2D_{i-1}(2\mu, \delta_r, 2\delta_r^2)u_i^j + f_{i+1}^j + 2f_{i-1}^j, \]

\[ \bar{M}_{i+1} - \bar{M}_{i-1} = \left( u_{i+1}^j - u_{i+1}^j \right) - \left( \bar{G}_{i+1} - \bar{G}_{i-1} \right) \]

\[ = \frac{\delta_t^2}{k^2} \left( u_{i+1}^j - u_{i+1}^j \right) - \frac{1}{2h} \left[ D_{i+1}(2\mu, \delta_r, 2\delta_r^2)u_i^j - D_{i-1}(2\mu, \delta_r, 2\delta_r^2)u_i^j \right] - \left[ f_{i+1}^j - f_{i-1}^j \right] \]

\[ = \frac{\delta_t^2}{k^2} \left( 2\mu, \delta_r \right) u_i^j - \frac{1}{2h} \left[ (D_{i+1} - D_{i-1})(2\mu, \delta_r)u_i^j + (D_{i+1} + D_{i-1})(2\delta_r^2)u_i^j \right] - \left[ f_{i+1}^j - f_{i-1}^j \right] \]

\[ \bar{M}_{i} + 2\bar{M}_{i+1} = \left( u_{i+1}^j + 2u_{i+1}^j \right) - \left( \bar{G}_{i} + 2\bar{G}_{i+1} \right) \]

\[ = \frac{\delta_t^2}{k^2} \left( u_{i+1}^j + 2u_{i+1}^j \right) - \frac{1}{2h} \left[ D_{i}(2\mu, \delta_r)u_i^j + 2D_{i+1}(2\mu, \delta_r, 2\delta_r^2)u_i^j \right] - \left[ f_{i+1}^j + 2f_{i+1}^j \right] \]

\[ \bar{M}_{i} + 2\bar{M}_{i-1} = \left( u_{i-1}^j + 2u_{i-1}^j \right) - \left( \bar{G}_{i} + 2\bar{G}_{i-1} \right) \]

\[ = \frac{\delta_t^2}{k^2} \left( u_{i-1}^j + 2u_{i-1}^j \right) - \frac{1}{2h} \left[ D_{i}(2\mu, \delta_r)u_i^j + 2D_{i-1}(2\mu, \delta_r, 2\delta_r^2)u_i^j \right] - \left[ f_{i+1}^j + 2f_{i+1}^j \right] \]
\[ \dot{m}_i' = \frac{u_{i+1} - u_{i-1}}{2h} - \frac{h}{12} \left[ \ddot{M}_{i+1} - \ddot{M}_{i-1} \right] \]

\[ = \frac{1}{2h} \left( 2\mu, \delta_x \right) u_i' - \frac{h}{12} \left[ \frac{\delta_x^2}{k^2} (2\mu, \delta_x) u_i' \right] - \frac{1}{2h} \left[ (D_{i+1} - D_{i-1})(2\mu, \delta_x) u_i' + (D_{i+1} + D_{i-1})(2\delta_x) u_i' \right] \]

\[ \dot{m}_{i-1}' = \frac{u_{i+1} - u_{i-1}}{h} + \frac{h}{6} \left[ M_i + 2\ddot{M}_{i+1} \right] \]

\[ = \frac{u_{i+1} - u_{i-1}}{h} + \frac{h}{6} \left[ \frac{\delta_x^2}{k^2} (u_i' + 2u_{i-1}') \right] - \frac{1}{2h} \left[ D_i(2\mu, \delta_x) u_i' + 2D_{i+1}(2\mu, \delta_x + 2\delta_x^2) u_i' \right] \]

\[ \dot{m}_{i+1}' = \frac{u_{i+1} - u_{i-1}}{h} - \frac{h}{6} \left[ \ddot{M}_i + 2\ddot{M}_{i+1} \right] \]

\[ = \frac{u_{i+1} - u_{i-1}}{h} - \frac{h}{6} \left[ \frac{\delta_x^2}{k^2} (u_i' + 2u_{i-1}') \right] - \frac{1}{2h} \left[ D_i(2\mu, \delta_x) u_i' + 2D_{i-1}(2\mu, \delta_x - 2\delta_x^2) u_i' \right] \]

Now,

\[ \dot{G}_i = D_i \dot{u}_i' + f_i' = D_i \dot{m}_i' + f_i' \]

\[ = \frac{D_i}{2h} \left( 2\mu, \delta_x \right) u_i' - \frac{D_i h}{12} \left[ \frac{\delta_x^2}{k^2} (2\mu, \delta_x) u_i' \right] - \frac{1}{2h} \left[ (D_{i+1} - D_{i-1})(2\mu, \delta_x) u_i' + (D_{i+1} + D_{i-1})(2\delta_x) u_i' \right] + f_i' \]

