Chapter-2

An Off-step Discretization for the Solution of 1-D Non-linear Wave Equations with Variable Coefficients*

2.1. Introduction

Consider the one-space dimensional nonlinear hyperbolic partial differential equation

\[
\frac{\partial^2 u}{\partial t^2} = A(x,t) \frac{\partial^2 u}{\partial x^2} + g\left(x,t,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial t}\right), \quad 0 < x < 1, \ t > 0
\]  

subject to the initial conditions

\[
u(x,0) = \varphi(x), \quad \frac{\partial u(x,0)}{\partial t} = \psi(x), \quad 0 \leq x \leq 1
\]  

and the Dirichlet boundary conditions

\[
u(0,t) = a(t), \quad \nu(1,t) = b(t), \quad t \geq 0
\]  

where the equation (2.1.1) is assumed to satisfy the hyperbolicity condition \( A(x, t) > 0 \) in the solution region \( \Omega \equiv \{(x,t): 0 < x < 1, t > 0\} \). Further we assume that \( u(x,t) \in C^6(\Omega) \), \( A(x,t) \in C^4(\Omega) \) and \( \varphi(x) \) and \( \psi(x) \) are sufficiently differentiable function of as higher order as possible.

In this chapter, we derive a new three-level compact finite difference method of $O(k^2 + k^2h^2 + h^4)$ based on arithmetic average discretization given by Chawla and Shivkumar [25] for the numerical solution of the differential equation (2.1.1) subject to the initial conditions (2.1.2) and the Dirichlet boundary conditions (2.1.3).

The numerical solution of one space dimensional second order non-linear hyperbolic equation plays an important role in many fields of engineering and sciences. Using boundary value technique Greenspan [55] has obtained approximate solution of wave equation. Later, Ciment and Leventhal [29, 30] have discussed operator compact implicit method to solve wave equation. It is well known that any explicit method for the wave equation is conditionally stable. Twizell [180] has derived a new explicit difference method for the wave equation with extended stability range. In the recent past, many researchers [39, 43, 46, 54, 91, 101, 109, 125, 133] have developed stable implicit finite difference methods for the solution of one space dimensional linear hyperbolic equation with significant first derivative terms. Using only three grid points, Chawla and Shivkumar [25] have originally developed arithmetic average discretization for the solution of two point non-linear boundary value problems. Using this method, one can solve singular problems, whether linear or non-linear, directly (without any modification in the original method) even in the vicinity of the singularity. Using their technique, later, Mohanty et al [121, 125] have developed arithmetic average discretizations for the solution of nonlinear elliptic and parabolic partial differential equations. In this Chapter, using nine-grid points (Fig. 2.1.1), we discuss a new three-level implicit scheme of order two in time and four in space based on arithmetic average discretization for the solution

![Schematic representation of three-level implicit scheme](image-url)
of non-linear hyperbolic equation (2.1.1). In this method we require only three evaluations of the function \( g \). In next section 2.3, we give the complete derivation of the method. In section 2.4, we see the application of the proposed method to the wave equation in polar coordinates, and discuss their stability analysis. In section 2.5, we apply the proposed method to one dimensional telegraphic equation and discuss the error analysis. In section 2.6, we examine our difference method over a set of linear and nonlinear second order hyperbolic equations whose exact solutions are known and compared the results with the results of other known methods. Concluding remarks are given in section 2.7.

## 2.2 Finite difference method based on off-step discretization

Let \( h > 0 \) and \( k > 0 \) be the mesh spacing in the space and time directions respectively, so that \( x_l = lh, \ l=0(1)N+1 \) and \( t_j = jk, \ 0<j<J, \ N \) and \( J \) being positive integers and \((N+1)h=1\). Let \( \lambda = \frac{h}{k} > 0 \) be the mesh ratio parameter. We replace the region \( \Omega \) by a set of grid points \((x_l, t_j)\). The exact values of \( A(x,t) \), \( A_x(x,t) \), \( A_{xx}(x,t) \), etc. at the grid point \((x_l, t_j)\), are denoted by \( A^j_l = A(x_l, t_j) \), \( A_{x}^{j} \), \( A_{xx}^{j} \), etc., respectively. Let \( U^j_l \) and \( u^j_l \) be the exact and the approximate solution of \( u(x,t) \) at the same grid point, respectively.

By Taylor series expansion of the function \( U(x,t) \) about the grid point \((x_l, t_j)\), we have

\[
U^j_{l+1} = U^j_l \pm h U^j_{x_l} + \frac{h^2}{2} U^j_{xx_{l}} \pm \frac{h^3}{6} U^j_{xxx_{l}} + O(h^4), \tag{2.2.1a}
\]

\[
U^j_{l-1} = U^j_l \pm k U^j_{t_l} + \frac{k^2}{2} U^j_{tt_{l}} \pm \frac{k^3}{6} U^j_{ttx_{l}} + O(k^4). \tag{2.2.1b}
\]

Using (2.2.1a) and (2.2.1b), we get the following

\[
U^j_{l+1} = \frac{1}{2} (U^j_{l+1} + U^j_l) + O(h^2) \approx \frac{1}{2} (U^j_{l+1} + U^j_l),
\]

\[
U^j_{l-1} = \frac{1}{2h} (U^j_{l+1} - U^j_{l-1}) + O(h^2) \approx \frac{1}{2h} (U^j_{l+1} - U^j_{l-1}),
\]

\[
U^j_{xx} = \frac{1}{h^2} (U^j_{l+1} - 2U^j_l + U^j_{l-1}) + O(h^2) \approx \frac{1}{h^2} (U^j_{l+1} - 2U^j_l + U^j_{l-1}),
\]

\[
U^j_{tt} = \frac{1}{k^2} (U^j_{l+1} - 2U^j_l + U^j_{l-1}) + O(k^2) \approx \frac{1}{k^2} (U^j_{l+1} - 2U^j_l + U^j_{l-1}),
\]

and so on.
Hence, at the grid point \((x_j, t_j)\), we consider the following approximations:

\[
\begin{align*}
\overline{U}_{x_{i+\frac{1}{2}}}^j &= \frac{1}{2} (U_{i+1}^j + U_i^j), \quad (2.2.2a) \\
\overline{U}_{t_{j+\frac{1}{2}}}^i &= \frac{1}{2} (U_{i+1}^j + U_i^j), \quad (2.2.2b) \\
\overline{U}_{x_{i+1}}^j &= \frac{1}{h} (U_{i+1}^j - U_i^j), \quad (2.2.3a) \\
\overline{U}_{t_{j+1}}^i &= \frac{1}{h} (U_{i+1}^j - U_i^j), \quad (2.2.3b) \\
\overline{U}_{x_{i+\frac{1}{2}}}^j &= \frac{1}{h} (U_{i+1}^j - U_i^j), \quad (2.2.3c) \\
\overline{U}_{x_{i+1}}^j &= \frac{1}{h} (U_{i+1}^j - 2U_i^j + U_{i-1}^j), \quad (2.2.3d) \\
\overline{U}_{t_{j+1}}^i &= \frac{1}{h} (U_{i+1}^j - U_i^j), \quad (2.2.4a) \\
\overline{U}_{x_{i+1}}^j &= \frac{1}{h} (U_{i+1}^j - U_i^j), \quad (2.2.4b) \\
\overline{U}_{t_{j+1}}^i &= \frac{1}{h} (U_{i+1}^j - U_i^j), \quad (2.2.4c) \\
\overline{U}_{x_{i+\frac{1}{2}}}^j &= \frac{1}{h} (U_{i+1}^j + U_i^j - U_{i+1}^{j+1}), \quad (2.2.4d) \\
\overline{U}_{x_{i+1}}^j &= \frac{1}{h} (U_{i+1}^j + U_i^j - U_i^{j+1}), \quad (2.2.4e) \\
\overline{U}_{t_{j+1}}^i &= \frac{1}{h} (U_{i+1}^j - 2U_i^j + U_{i-1}^j), \quad (2.2.5a) \\
\overline{U}_{x_{i+1}}^j &= \frac{1}{h} (U_{i+1}^j - 2U_i^j + U_{i+1}^j), \quad (2.2.5b) \\
\overline{U}_{t_{j+1}}^i &= \frac{1}{h} (U_{i-1}^j - 2U_i^j + U_{i-1}^j). \quad (2.2.5c)
\end{align*}
\]