\[ \left[ f_{i+1}' - f_{i-1}' \right] \]
\[
\begin{align*}
\hat{G}_{i+1} &= D_{i+1} \hat{u}_{i+1}^j + f_{i+1}^j = D_{i+1} \hat{m}_{i+1}^j + f_{i+1}^j \\
&= D_{i+1} \left( \frac{u_{i+1}^j - u_i^j}{h} \right) + D_{i+1} \frac{h}{6} \left[ \frac{\delta^2}{k^2} \left( u_i^j + 2 u_{i+1}^j \right) - \frac{1}{2h} \left[ D_i (2 \mu, \delta_i) u_i^j + 2 D_{i+1} (2 \mu, \delta_i + 2 \delta_i^2) u_i^j \right] \right] + f_{i+1}^j \\
\hat{G}_{i-1} &= D_{i-1} \hat{u}_{i-1}^j + f_{i-1}^j = D_{i-1} \hat{m}_{i-1}^j + f_{i-1}^j \\
&= D_{i-1} \left( \frac{u_{i-1}^j - u_i^j}{h} \right) - D_{i-1} \frac{h}{6} \left[ \frac{\delta^2}{k^2} \left( u_i^j + 2 u_{i-1}^j \right) - \frac{1}{2h} \left[ D_i (2 \mu, \delta_i) u_i^j + 2 D_{i-1} (2 \mu, \delta_i - 2 \delta_i^2) u_i^j \right] \right] + f_{i-1}^j \\
\text{Hence,} \\
\hat{G}_{i+1} + \hat{G}_{i-1} + 10\hat{G}_i &= D_{i+1} \left( \frac{u_{i+1}^j - u_i^j}{h} \right) + D_{i-1} \left( \frac{u_{i-1}^j - u_i^j}{h} \right) + 10D_i \left( \frac{2 \mu, \delta_i \right) u_i^j \\
&+ D_{i+1} \frac{h}{6} \left[ \frac{\delta^2}{k^2} \left( u_i^j + 2 u_{i+1}^j \right) - \frac{1}{2h} \left[ D_i (2 \mu, \delta_i) u_i^j + 2 D_{i+1} (2 \mu, \delta_i + 2 \delta_i^2) u_i^j \right] \right] + f_{i+1}^j \\
&- D_{i-1} \frac{h}{6} \left[ \frac{\delta^2}{k^2} \left( u_i^j + 2 u_{i-1}^j \right) - \frac{1}{2h} \left[ D_i (2 \mu, \delta_i) u_i^j + 2 D_{i-1} (2 \mu, \delta_i - 2 \delta_i^2) u_i^j \right] \right] + f_{i-1}^j \\
&= D_i \left( \frac{2 \mu, \delta_i \right) u_i^j + 10D_i \left( \frac{2 \mu, \delta_i \right) u_i^j \\
&+ \frac{h}{6} \left[ \frac{\delta^2}{k^2} \left( u_i^j + 2 u_{i+1}^j \right) - \frac{1}{2h} \left[ D_i (2 \mu, \delta_i) u_i^j + 2 D_{i+1} (2 \mu, \delta_i + 2 \delta_i^2) u_i^j \right] \right] + f_{i+1}^j \\
&- \frac{h}{6} \left[ \frac{\delta^2}{k^2} \left( u_i^j + 2 u_{i-1}^j \right) - \frac{1}{2h} \left[ D_i (2 \mu, \delta_i) u_i^j + 2 D_{i-1} (2 \mu, \delta_i - 2 \delta_i^2) u_i^j \right] \right] + f_{i-1}^j \\
&= D_i \left( \frac{2 \mu, \delta_i \right) u_i^j + 10D_i \left( \frac{2 \mu, \delta_i \right) u_i^j \\
&+ \frac{h}{6} \left[ \frac{\delta^2}{k^2} \left( u_i^j + 2 u_{i+1}^j \right) - \frac{1}{2h} \left[ D_i (2 \mu, \delta_i) u_i^j + 2 D_{i+1} (2 \mu, \delta_i + 2 \delta_i^2) u_i^j \right] \right] + f_{i+1}^j \\
&- \frac{h}{6} \left[ \frac{\delta^2}{k^2} \left( u_i^j + 2 u_{i-1}^j \right) - \frac{1}{2h} \left[ D_i (2 \mu, \delta_i) u_i^j + 2 D_{i-1} (2 \mu, \delta_i - 2 \delta_i^2) u_i^j \right] \right] + f_{i-1}^j \\
&= D_i \left( \frac{2 \mu, \delta_i \right) u_i^j + 10D_i \left( \frac{2 \mu, \delta_i \right) u_i^j \
\end{align*}
\]
\[
\begin{align*}
- \frac{10D_h}{12} & \begin{bmatrix}
\frac{\delta^2}{k^2}(2\mu, \delta)u_i^j \\
- \frac{1}{2h}\left[(D_{i+1} - D_{i-1})(2\mu, \delta)u_i^j + (D_{i+1} + D_{i-1})(2\delta^2)u_i^j\right] \\
& \left[f_{i+1}^j - f_{i-1}^j\right]
\end{bmatrix} + 10f_i^j \\
= \frac{1}{h}\left[(D_{i+1}u_i^j - D_{i-1}u_i^j) - (D_{i+1} - D_{i-1})\right]u_i^j + \frac{5D_h}{h}(2\mu, \delta)u_i^j \\
+ \frac{h \delta^2}{6k^2}\left[2(D_{i+1}u_i^j - D_{i-1}u_i^j) + (D_{i+1} - D_{i-1})u_i^j\right] \\
+ \frac{D_{i+1}h}{6} & \left[-\frac{1}{2}\left[D_i(2\mu, \delta)u_i^j + 2D_{i+1}(2\mu, \delta + 2\delta^2)u_i^j\right] \\
- \frac{D_{i-1}h}{6} & \left[-\frac{1}{2}\left[D_i(2\mu, \delta)u_i^j + 2D_{i-1}(2\mu, \delta - 2\delta^2)u_i^j\right] \\
- \frac{10D_h}{12} & \left[-\frac{1}{2}\left[(D_{i+1} - D_{i-1})(2\mu, \delta)u_i^j + (D_{i+1} + D_{i-1})(2\delta^2)u_i^j\right] \\
- \frac{5D_h}{6k^2} & \left[2\mu, \delta)u_i^j - \frac{h}{6}\left[2(D_{i+1}f_{i+1}^j - D_{i-1}f_{i-1}^j) + (D_{i+1} - D_{i-1})f_i^j\right] \\
+ \frac{5D_h}{6} & \left[f_{i+1}^j - f_{i-1}^j\right] + [f_i^j + f_{i+1} + 10f_i^j]
\end{align*}
\]
Also,
\[\left[\tilde{u}_{n+1} + \tilde{u}_{n-1} + 10\tilde{u}_n\right] = \left[\tilde{u}_{n+1} - 2\tilde{u}_n + \tilde{u}_{n-1} + 12\tilde{u}_n\right]\]
\[= \left[\delta_i^2 + 12\right]\frac{k^2}{\delta_i^2}\tilde{u}_n\]
\[= \frac{12}{k^2}\left[1 + \frac{\delta_i^2}{12}\right]\delta_i^2\tilde{u}_n.
\]

where, \(\delta_i u^i = (u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}})\) and \(\mu_i u^i = \frac{1}{2}(u_{i+\frac{1}{2}} + u_{i-\frac{1}{2}})\) are averaging and central difference operators with respect to \(r\)-direction etc. and hence, \(2\mu_i \delta_i u^i = (u_{i+1} - u_{i-1})\), \(\delta_i^2 u^i = (u_{i+1} - 2u_i + u_{i-1})\). Also, \(\delta_i^2 u^i = (u^i_{i+1} - 2u^i_i + u^i_{i-1})\).

Hence, the scheme (5.4.4) may be written as

\[6\lambda^2 \delta_i^2 u^i = 6\left(1 + \frac{\delta_i^2}{12}\right)\delta_i^2 u^i\]
\[\left[\frac{1}{h}\left[(D_{i+1}u^i_{i+1} - D_{i-1}u^i_{i-1}) - (D_{i+1} - D_{i-1})\right]u^i_i + \frac{5D_i}{h}(2\mu_i \delta_i u^i)\right]
+ \frac{h}{6k^2}\left[2(D_{i+1}u^i_{i+1} - D_{i-1}u^i_{i-1}) + (D_{i+1} - D_{i-1})\right]u^i_i
- \frac{1}{12}\left[(D_{i+1} + D_{i-1})(2\mu_i \delta_i u^i) + 2((D_{i+1})^2 + (D_{i-1})^2)(2\mu_i \delta_i u^i)\right]
+ \frac{5D_i}{12}\left[(D_{i+1} - D_{i-1})(2\mu_i \delta_i u^i) + (D_{i+1} + D_{i-1})(2\delta_i^2 u^i)\right]
- \frac{5D_i h}{6}\left[\frac{\delta_i^2}{k^2}(2\mu_i \delta_i u^i) - \frac{h}{6}(2(D_{i+1}f_{i+1} - D_{i-1}f_{i-1}) + (D_{i+1} - D_{i-1})f_i)\right]
+ \frac{5D_i}{6}\left[f_{i+1} - f_{i-1}\right] + \left[5D_i + f_{i+1} - f_{i-1} + 10f_i\right]
\]
(5.4.5)

Note that the scheme (5.4.5) is of \(O(k^2 + k^2 h^2 + h^4)\) accuracy for the solution of wave equation (5.4.1). For \(r_i = 0\), the scheme (5.4.5) fails to compute at \(l = 1\) due to zero division. To overcome this problem and in order to get a stable cubic spline scheme of \(O(k^2 + k^2 h^2 + h^4)\) accuracy, we need the following approximations:
\[ D_{t+1} = D_{00} + hD_{10} + \frac{h^2}{2} D_{20} + O(h^3), \]  
\[ D_{t-1} = D_{00} - hD_{10} + \frac{h^2}{2} D_{20} - O(h^3), \]  
\[ f_{i+1}^j = f_i^j + hf_{10} + \frac{h^2}{2} f_{20} + O(h^3), \]  
\[ f_{i-1}^j = f_i^j - hf_{10} + \frac{h^2}{2} f_{20} - O(h^3), \]

where

\[ f_i^j = f(r_t, t_j) = f_{00}, \quad f_i^j = f_j(r_t, t_j) = f_{10}, \quad f_{i+1}^j = f(r_{t+1}, t_j) = f_{20}, \ldots \text{ etc.} \]

Using (5.4.6a)-(5.4.6d), we have

\[
D_{t+1} - D_{t-1} = \left( D_{00} + hD_{10} + \frac{h^2}{2} D_{20} + O(h^3) \right) - \left( D_{00} - hD_{10} + \frac{h^2}{2} D_{20} - O(h^3) \right) \\
= 2hD_{10} + O(h^3)
\]

\[
D_{t+1} + D_{t-1} = \left( D_{00} + hD_{10} + \frac{h^2}{2} D_{20} + O(h^3) \right) + \left( D_{00} - hD_{10} + \frac{h^2}{2} D_{20} - O(h^3) \right) \\
= 2D_{00} + h^2 D_{20} + O(h^3)
\]

\[
(D_{t+1})^2 = \left( D_{00} + hD_{10} + \frac{h^2}{2} D_{20} + O(h^3) \right)^2 \\
= (D_{00})^2 + 2hD_{00}D_{10} + h^2 \left( (D_{10})^2 + D_{00}D_{20} \right) + O(h^5)
\]

\[
(D_{t-1})^2 = \left( D_{00} - hD_{10} + \frac{h^2}{2} D_{20} - O(h^3) \right)^2 \\
= (D_{00})^2 - 2hD_{00}D_{10} + h^2 \left( (D_{10})^2 + D_{00}D_{20} \right) - O(h^5)
\]

\[
(D_{t+1})^2 + (D_{t-1})^2 = 2(D_{00})^2 + 2h^2 \left( (D_{10})^2 + D_{00}D_{20} \right) + O(h^5)
\]

\[
(D_{t+1})^2 - (D_{t-1})^2 = 4hD_{00}D_{10} + O(h^3),
\]
\[ f_{i+1}^{'} + f_{i-1}^{'} = \left( f_i^{'} + hf_{10} + \frac{h^2}{2} f_{20} + O(h^3) \right) + \left( f_i^{'} - hf_{10} + \frac{h^2}{2} f_{20} - O(h^3) \right) \\
= \left( 2f_i^{'} + h^2 f_{20} + O(h^3) \right) \\
\]

\[ f_{i+1}^{'} - f_{i-1}^{'} = \left( f_i^{'} + hf_{10} + \frac{h^2}{2} f_{20} + O(h^3) \right) - \left( f_i^{'} - hf_{10} + \frac{h^2}{2} f_{20} - O(h^3) \right) \\
= 2hf_{10} + O(h^3) \\
\]

\[ D_{i+1} f_{i+1}^{'} = \left( D_{00} + hD_{10} + \frac{h^2}{2} D_{20} + O(h^3) \right) \left( f_i^{'} + hf_{10} + \frac{h^2}{2} f_{20} + O(h^3) \right) \\
= D_{00} f_{00} + h(D_{00} f_{10} + D_{10} f_{00}) + \frac{h^2}{2} (D_{00} f_{20} + 2D_{10} f_{10} + D_{20} f_{00}) + O(h^3) \\
\]

\[ D_{i-1} f_{i-1}^{'} = \left( D_{00} - hD_{10} + \frac{h^2}{2} D_{20} - O(h^3) \right) \left( f_i^{'} - hf_{10} + \frac{h^2}{2} f_{20} - O(h^3) \right) \\
= D_{00} f_{00} - h(D_{00} f_{10} + D_{10} f_{00}) + \frac{h^2}{2} (D_{00} f_{20} + 2D_{10} f_{10} + D_{20} f_{00}) - O(h^3) \\
\]

\[ D_{i+1} f_{i+1}^{'} - D_{i-1} f_{i-1}^{'} = \left( D_{00} f_{00} + h(D_{00} f_{10} + D_{10} f_{00}) + \frac{h^2}{2} (D_{00} f_{20} + 2D_{10} f_{10} + D_{20} f_{00}) + O(h^3) \right) \\
- \left( D_{00} f_{00} - h(D_{00} f_{10} + D_{10} f_{00}) + \frac{h^2}{2} (D_{00} f_{20} + 2D_{10} f_{10} + D_{20} f_{00}) - O(h^3) \right) \\
= 2h(D_{00} f_{10} + D_{10} f_{00}) + O(h^3) \\
\]

\[ (D_{i+1} u_{i+1}^{'} - D_{i-1} u_{i-1}^{'}) = \left( D_{00} + hD_{10} + \frac{h^2}{2} D_{20} + O(h^3) \right) u_{i+1}^{'} \\
- \left( D_{00} - hD_{10} + \frac{h^2}{2} D_{20} - O(h^3) \right) u_{i-1}^{'} \\
= \left( D_{00} + \frac{h^2}{2} D_{20} \right) (u_{i+1}^{'} - u_{i-1}^{'}) + hD_{10} (u_{i+1}^{'} + u_{i-1}^{'}) + O(h^3) \\
= \left( D_{00} + \frac{h^2}{2} D_{20} \right) (u_{i+1}^{'} - u_{i-1}^{'}) + hD_{10} (u_{i+1}^{'} - 2u_i^{'} + u_{i-1}^{'}) + 2u_i^{'} + O(h^3) \\
= \left[ \left( D_{00} + \frac{h^2}{2} D_{20} \right) (2\mu \delta_i + hD_{10} (\delta_i^2 + 2) \right] u_i^{'} \\
\]
\[
\left( D_{i+1}u'_{i+1} - D_{i-1}u'_{i-1} \right) - (D_{i+1} - D_{i-1})u'_i = \begin{pmatrix} D_{00} + hD_{10} + \frac{h^2}{2} D_{20} + O(h^3) \\ D_{00} - hD_{10} + \frac{h^2}{2} D_{20} - O(h^3) \\ (2hD_{10} + O(h^3))u'_1 \end{pmatrix} \]

Hence, the scheme (5.4.5) may be written as

\[
6\lambda^2 \delta^2 u'_i = 6 \left(1 + \frac{\delta^2}{12} \right) \delta^2 u'_i + \frac{1}{h} \left[ D_{00} + \frac{h^2}{2} D_{20} \right] \left( 2\mu, \delta_r \right) + hD_{10} \delta^2 \left( 2\mu, \delta_r \right) + \frac{5D_{10}}{h} \left( 2\mu, \delta_r \right) \left( 2\mu, \delta_r \right)
\]

\[
+ \frac{h}{6} \delta^2 \left[ 2 \left( D_{00} + \frac{h^2}{2} D_{20} \right) \left( 2\mu, \delta_r \right) + hD_{10} \left( 2\mu, \delta_r \right) \right] u'_i \]

\[
- \frac{k^2}{2} \left[ \frac{1}{12} \delta^2 \right] \left[ \left( D_{00} + \frac{h^2}{2} D_{20} \right) \left( 2\mu, \delta_r \right) \right] u'_i \]

\[
+ \left[ 4(D_{00})^2 + h^2 \left( \left( D_{10} \right)^2 + D_{00}D_{20} \right) \right] \left( 2\mu, \delta_r \right) u'_i \]

\[
+ \frac{5D_{10}}{12} \left[ \left( 2\mu, \delta_r \right) u'_i + \left( 2D_{00} + h^2 D_{20} \right) \delta^2 u'_i \right] \]

\[
+ \left[ (2D_{10})^2 \right] \left( 2\mu, \delta_r \right) u'_i \]

\[
- \frac{5D_{10} h}{6} \delta^2 \left( 2\mu, \delta_r \right) u'_i - \frac{h}{6} \left[ 4h(D_{00}f_{10} + D_{10}f_{00}) + (2D_{10})f_{10}' \right] \]

\[
+ \frac{5D_{10} h}{6} \left[ 2hf_{10} + [12f_{00} + h^2 f_{20}] \right] \]
\[ \Rightarrow \lambda^2 \delta^2 u_i' = \left( 1 + \frac{\Delta^2}{12} \right) \delta^2 u_i' \]

\[ = \frac{1}{h} \left[ \left( D_{00} + \frac{h^2}{2} D_{20} \right) (2 \mu, \delta_j) + hD_{10} \delta_j^2 \right] u_i' + \frac{5D_j}{h} (2 \mu, \delta_j) u_i' \]

\[ + \frac{h}{6} \delta^2 \left[ \frac{\Delta^2}{2} \left( D_{00} + \frac{h^2}{2} D_{20} \right) (2 \mu, \delta_j) + hD_{10} (\delta_j^2 + 2) \right] u_i' + (2hD_{10}) u_i' \]

\[ - \frac{k^2}{12} \left[ D_j (2D_{00} + h^2 D_{20}) (2\mu, \delta_j) u_i' \right] \]

\[ - \frac{1}{12} \left[ 4 (D_{00})^2 + h^2 (D_{10})^2 + D_{00} D_{20} \right] (2 \mu, \delta_j) u_i' \]

\[ + (4hD_{10}) (2 \delta_j^2) u_i' \]

\[ + \frac{5D_j}{12} \left[ (2hD_{10}) (2 \mu, \delta_j) u_i' + (2D_{00} + h^2 D_{20}) (2 \delta_j^2) u_i' \right] - \frac{5D_j}{6} \delta^2 (2 \mu, \delta_j) u_i' \]

\[ - \frac{k^2}{12} \left[ 12 f_{00} + h^2 (f_{20} + D_{00} f_{10} - D_{10} f_{00}) \right] \]