Next, we define

\[
\begin{align*}
\overline{G}_{x_{i+\frac{1}{2}}} &= g(x_{i+\frac{1}{2}}, t_j, \overline{U}_{x_{i+\frac{1}{2}}}^j, \overline{U}_{t_{j+\frac{1}{2}}}^i, \overline{U}_{x_{i+\frac{1}{2}}}^j, \overline{U}_{t_{j+\frac{1}{2}}}^i), \quad (2.2.6) \\
\overline{G}_{t_{j+\frac{1}{2}}} &= g(x_{i+\frac{1}{2}}, t_j, \overline{U}_{x_{i+\frac{1}{2}}}^j, \overline{U}_{t_{j+\frac{1}{2}}}^i, \overline{U}_{x_{i+\frac{1}{2}}}^j, \overline{U}_{t_{j+\frac{1}{2}}}^i). \quad (2.2.7)
\end{align*}
\]

Further, we need the following combinations

\[
\begin{align*}
\widehat{U}_i^j &= U_i^j - \frac{h^2}{4} U_{xx}^j, \quad (2.2.8a)
\end{align*}
\]
\[ \hat{U}_{i/l} = \tilde{U}_{i/l} + \frac{h}{4A_l} (\tilde{G}_{i+1/l} - \tilde{G}_{i-1/l}) - \frac{h}{8A_l} (\tilde{U}_{i+1/l} - \tilde{U}_{i-1/l}) + \frac{A_l^2 h^2}{4A_l^2} \tilde{U}_{xx/l}, \]  
(2.2.8b)

\[ \hat{U}_{i/l} = \tilde{U}_{i/l} - \frac{1}{4} (\tilde{U}_{i+1/l} - 2\tilde{U}_{i/l} + \tilde{U}_{i-1/l}). \]  
(2.2.8c)

Finally, let

\[ \tilde{G}_l = g(x_i, t_j, \hat{U}_l, \hat{U}_{x/l}, \hat{U}_{xx/l}). \]  
(2.2.9)

Now using the technique given by Chawla and Shivkumar [25] an arithmetic average discretization for the solution of the nonlinear hyperbolic differential equation (2.1.1) may be written as

\[
\lambda^2 \left[ A_l' - \frac{h^2}{6} \left( \frac{A_{x/l}}{A_l'} \right) A_{x/l} + \frac{h^2}{12} A_{xx/l} \right] (U_{i+1/l} - 2U_{i/l} + U_{i-1/l}) \\
= \frac{k^2}{12} \left[ \left( 1 - \frac{hA_{x/l}}{A_l'} \right) \tilde{U}_{n+1/l} + \left( 1 + \frac{hA_{x/l}}{A_l'} \right) \tilde{U}_{n-1/l} + 10\tilde{U}_{n/l} \right] \\
- \frac{k^2}{3} \left[ \left( 1 - \frac{hA_{x/l}}{2A_l'} \right) \tilde{G}_{i+1/2} + \left( 1 + \frac{hA_{x/l}}{2A_l'} \right) \tilde{G}_{i-1/2} + \tilde{G}_l \right] + \hat{T}_l, \quad l = 1(1)N; \ j = 0,1,2,...  
\]  
(2.2.10)

where

\[ \hat{T}_l = O(k^4 + k^2 h^2 + k^2 h^4). \]