(5.4.7)

Neglecting the high order terms, we can re-write the scheme (5.4.7) in three-level operator compact implicit form

\[ [R_0 + \frac{1}{12} (\delta_j^2 + R_i (2 \mu, \delta_j))] \delta_j^2 u_i' = \lambda^2 [R_j \delta_j^2 + R_i (2 \mu, \delta_j)] u_i' + \sum f; \quad l=1(1)N, \quad j=1(1)J \quad (5.4.8) \]

where

\[ R_0 = 1 - \frac{h^2}{12} D_{10}, \]

\[ R_1 = \frac{h}{2} D_{00}, \]

\[ R_2 = 1 + \frac{h^2}{12} [(D_{00})^2 - D_{10}], \]

\[ R_3 = \frac{h}{2} [D_{00} + \frac{h^2}{12} D_{20}], \quad \text{and} \]

\[ \sum f = - \frac{k^2}{2} \left[ 12 f_{00} + h^2 [f_{20} + D_{00} f_{10} - D_{10} f_{00}] \right] \]
where \((2\mu, \delta)u_i^j = u_{i+1}^j - u_{i-1}^j, \ \delta_r^2 u_i^j = u_{i+1}^j - 2u_i^j + u_{i-1}^j, \ \delta_r^2 u_i^j = u_{i+1}^j - 2u_i^j + u_{i-1}^j, \) etc. The cubic spline finite difference scheme \((5.4.8)\) has a local truncation error of \(O(k^2 + k^2 h^2 + h^4)\) and is free from the terms \(\frac{1}{111}\) and hence, it can be solved for \(l=1(1)N, j=1(1)J\) in the region \(0<r<1, t>0\).

For stability of the method \((5.4.8)\), we follow the technique used by Mohanty [122]. We may re-write \((5.4.8)\) as

\[
[R_0 + \frac{1}{12}(R_2\delta_r^2 + R_3(2\mu, \delta))]|\delta_r^2 u_i^j = \lambda^2[R_2\delta_r^2 + R_3(2\mu, \delta)]u_i^j + \sum f \quad (5.4.9)
\]

The additional terms are of high orders and do not affect the accuracy of the scheme. The exact value \(U_i^j = u(r_i, t_j)\) satisfies

\[
[R_0 + \frac{1}{12}(R_2\delta_r^2 + R_3(2\mu, \delta))]|\delta_r^2 U_i^j = \lambda^2[R_2\delta_r^2 + R_3(2\mu, \delta)]U_i^j + \sum f + O(k^4 + k^4 h^2 + k^2 h^4). \quad (5.4.10)
\]

We assume that there exists an error \(e_i^j = U_i^j - u_i^j\) at the grid point \((x_i, t_j)\). Subtracting \((5.4.10)\) from \((5.4.11)\), we obtain the error equation

\[
[R_0 + \frac{1}{12}(R_2\delta_r^2 + R_3(2\mu, \delta))]|\delta_r^2 e_i^j = \lambda^2[R_2\delta_r^2 + R_3(2\mu, \delta)]e_i^j + O(k^4 + k^4 h^2 + k^2 h^4). \quad (5.4.11)
\]

For stability of the modified scheme \((5.4.9)\), we assume that \(e_i^j = A e^{i\phi} e^{i\theta}\) (where \(\xi = e^{i\phi}\) such that \(|\xi|=1\)) at the grid point \((r_i, t_j)\), where \(\xi\) is in general complex, \(\theta\) is an arbitrary real number and \(A\) is a non-zero real parameter to be determined. Substituting \(e_i^j = A e^{i\phi} e^{i\theta}\) in the homogeneous part of the error equation \((5.4.11)\), we obtain the amplification factor
\[ -4\sin^2\left(\frac{\phi}{2}\right) = \frac{\lambda^2 \left[ R_2 \left\{ (A + A^{-1})\cos \theta - 2 + i(A - A^{-1})\sin \theta \right\} 
+ R_1 \left\{ (A - A^{-1})\cos \theta + i(A + A^{-1})\sin \theta \right\} \right] 
}{R_0 + \frac{1}{4} \left[ R_2 \left\{ (A + A^{-1})\cos \theta - 2 + i(A - A^{-1})\sin \theta \right\} 
+ R_1 \left\{ (A - A^{-1})\cos \theta + i(A + A^{-1})\sin \theta \right\} \right]} \]  

(5.4.12)

Since left-hand side of (5.4.12) is a real quantity, hence the imaginary part of right-hand side of (5.4.12) must be zero, from which we obtain

\[ R_2(A - A^{-1}) + R_1(A + A^{-1}) = 0 \]

or,

\[ A = \sqrt{\frac{R_2 - R_1}{R_2 + R_1}} \]  

(5.3.13)

where \( R_2 \pm R_1 > 0 \). Substituting the values of \( A \) and \( A^{-1} \) in (5.4.12), we get

\[ \sin^2\left(\frac{\phi}{2}\right) = \frac{\lambda^2 [R_2 + \sqrt{(R_2^2 - R_1^2)(2\sin^2\left(\frac{\phi}{2}\right) - 1)}]}{2R_0 - \frac{1}{4}[R_2 + \sqrt{(R_2^2 - R_1^2)(2\sin^2\left(\frac{\phi}{2}\right) - 1)}]} \]  

(5.4.14)

Since \( 0 \leq \sin^2\left(\frac{\phi}{2}\right) \leq 1 \), \( \max\sin^2\left(\frac{\phi}{2}\right) = 1 \), \( \min\sin^2\left(\frac{\phi}{2}\right) = 0 \), it follows from (5.4.14) that the cubic spline finite difference scheme (5.4.9) is stable if

\[ 0 < \lambda^2 \leq \frac{2R_0 - \frac{1}{4}[R_2 - \sqrt{R_2^2 - R_1^2}]}{R_2 + \sqrt{R_2^2 - R_1^2}} \]  

(5.4.15)

leading to \( |\xi| = 1 \). It is easy to verify that as \( l \to \infty \), \( 0 < \lambda^2 \leq 1 \).

### 5.5 Application to one space dimensional Telegraph equation and error analysis

We discuss the application of cubic spline finite difference method (5.2.18) to the telegraphic equation with forcing function...
\[ u_t + 2\alpha u_t + \beta^2 u = u_{xx} + f(x,t), \quad 0 < x < 1, \quad t > 0 \]  
(5.5.1)

with the following initial conditions

\[ u(x,0) = \phi(x), \quad u_t(x,0) = \varphi(x), \quad 0 \leq x \leq 1 \]  
(5.5.2)

and the boundary conditions

\[ u(0,t) = a(t), \quad u(1,t) = b(t), \quad t \geq 0 \]  
(5.5.3)

where \( \alpha > 0, \quad \beta \geq 0 \) are real parameters. For \( \beta = 0 \), the equation above represents damped wave equation.

Equation (5.5.1) can be written as

\[ \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + g(x,t,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial t}) \]  
(5.5.4)

where \( g(x,t,u,u_x,u_t) = -2\alpha u_t - \beta^2 u + f(x,t) \equiv G \).

Applying method (5.2.18) to equation (5.5.4), we have

\[ \lambda^2 \delta^2 U/ = \frac{k^2}{12} \left[ \tilde{u}_{n+1} + \tilde{u}_{n-1} + 10\tilde{u}_n \right] - \frac{k^2}{3} \left[ \hat{G}_{n+1} + \hat{G}_{n-1} + 10\hat{G}_n \right] \]  
(5.5.5)

where

\[ \left[ \tilde{u}_{n+1} + \tilde{u}_{n-1} + 10\tilde{u}_n \right] = \left[ \tilde{u}_{n+1} - 2u_n + \tilde{u}_{n-1} + 12u_n \right] \]  

\[ = \left[ \delta^2 + 12 \right] \frac{\delta^2}{k^2} \tilde{u}_n \]  

\[ = \frac{12}{k^2} \left[ 1 + \frac{\delta^2}{12} \right] \delta^2 \tilde{u}_n \]
where, $\delta_i u_1' = (u_{i+1} - u_{i-1})$ and $\mu_i u_1' = \frac{1}{2} (u_{i+1} + u_{i-1})$ are averaging and central difference operators with respect to $x$-direction etc. and hence, $(2 \mu_i \delta_i) u_1' = (u_{i+1} - u_{i-1})$, $\delta_i^2 u_1' = (u_{i+1} - 2u_i' + u_{i-1})$. Also, $\delta_i^2 u_1' = (u_{i+1} - 2u_i' + u_{i+1})$.

Also,
\[
\hat{G}_i = -2\alpha \hat{u}_i - \beta^2 \hat{u}_i + f_i,
\]
\[
\hat{G}_i = -2\alpha \hat{u}_i - \beta^2 \hat{u}_i + f_i,
\]
\[
\hat{G}_i = -2\alpha \hat{u}_i - \beta^2 \hat{u}_i + f_i
\]

Hence, the scheme (5.5.5) becomes
\[
\left(1 + \frac{\delta^2}{12}\right) \delta_i^2 u_1' = \lambda^2 \delta_i^2 u_1' + \frac{k^2}{12} \left[ \frac{-2a}{\Delta x} (2 \mu_i \delta_i) u_{i+1} - \beta^2 u_{i+1} + f_{i+1} \right]
\]
\[
\Rightarrow \left(1 + \frac{\delta^2}{12}\right) \delta_i^2 u_1' = \lambda^2 \delta_i^2 u_1' + \frac{k^2}{12} \left[ \frac{-2a}{\Delta x} (2 \mu_i \delta_i) u_{i+1} - \beta^2 u_{i+1} + f_{i+1} \right]
\]
\[
\Rightarrow \delta_i^2 u_1' + \alpha k (2 \mu_i \delta_i) u_1' + \frac{\alpha}{12} (\delta_i^2 2 \mu_i \delta_i) u_1' + \left( \frac{\beta^2 k^2}{12} - \lambda^2 \right) \delta_i^2 u_1' + \beta^2 \delta_i^2 u_1' + \frac{1}{2} \delta_i^2 \delta_i^2 u_1' + \frac{k^2}{12} \left[ f_{i+1} + f_{i-1} + 10 f_i \right]
\]

In this section, we denote $a = \alpha^2 k^2$, $b = \beta^2 k^2$. 

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Hence, we obtain a numerical approximation of $O(k^2 + k^2h^2 + h^4)$ as

$$\delta^2_i u'_j + \sqrt{a} (2\mu_i \delta_i) u'_j + \sqrt{a} \left( \delta^2_i 2\mu_i \delta_i \right) u'_j + \left( \frac{b}{12} - \lambda^2 \right) \delta^2_i u'_j$$

$$+ bu'_j + \frac{\delta^2_i \delta^2_i - u'_j}{12} = \frac{k^2}{12} \left[ f''_{i+1} + f''_{i-1} + 10f'_j \right].$$

This scheme is conditionally stable [100]. (Similar argument is given in Chapter 1).

In order to obtain an unconditionally stable scheme, we may re-write the above scheme as

$$(1 + \eta b^2) \delta^2_i u'_j + \sqrt{a} (2\mu_i \delta_i) u'_j + \sqrt{a} \left( \delta^2_i 2\mu_i \delta_i \right) u'_j + \left( \frac{b}{12} - \lambda^2 \right) \delta^2_i u'_j + bu'_j$$

$$+ \frac{\delta^2_i \delta^2_i - u'_j}{12} - \gamma \lambda^2 \delta^2_i \delta^2_i u'_j = \frac{k^2}{12} \left[ f''_{i+1} + f''_{i-1} + 10f'_j \right],$$

where $\eta$ and $\gamma$ are free parameters to be determined. The additional terms are of high-orders and do not affect the accuracy of the scheme.

The exact value $U'_j = u(x_j, t_j)$ satisfies

$$(1 + \eta b^2) \delta^2_i U'_j + \sqrt{a} (2\mu_i \delta_i) U'_j + \sqrt{a} \left( \delta^2_i 2\mu_i \delta_i \right) U'_j + \left( \frac{b}{12} - \lambda^2 \right) \delta^2_i U'_j + bU'_j$$

$$+ \frac{\delta^2_i \delta^2_i - U'_j}{12} - \gamma \lambda^2 \delta^2_i \delta^2_i U'_j = \frac{k^2}{12} \left[ f''_{i+1} + f''_{i-1} + 10f'_j \right] + O(k^4 + k^4h^2 + k^2h^4)$$

(5.5.8)

We assume that there exists an error $\varepsilon'_j = U'_j - u'_j$ at the grid point $(x_j, t_j)$. Subtracting (5.5.7) from (5.5.8), we obtain the error equation

$$(1 + \eta b^2) \delta^2_i \varepsilon'_j + \sqrt{a} (2\mu_i \delta_i) \varepsilon'_j + \sqrt{a} \left( \delta^2_i 2\mu_i \delta_i \right) \varepsilon'_j + \left( \frac{b}{12} - \lambda^2 \right) \delta^2_i \varepsilon'_j$$

$$+ b\varepsilon'_j + \frac{\delta^2_i \delta^2_i - \varepsilon'_j}{12} - \gamma \lambda^2 \delta^2_i \delta^2_i \varepsilon'_j = O(k^4 + k^4h^2 + k^2h^4)$$

(5.5.9)
Now, let us assume \( \epsilon_i^j = \xi^j e^{j0i} \) (where \( \xi = e^{j0} \) such that \( |\xi| = 1 \)) at the grid point \((x_l, t_j)\), where \( \xi \) is in general complex, \( \theta \) is an arbitrary real number, then

\[
\delta_t^2 \epsilon_i^j = \epsilon_{i+1}^j - 2\epsilon_i^j + \epsilon_{i-1}^j = (\xi - 2 + \xi^{-1}) \xi^j e^{j0i}
\]

\[
\delta_t^2 \epsilon_i^j = \epsilon_{i+1}^j - 2\epsilon_i^j + \epsilon_{i-1}^j = \xi^j (e^{j0} - 2 + e^{-j0}) e^{j0i}
\]

\[
(2\mu, \delta_t^2) \epsilon_i^j = \epsilon_{i+1}^j - \epsilon_{i-1}^j = (\xi - \xi^{-1}) \xi^j e^{j0i}
\]

\[
\delta_t^2 (2\mu, \delta_t^2) \epsilon_i^j = \delta_x^2 (\epsilon_{i+1}^j - \epsilon_{i-1}^j)
\]

\[
= (\epsilon_{i+1}^j - 2\epsilon_i^j + \epsilon_{i-1}^j) - (\epsilon_{i+1}^j - 2\epsilon_i^j + \epsilon_{i-1}^j)
\]

\[
= (\xi - \xi^{-1}) \xi^j (e^{j0} - 2 + e^{-j0}) e^{j0i}
\]

\[
\delta_t^2 \delta_x^2 \epsilon_i^j = \delta_x^2 (\epsilon_{i+1}^j - 2\epsilon_i^j + \epsilon_{i-1}^j)
\]

\[
= (\epsilon_{i+1}^j - 2\epsilon_i^j + \epsilon_{i-1}^j) - 2(\epsilon_{i+1}^j - 2\epsilon_i^j + \epsilon_{i-1}^j) + (\epsilon_{i+1}^j - 2\epsilon_i^j + \epsilon_{i-1}^j)
\]

\[
= (\xi - 2 + \xi^{-1}) \xi^j (e^{j0} - 2 + e^{-j0}) e^{j0i}
\]