### 2.3. Derivation of the method

At the grid point \((x_i, t_j)\), we may write the differential equation (2.1.1) as

\[ U_{i/l} - A_l/U_{xx/l} = g(x_i, t_j, U_{i/l}, U_{x/l}, U_{xx/l}) \equiv G_l, \text{(say)} \]  
(2.3.1)

and at the same point \((x_i, t_j)\), let us denote

\[ \alpha_l' = \left( \frac{\partial g}{\partial U_l} \right)_l, \quad \beta_l' = \left( \frac{\partial g}{\partial U_x} \right)_l, \quad \gamma_l' = \left( \frac{\partial g}{\partial U_{xx/l}} \right)_l, \quad U_{pq} = \frac{\partial^p g}{\partial x^p \partial t^q}. \]  
(2.3.2)
By Taylor series expansion, we obtain

\[\lambda^2 \left[ A_i^j - \frac{h^2}{6} \left( \frac{A_{il}}{A_i^j} \right) A_{il}^j + \frac{h^2}{12} A_{ii}^j \right] \left( U_{i+1}^j - 2U_i^j + U_{i-1}^j \right)\]

\[= \frac{k^2}{12} \left[ 1 - \frac{hA_{il}}{A_i^j} \right] U_{li+1}^j + \left[ 1 + \frac{hA_{il}}{A_i^j} \right] U_{li-1}^j + 10U_{li}^j \]

\[-\frac{k^2}{3} \left[ 1 - \frac{hA_{il}}{2A_i^j} \right] G_{li+1}^j + \left[ 1 + \frac{hA_{il}}{2A_i^j} \right] G_{li-1}^j + O(k^4 + k^4h^2 + k^2h^4), \quad (2.3.3)\]

Now simplifying the approximations (2.2.2a) to (2.2.5c) defined in section (2.2), we obtain

\[\overline{U}_{l+1}^{i+1} = U_{i+1}^{j+1} + \frac{h^2}{8} U_{20} + O(h^3), \quad (2.3.4a)\]

\[\overline{U}_{l-1}^{i+1} = U_{i-1}^{j+1} + \frac{h^2}{8} U_{20} - O(h^3), \quad (2.3.4b)\]

\[\overline{U}_{li}^{i+1} = U_{li}^{j+1} + \frac{h^2}{4} U_{30} + O(h^3), \quad (2.3.5a)\]

\[\overline{U}_{li}^{i-1} = U_{li}^{j-1} + \frac{h^2}{4} U_{30} - O(h^3), \quad (2.3.5b)\]

\[-k^2 \left[ 1 - \frac{hA_{il}}{2A_i^j} \right] G_{li+1}^j + \left[ 1 + \frac{hA_{il}}{2A_i^j} \right] G_{li-1}^j + O(k^4 + k^4h^2 + k^2h^4), \quad (2.3.3)\]

\[l = 1, 2, \ldots, N; j = 0, 1, 2, \ldots. \]
\[
\overline{U}_{n,i} = U_{n,i} + \frac{h^3}{12} U_{04} - \frac{h^3}{12} U_{14} + \frac{h^5}{24} U_{24} - O(h^7k^2). \quad (2.3.7c)
\]

By the help of the approximations (2.3.4a), (2.3.5b) and (2.3.6d), we can write (2.2.6) as

\[
\overline{G}_{i,j} = g \left( x_{i-1,j}, t_j, U_{i-1,j} + \frac{k^2}{8} U_{20} + O(h^3), U_{i,j} + \frac{k^2}{24} U_{30} + O(h^3), \right.
\]
\[
 \left. \left( U_{i+1,j} + \frac{k^2}{6} U_{03} + \frac{k^2}{6} U_{21} + O(h^3 + h^2 + h^4) \right) \right)
\]
\[
= g \left( x_{i-1,j}, t_j, U_{i-1,j} + U_{i,j} + U_{i+1,j} \right) + \left( \frac{k^2}{8} U_{20} + O(h^3) \right) \alpha_{i,j}^{(1)}
\]
\[
+ \left( \frac{k^2}{24} U_{30} + O(h^3) \right) \beta_{i,j}^{(1)} + \left( \frac{k^2}{6} U_{21} + O(h^3) \right) \gamma_{i,j}^{(1)} + O(k^2 + h^3 + h^2 + h^4)
\]
\[
= G_{i,j} + \frac{k^2}{24} T_i + O(k^2 + h^2 + h^3 + h^4), \quad (2.3.8a)
\]

where we have used
\[
\alpha_{i,j}^{(1)} = \alpha^{(1)} + O(h),
\]
\[
\beta_{i,j}^{(1)} = \beta^{(1)} + O(h),
\]
\[
\gamma_{i,j}^{(1)} = \gamma^{(1)} + O(h),
\]

and
\[
T_i = 3U_{20}\alpha_{i}^{(1)} + U_{30}\beta_{i}^{(1)} + 3U_{21}\gamma_{i}^{(1)}.
\]