For stability, we put \( \epsilon_i^j = \xi^j e^{j0i} \) in the homogeneous part of the error equation (5.5.9) and using the above approximations, we get

\[
\left[
(1 + \eta b^2)(\xi - 2 + \xi^{-1}) + \sqrt{a} \left(\xi - \xi^{-1}\right) + \frac{\sqrt{a}}{12} \left(\xi - \xi^{-1}\right)(e^{j0} - 2 + e^{-j0}) + \left(\frac{b}{12} - \lambda^2\right)(e^{j0} - 2 + e^{-j0}) + b
\right]
\]

\[
= 0
\]

\[
\Rightarrow
\left[
(1 + \eta b^2)(\xi - 2 + \xi^{-1}) + \sqrt{a} \left(\xi - \xi^{-1}\right) + \frac{\sqrt{a}}{12} \left(\xi - \xi^{-1}\right)(-4\sin^2 \frac{\theta}{2}) + \left(\frac{b}{12} - \lambda^2\right)(-4\sin^2 \frac{\theta}{2}) + b
\right]
\]

\[
= 0
\]
Hence, the characteristic equation is given by

\[ P_1 \xi^2 + P_2 \xi + P_3 = 0 \] (5.5.10)

where

\[ P_1 = \left( 1 + \eta b^2 + \sqrt{a} - \frac{\sqrt{a}}{3} \sin^2 \left( \frac{\theta}{2} \right) - \frac{1}{3} \sin^2 \left( \frac{\theta}{2} \right) + 4 \gamma \lambda^2 \sin^2 \left( \frac{\theta}{2} \right) \right), \]

\[ P_2 = \left( -2 - 2 \eta b^2 + 4 \left( \lambda^2 - \frac{b}{12} \sin^2 \left( \frac{\theta}{2} \right) + b + \frac{2}{3} \sin^2 \left( \frac{\theta}{2} \right) - 8 \gamma \lambda^2 \sin^2 \left( \frac{\theta}{2} \right) \right) \right), \]

\[ P_3 = \left( 1 + \eta b^2 - \sqrt{a} + \frac{\sqrt{a}}{3} \sin^2 \left( \frac{\theta}{2} \right) - \frac{1}{3} \sin^2 \left( \frac{\theta}{2} \right) + 4 \gamma \lambda^2 \sin^2 \left( \frac{\theta}{2} \right) \right). \]

Using the transformation \( \xi = \frac{1 + z}{1 - z} \), the characteristic equation (5.5.10) reduces to

\[ (P_1 - P_2 + P_3) z^2 + 2(P_1 - P_3) z + (P_1 + P_2 + P_3) = 0 \] (5.5.11)

The necessary and sufficient condition for \(|\xi| < 1\) is that
Thus for stability, we must have the conditions
\[
P_1 + P_2 + P_3 = b \cos^2 \left( \frac{x}{2} \right) + 4 \lambda^2 \sin^2 \left( \frac{x}{2} \right) + \frac{2b}{3} \sin^2 \left( \frac{x}{2} \right) > 0,
\]
\[
P_1 - P_3 = 2a \left( \cos^2 \left( \frac{x}{2} \right) + \frac{2}{3} \sin^2 \left( \frac{x}{2} \right) \right) > 0,
\]
\[
P_1 - P_2 + P_3 = 4 + 4 \eta b^2 - b + \frac{b}{3} \sin^2 \left( \frac{x}{2} \right) + 4 \left( 4 \gamma - 1 \right) \lambda^2 - \frac{1}{3} \right) \sin^2 \left( \frac{x}{2} \right) > 0.
\]
First two conditions are satisfied for all choices of variable angle \( \theta \). Multiplying third condition by \( 16 \eta \), we get
\[
(64 \eta - 1) + (8 \eta b - 1)^2 + \frac{16 \eta b^2}{3} \sin^2 \theta + 64 \eta \left( 4 \gamma - 1 \right) \lambda^2 - \frac{1}{3} \right) \sin^2 \theta > 0 \tag{5.5.13}
\]
Thus the scheme is stable if \( \eta \geq \frac{1}{64} \), \( \gamma \geq \frac{1 + 3 \lambda^2}{12 \lambda^2} \), \( \alpha > 0 \) and \( \beta \geq 0 \) for all \( \theta \) except \( \theta = 0 \) and \( 2\pi \) (when \( b = 0 \)). We treat this case separately.

For \( \theta = 0 \) or \( 2\pi \) and \( b = 0 \), we have the characteristic equation
\[
\left( 1 + \sqrt{a} \right) \xi^2 - 2 \xi + \left( 1 - \sqrt{a} \right) = 0 \tag{5.5.14}
\]
where \( P_1 = 1 + \sqrt{a}, P_2 = -2 \) and \( P_3 = 1 - \sqrt{a} \).

The roots of (5.5.14) are \( \xi_{1,2} = 1, \frac{1 - \sqrt{a}}{1 + \sqrt{a}} \). In this case also \( |\xi| \leq 1 \).

Hence for \( \alpha > 0, \beta \geq 0, \eta \geq \frac{1}{64}, \gamma \geq \frac{1 + 3 \lambda^2}{12 \lambda^2} \), the scheme (5.5.7) is unconditionally stable.

### 5.6. Numerical illustrations

In this section, we have solved some benchmark problems using the method described by equation (5.2.18) and compared our results with those obtained by the numerical method...
of \( O(k^2 + k^2 h^2 + h^4) \) accuracy based on cubic spline approximations for the solution of 1-D non-linear wave equations. The exact solutions are provided in each case. The linear difference equation has been solved using a tri-diagonal solver, whereas non-linear difference equations have been solved using the Newton-Raphson method. While using the Newton-Raphson method, the iterations were stopped when absolute error tolerance \( \leq 10^{-10} \) was achieved. In order to demonstrate the fourth order convergence of the proposed method, throughout the computation we have chosen the fixed value of the parameter \( \sigma = \frac{1}{h^2} = 3.2 \). All computations were carried out using double precision arithmetic.

Note that, the proposed cubic spline method (5.2.18) for second order hyperbolic equations is a three-level scheme. The value of \( u \) at \( t=0 \) is known from the initial condition. To start any computation, it is necessary to know the numerical value of \( u \) of required accuracy at \( t=k \). In this section, we discuss an explicit scheme of \( O(k^2) \) for \( u \) at first time level, i.e., at \( t=k \) in order to solve the differential equation (5.1.1) using the method (5.2.18), which is applicable to problems in cartesian and polar coordinates.

Since the values of \( u \) and \( u_t \) are known explicitly at \( t=0 \), this implies all their successive tangential derivatives are known at \( t=0 \), i.e. the values of \( u, u_x, u_{xx}, \ldots, u_t, u_{tx}, \ldots, \) etc. are known at \( t=0 \).