Similarly,
\[
\overline{G}_{i-1,j} = G_{i-1,j} + \frac{k^2}{24} T_i + O(k^2 - h^2 - h^3 + h^4). \quad (2.3.8b)
\]

Now, let
\[
\hat{U}_{i} = U_{i} + a_i h^2 \overline{U}_{i+1}, \quad (2.3.9a)
\]
\[
\hat{U}_{i+1} = \overline{U}_{i+1} + b_i h(\overline{G}_{i+1} - \overline{G}_{i-1}) + b_i h(\overline{U}_{n+1} - \overline{U}_{n-1}) + b_i h^2 \overline{U}_{i+1}, \quad (2.3.9b)
\]
\[
\hat{U}_{i} = \overline{U}_{i} + c_i (\overline{U}_{i+1} - 2\overline{U}_{i} + \overline{U}_{i-1}), \quad (2.3.9c)
\]

where \(a_i, b_i, b_i, b_i, c_i\) are unknown parameters to be determined.
Using (2.3.5d), the approximation (2.3.9a) can be written as

\[
\hat{U}_i^j = U_i^j + a_i h^2 \left( U_{x_i^j} + \frac{k_i^2}{24} U_{40} + O(h^4) \right)
\]

\[
= U_i^j + a_i h^2 U_{20} + O(h^4).
\]

Also,

\[
\bar{G}_{i+\frac{1}{2}} - \bar{G}_{i-\frac{1}{2}} = \left( G_{i+\frac{1}{2}} + \frac{k_i^2}{24} T_i + O(k^2 + h k^2 + h^3 + h^4) \right)
\]

\[
- \left( G_{i-\frac{1}{2}} + \frac{k_i^2}{24} T_i + O(k^2 - h k^2 - h^3 + h^4) \right)
\]

\[
= G_{i+\frac{1}{2}} - G_{i-\frac{1}{2}} + O(h k^2 + h^3)
\]

\[
= h \left( U_{12} - A_i U_{30} - A_i U_{20} \right) + O(h k^2 + h^3) \quad (\therefore \text{G}_i^j = U_{12} - A_i U_{20}), \quad (2.3.11a)
\]

\[
\bar{U}_{i+1}^j - \bar{U}_{i-1}^j = \left( U_{i+1}^j + \frac{k_i^2}{12} U_{10} + \frac{4k_i^2}{12} U_{14} + \frac{k_i^2}{24} U_{24} + O(h^3 k^2) \right)
\]

\[
- \left( U_{i-1}^j + \frac{k_i^2}{12} U_{10} - \frac{4k_i^2}{12} U_{14} + \frac{k_i^2}{24} U_{24} - O(h^3 k^2) \right)
\]

\[
= U_{i+1}^j - U_{i-1}^j + \frac{h k_i^2}{6} U_{14} + O(h^3 k^2)
\]

\[
= 2h U_{12} + \frac{h k_i^2}{3} U_{32} + \frac{h k_i^2}{6} U_{14} + O(h^3 k^2). \quad (2.3.11b)
\]

Using (2.3.5a), (2.3.5d), (2.11a) and (2.11b) in the approximation (2.3.9b), we obtain

\[
\hat{U}_{x_i^j} = \left( U_{x_i^j} + \frac{k_i^2}{6} U_{30} + O(h^4) \right) + b_i h \left( U_{12} - A_i U_{30} - A_i U_{20} \right) + O(h k^2 + h^3)
\]

\[
+ b_i h \left( 2h U_{12} + \frac{k_i^2}{3} U_{32} + \frac{h k_i^2}{6} U_{14} + O(h^3 k^2) \right) + b_i h^2 \left( U_{x_i^j} + \frac{k_i^2}{12} U_{40} + O(h^4) \right)
\]

\[
= U_{x_i^j} + \frac{k_i^2}{6} U_{30} + b_i h^2 \left( U_{12} - A_i U_{30} - A_i U_{20} \right) + 2b_i h^2 U_{12} + b_i h^2 U_{20} + O(k^2 + k^2 h^2 + h^4)
\]

\[
= U_{x_i^j} + \frac{k_i^2}{6} T_2 + O(k^2 + k^2 h^2 + h^4), \quad (2.3.12)
\]

where

\[
T_2 = (1 - 6b_i A_i^j) U_{30} + 6(b_1 + 2b_2) U_{12} + 6(b_3 - b_1 A_i^j) U_{20}.
\]
Using (2.3.7a)-(2.3.7c) in (2.3.9c), we obtain

\[ \hat{U}_{l+} = U_{l+} + c_i h^2 U_{21} + O(k^2 + k^2 h^2 + h^4). \]  

(2.3.13)

Finally, by the help of (2.3.10), (2.3.12)-(2.3.13) and (2.3.2), the approximation (2.2.9) becomes

\[
\hat{G}_l = g \left( x_i, t_j, U_{l+} + a_i h^2 U_{20} + O(h^4), U_{l+} + \frac{k^2}{6} T_2 + O(k^2 + k^2 h^2 + h^4),
\right.
\]

\[
\left. U_{l+} + c_i h^2 U_{21} + O(k^2 + k^2 h^2 + h^4) \right)
\]

\[
= g \left( x_i, t_j, U_{l+}, U_{l+} \right) + a_i h^2 U_{20} \alpha_i^l
\]

\[
+ \frac{k^2}{6} T_2 \beta_i^l + c_i h^2 U_{21} \gamma_i^l + O(k^2 + h^3 + h^2 + h^4)
\]

\[
= G_i^l + \frac{k^2}{6} T_3 + O(k^2 + k^2 h^2 + h^4),
\]

(2.3.14)

where

\[ T_3 = 6a_i U_{20} \alpha_i^l + T_2 \beta_i^l + 6c_i U_{21} \gamma_i^l. \]

Now, from (2.2.10) and (2.3.3), we have

\[
- \frac{k^2}{3} \left[ \left( 1 - \frac{h A_i^l}{2 A_i^l} \right) \bar{G}_{l+} + \left( 1 + \frac{h A_i^l}{2 A_i^l} \right) \bar{G}_{l+} + \hat{G}_l \right] + \hat{T}_i
\]

\[
= - \frac{k^2}{3} \left[ \left( 1 - \frac{h A_i^l}{2 A_i^l} \right) G_{l+} + \left( 1 + \frac{h A_i^l}{2 A_i^l} \right) G_{l+} + G_i \right] + O(k^4 + k^4 h^2 + k^2 h^4),
\]

for \( l = 1, 2, ..., N; j = 0, 1, 2, .... \)

Using (2.3.8a)-(2.3.8b) and (2.14) in above equation, we have

\[
\left[ \left( 1 - \frac{h A_i^l}{2 A_i^l} \right) \left( G_{l+} + \frac{k^2}{2} T_1 + O(k^2 + h k^2 + h^3 + h^4) \right) \right]
\]

\[
- \frac{k^2}{3} \left[ \left( 1 + \frac{h A_i^l}{2 A_i^l} \right) \left( G_{l+} + \frac{k^2}{2} T_1 + O(k^2 + h k^2 - h^3 + h^4) \right) \right] + \hat{T}_i
\]

\[
+ \left( G_i + \frac{k^2}{6} T_3 + O(k^2 + k^2 h^2 + h^4) \right)
\]
\[ = -\left(1 - \frac{A_i}{2A_j}\right)G_{l_{i+\frac{1}{2}}} + \left(1 + \frac{A_i}{2A_j}\right)G_{l_{i-\frac{1}{2}}} + O(k^4 + k^4h^2 + k^2h^4), \]

\[ l = 1, 2, \ldots, N; j = 0, 1, 2, \ldots. \]