An approximation for \( u \) of \( O(k^2) \) at \( t=k \) may be written as
\[
\begin{align*}
u^i_l &= u^0_l + ku^0_t + \frac{k^2}{2}(u^0_{xx})^i_l + O(k^3) \tag{5.6.1}\end{align*}
\]
From equation (5.1.1), we have
\[
(u^0_n)^i_l = [u_{xx} + g(x,t,u,u_x,u_{xx},u_{tx})]^0_l \tag{5.6.2}
\]
Thus using the initial values and their successive tangential derivative values, from (5.6.2) we can obtain the value of \( (u_n^0)^i_l \), and then ultimately, from (5.6.1) we can compute the value of \( u \) at first time level, i.e. at \( t=k \). Replacing the variable \( x \) by \( r \) in (5.6.1), we can also obtain an approximation of \( O(k^2) \) for \( u \) at \( t=k \) in polar coordinates.
Example 5.6.1: (Wave equation in polar coordinates)

\[
\frac{\partial^2 u}{\partial t^2} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial r} \left( \frac{\partial u}{\partial r} \right) - \left( 2 \cosh r + \frac{\gamma}{r} \sinh r \right) \sin t, \quad 0 < r < 1, \quad t > 0.
\]

The initial and boundary conditions are given by

\[
\begin{align*}
  u(r,0) &= 0, \quad u_t(r,0) = \cosh r, \quad 0 \leq r \leq 1, \\
  u(0,t) &= \sin t, \quad u(1,t) = \frac{1}{2} (e + e^{-1}) \sin t, \quad t \geq 0.
\end{align*}
\]

The exact solution is given by \( u(r,t) = \cosh r \cdot \sin t \). The maximum absolute errors (MAE) are tabulated in Table-5.6.1 at \( t = 1.0, t = 2.0 \) and for \( \gamma = 1, \gamma = 2 \). The exact and numerical solution curves using proposed method are plotted in Fig 5.6.1a and Fig 5.6.1b at \( t = 2 \) for \( \gamma = 1 \) and \( \gamma = 2 \), respectively. Also, the graph of the analytical and the numerical solution are given in Fig. 5.6.1a and Fig. 5.6.1b at \( \gamma = 1 \) and \( \gamma = 2 \) for \( t = 2 \), respectively.

**Table-5.6.1**

<table>
<thead>
<tr>
<th>( h )</th>
<th>( O(k^2 + k^2h^2 + h^4) ) -method</th>
<th>( O(k^2 + k^2h^2 + h^4) ) -method</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma = 1 )</td>
<td>( \gamma = 2 )</td>
<td>( \gamma = 1 )</td>
</tr>
<tr>
<td>( t = 1 )</td>
<td>( t = 2 )</td>
<td>( t = 1 )</td>
</tr>
<tr>
<td>( \frac{1}{8} )</td>
<td>.2324(-04)</td>
<td>.3786(-04)</td>
</tr>
<tr>
<td>( \frac{1}{16} )</td>
<td>.1378(-05)</td>
<td>.2359(-05)</td>
</tr>
<tr>
<td>( \frac{1}{32} )</td>
<td>.8163(-07)</td>
<td>.1443(-06)</td>
</tr>
<tr>
<td>( \frac{1}{64} )</td>
<td>.4828(-08)</td>
<td>.8679(-08)</td>
</tr>
</tbody>
</table>
Fig. 5.6.1a: Exact and numerical solution for $\gamma = 1$ at $t=2$

Fig. 5.6.1b: Exact and numerical solution for $\gamma = 2$ at $t=2$
Example 5.6.2: (Telegraphic wave equation)

\[
\frac{\partial^2 u}{\partial t^2} + 2\alpha \frac{\partial u}{\partial t} + \beta^2 u = \frac{\partial^2 u}{\partial x^2} + (3 + \beta^2 - 4\alpha)e^{-2t}\sinh x, \quad 0 < x < 1, \quad t > 0.
\]

The initial and boundary conditions are given by

\[
\begin{align*}
&u(x,0) = \sinh x, \quad u_t(x,0) = -2\sinh x, \quad 0 \leq x \leq 1, \\
&u(0,t) = 0, \quad u(1,t) = \frac{1}{2}(e^{-t} - e^{-t})e^{-2t}, \quad t \geq 0.
\end{align*}
\]

The exact solution is given by \( u(x,t) = e^{-2t}\sinh x \). The maximum absolute errors are tabulated in Table-5.6.2 at \( t = 2.0 \) for various values of \( \alpha > 0, \beta \geq 0 \). Also, the graph of the analytical and the numerical solution are given in Fig.5.6.2a and Fig.5.6.2b at \( \alpha = 10, \beta = 5, \eta = 0.5, \gamma = 1 \) for \( t = 2.0 \) and at \( \alpha = 20, \beta = 10, \eta = 1, \gamma = 1 \) for \( t = 2 \).

\begin{center}
Table-5.6.2
\end{center}

Example 5.6.2: The Maximum Absolute Error

<table>
<thead>
<tr>
<th>( h )</th>
<th>( \alpha = 10, \beta = 5, \eta = 0.5, \gamma = 1 )</th>
<th>( \alpha = 20, \beta = 10, \eta = 1, \gamma = 1 )</th>
<th>( \alpha = 40, \beta = 4, \eta = 10, \gamma = 20 )</th>
<th>( \alpha = 50, \beta = 5, \eta = 0.25, \gamma = 0.75 )</th>
<th>( \alpha = 50, \beta = 2, \eta = 10, \gamma = 5 )</th>
<th>( \alpha = 10, \beta = 0, \eta = 5, \gamma = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{8} )</td>
<td>0.1036(-03) *0.8675(-03)</td>
<td>0.3693(-04) *0.3930(-02)</td>
<td>0.1077(-02) *0.7688(-02)</td>
<td>0.7157(-03) *0.2929(-01)</td>
<td>0.1092(-02) *0.3633(-01)</td>
<td>0.8913(-03) *0.8500(-01)</td>
</tr>
<tr>
<td>( \frac{1}{16} )</td>
<td>0.7303(-05) *0.1938(-03)</td>
<td>0.2652(-05) *0.7960(-03)</td>
<td>0.8153(-04) *0.1535(-02)</td>
<td>0.4704(-04) *0.5937(-02)</td>
<td>0.7214(-04) *0.8827(-02)</td>
<td>0.5783(-04) *0.1003(-01)</td>
</tr>
<tr>
<td>( \frac{1}{32} )</td>
<td>0.4619(-06) *0.7548(-04)</td>
<td>0.1858(-06) *0.2606(-04)</td>
<td>0.5175(-05) *0.7485(-03)</td>
<td>0.2978(-05) *0.1800(-02)</td>
<td>0.4563(-05) *0.2652(-02)</td>
<td>0.3640(-05) *0.2523(-02)</td>
</tr>
<tr>
<td>( \frac{1}{64} )</td>
<td>0.2890(-07) *0.2054(-04)</td>
<td>0.1169(-07) *0.4582(-05)</td>
<td>0.3242(-06) *0.2018(-03)</td>
<td>0.1864(-06) *0.4826(-03)</td>
<td>0.2860(-06) *0.7276(-03)</td>
<td>0.2280(-06) *0.6434(-03)</td>
</tr>
</tbody>
</table>

* Result obtained by Mohanty [109].
Fig. 5.6.2a: Exact and numerical solution at $\alpha = 10, \beta = 5, \eta = 0.5, \gamma = 1$ for $t = 2$

Fig. 5.6.2b: Exact and numerical solution at $\alpha = 10, \beta = 10, \eta = 1, \gamma = 1$ for $t = 2$
**Example 5.6.3:** (Dissipative non-linear wave equation)

\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - 2u \frac{\partial u}{\partial t} + (\pi^2 - 1 - 2 \sin(\pi x) \sin t) \sin(\pi x) \cos t , \quad 0 < x < 1, \quad t > 0.
\]

The initial and boundary conditions are, respectively, given by

\[
u(x,0) = \sin(\pi x), \quad u_t(x,0) = 0, \quad 0 \leq x \leq 1,
\]
\[
u(0,t) = 0, \quad u(1,t) = 0, \quad t \geq 0.
\]

The exact solution is given by \(u(x,t) = \sin(\pi x) \cos t\). The maximum absolute errors are tabulated in Table-5.6.3 at \(t = 1.0\) and \(t = 2.0\). Also, the graph of the analytical and the numerical solution are given in Fig.5.6.3a and Fig.5.6.3b at \(t = 1.0\) and \(t = 2.0\).

**Table-5.6.3**

<table>
<thead>
<tr>
<th>(h)</th>
<th>(O(k^2 + k^2h^2 + h^4))-method</th>
<th>(O(k^2 + k^2h^2 + h^4))-method</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(t=1)</td>
<td>(t=2)</td>
</tr>
<tr>
<td>(\frac{1}{8})</td>
<td>0.6138(-04)</td>
<td>0.5655(-04)</td>
</tr>
<tr>
<td>(\frac{1}{16})</td>
<td>0.3820(-05)</td>
<td>0.3543(-05)</td>
</tr>
<tr>
<td>(\frac{1}{32})</td>
<td>0.2383(-06)</td>
<td>0.2214(-06)</td>
</tr>
<tr>
<td>(\frac{1}{64})</td>
<td>0.1445(-07)</td>
<td>0.1378(-07)</td>
</tr>
</tbody>
</table>
Fig 5.6.3a: The analytic and numerical solution for Example 5.6.3 for $t = 1$

Fig 5.6.3b: The analytic and numerical solution for Example 5.6.3 for $t = 2$
Example 5.6.4: (Vander Pol type non-linear wave equation)

\[
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \gamma(u^2 - 1) \frac{\partial u}{\partial t} + (\pi^2 + \gamma^2 e^{-2\gamma x} \sin^2(\pi x))e^{-\gamma x}\sin(\pi x), \quad 0 < x < 1, \quad t > 0.
\]

The initial and boundary conditions are given by

\[
\begin{align*}
    u(x,0) &= \sin(\pi x), & u_t(x,0) &= -\gamma \sin(\pi x), & 0 \leq x \leq 1, \\
    u(0,t) &= 0, & u(1,t) &= 0, & t \geq 0.
\end{align*}
\]

The exact solution is given by \( u(x,t) = e^{-\gamma x}\sin(\pi x) \). The maximum absolute errors are tabulated in Table-5.6.4 at \( t = 2.0 \) for \( \gamma = 1, 2 \) and 3. Also, the graph of the analytical and the numerical solution are given in Fig.5.6.2a and Fig.5.6.2b at \( \gamma = 1 \) for \( t = 2.0 \) and \( \gamma = 3 \) for \( t = 2.0 \), respectively.

**Table-5.6.4**

<table>
<thead>
<tr>
<th>( h )</th>
<th>( \gamma = 1 )</th>
<th>( \gamma = 2 )</th>
<th>( \gamma = 3 )</th>
<th>( \gamma = 1 )</th>
<th>( \gamma = 2 )</th>
<th>( \gamma = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{8} )</td>
<td>0.2323(-04)</td>
<td>0.2381(-04)</td>
<td>0.4279(-04)</td>
<td>0.3390(-02)</td>
<td>0.1580(-02)</td>
<td>0.6629(-03)</td>
</tr>
<tr>
<td>( \frac{1}{16} )</td>
<td>0.1446(-05)</td>
<td>0.1461(-05)</td>
<td>0.2551(-05)</td>
<td>0.8434(-03)</td>
<td>0.3986(-03)</td>
<td>0.1662(-03)</td>
</tr>
<tr>
<td>( \frac{1}{32} )</td>
<td>0.9034(-07)</td>
<td>0.9106(-07)</td>
<td>0.1575(-06)</td>
<td>0.2103(-03)</td>
<td>0.9976(-04)</td>
<td>0.4156(-04)</td>
</tr>
<tr>
<td>( \frac{1}{64} )</td>
<td>0.5417(-08)</td>
<td>0.5831(-08)</td>
<td>0.1002(-07)</td>
<td>0.5254(-04)</td>
<td>0.2494(-04)</td>
<td>0.1039(-04)</td>
</tr>
</tbody>
</table>
Fig. 5.6.4a: Analytical and numerical solution at $\gamma = 1$ for $t = 2$

Fig. 5.6.4b: Analytical and numerical solutions at $\gamma = 3$ for $t = 2$
The order of convergence may be obtained by using the formula

\[
\frac{\log(e_{h_1}) - \log(e_{h_2})}{\log(h_1) - \log(h_2)}
\]

where \( e_{h_1} \) and \( e_{h_2} \) are maximum absolute errors for two uniform mesh widths \( h_1 \) and \( h_2 \), respectively. For computation of order of convergence of the proposed method, we have considered \( h_1 = \frac{1}{32} \) and \( h_2 = \frac{1}{64} \) for all cases and results are reported in Table-5.6.5.

**Table-5.6.5**

Order of convergence: \( h_1 = \frac{1}{32}, h_2 = \frac{1}{64} \)

<table>
<thead>
<tr>
<th>Example</th>
<th>Parameters</th>
<th>Order of the method</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.6.1</td>
<td>( \gamma = 1 ) at t=1</td>
<td>4.07</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 1 ) at t=2</td>
<td>4.05</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 2 ) at t=1</td>
<td>4.00</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 2 ) at t=2</td>
<td>4.05</td>
</tr>
<tr>
<td>5.6.2</td>
<td>( \alpha = 10, \beta = 5, \eta = 0.5, \gamma = 1 ) at t=2</td>
<td>4.00</td>
</tr>
<tr>
<td></td>
<td>( \alpha = 20, \beta = 10, \eta = 1, \gamma = 1 ) at t=2</td>
<td>3.99</td>
</tr>
<tr>
<td></td>
<td>( \alpha = 40, \beta = 4, \eta = 10, \gamma = 20 ) at t=2</td>
<td>4.00</td>
</tr>
<tr>
<td></td>
<td>( \alpha = 50, \beta = 5, \eta = 0.25, \gamma = 0.75 ) at t=2</td>
<td>4.00</td>
</tr>
<tr>
<td></td>
<td>( \alpha = 50, \beta = 2, \eta = 10, \gamma = 5 ) at t=2</td>
<td>4.00</td>
</tr>
<tr>
<td></td>
<td>( \alpha = 10, \beta = 0, \eta = 5, \gamma = 5 ) at t=2</td>
<td>4.00</td>
</tr>
<tr>
<td>5.6.3</td>
<td>at t=1</td>
<td>4.04</td>
</tr>
<tr>
<td></td>
<td>at t=2</td>
<td>4.00</td>
</tr>
<tr>
<td>5.6.4</td>
<td>( \gamma = 1 ) at t=2</td>
<td>4.05</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 2 ) at t=2</td>
<td>3.97</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 3 ) at t=2</td>
<td>3.97</td>
</tr>
</tbody>
</table>
5.7. Final remarks

Available numerical methods based on cubic spline approximations for the numerical solution of non-linear wave equations are of $O(k^2 + k^2 h^2 + h^2)$ accuracy only and require 9-grid points. In this chapter, using the same number of grid points and three evaluations of the function $g$ (as compared to five and nine evaluations of the function $g$ discussed in [101] and [107]), we have derived a new stable cubic spline finite difference method of $O(k^2 + k^2 h^2 + h^2)$ accuracy for the solution of non-linear wave equation (5.1.1). For a fixed parameter $\sigma = \frac{k}{h^2}$, the proposed method behaves like a fourth order method, which is exhibited from the computed results. The proposed numerical method for the wave equation in polar coordinates is conditionally stable, whereas for the damped wave equation and telegraphic equation the method is shown to be unconditionally stable. From the Table-5.6.5, we found that the order of the method is nearly equal to four.