Simplifying further, we obtain the local truncation error

\[ \tilde{T}_i = \frac{k^2h^2}{36} \left[ 3(1 + 4a_i)U_{20,\alpha_i} + (3 - 12b_iA_i)U_{30,\beta_i} + 12(b_i + 2b_2)U_{12,\beta_i} \right] + O(k^4 + k^4h^2 + k^2h^4). \]

\[ \text{(2.3.13)} \]

The proposed method (2.2.10) to be of \( O(k^2 + k^2h^2 + h^4) \), the coefficients of \( k^2h^2 \) in (2.3.13) must be zero, which implies

\[ 1 + 4a_i = 0, \]
\[ 3 - 12b_iA_i = 0, \]
\[ b_i + 2b_2 = 0, \]
\[ b_3 - b_iA_i = 0, \]
\[ 1 + 4c_i = 0. \]

and we obtain the values of parameters as

\[ a_i = c_i = -\frac{1}{4}, \quad b_i = \frac{1}{4A_j}, \quad b_2 = -\frac{1}{8A_j}, \quad b_3 = \frac{A_i}{4A_j}, \]

\[ \text{(2.3.14)} \]

and hence, the local truncation error (2.3.13) reduces to \( \tilde{T}_i = O(k^4 + k^4h^2 + k^2h^4) \).

Note that, the initial and Dirichlet boundary conditions are given by (2.1.2) and (2.1.3), respectively. Incorporating the initial and boundary conditions, we can write the method (2.2.10) in a tri-diagonal matrix form. If the differential equation (2.1.1) is linear, we can solve the linear system using Gauss-elimination (tri-diagonal solver) method; in the non-
linear case, we can use Newton-Raphson iterative method to solve the non-linear system [58,82,163,182].

**2.4. Application to the wave equation in polar coordinates and stability analysis**

Let us consider the linear hyperbolic equation of the form

\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \gamma \frac{\partial u}{\partial r} + f(r,t), \quad 0 < r < 1, \quad t > 0
\]  

(2.4.1)

subject to the initial conditions

\[u(r,0) = \varphi(r), \quad \frac{\partial u(r,0)}{\partial t} = \psi(r), \quad 0 \leq r \leq 1\]

and the boundary conditions

\[u(0,t) = a(t), \quad u(1,t) = b(t), \quad t \geq 0.\]

For \(\gamma = 1\) and 2, the equation above represents linear hyperbolic equation in cylindrical and spherical polar coordinates respectively.

Denote \(\gamma = D(r)\) in (2.4.1). We have

\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + D(r) \frac{\partial u}{\partial r} + f(r,t), \quad 0 < r < 1, \quad t > 0
\]  

(2.4.2)

Replacing the variable \(x\) by \(r\) and applying the method (2.2.10) to the differential equation (2.4.2), we obtain the difference scheme

\[
\lambda^2 \delta_j^2 u_j^l = \frac{k^2}{12} \left[ u_{n+1} + u_{n-1} + 10u_n \right] - \frac{k^2}{3} \left[ \bar{G}^j_{i+2} + \bar{G}^j_{i-2} + \bar{G}^j_i \right]
\]  

(2.4.3)

where
\[ G'_i = g(r_i, t_j, u^l_i, u^r_i, u^u_i) = D(r_j)u^l_i + f(r_i, t_j) = D_{u^l_i} + f^l_i. \]

Now,
\[ \tilde{G}'_{i, l} = g(r_{i, l}, t_{j, l}, \tilde{u}^l_{i, l}, \tilde{u}^r_{j, l}, \tilde{u}^u_{i, l}) = D_{\tilde{u}^l_{i, l}} + f^l_l, \]
\[ \tilde{G}'_{i, r} = g(r_{i, r}, t_{j, r}, \tilde{u}^l_{i, r}, \tilde{u}^r_{j, r}, \tilde{u}^u_{i, r}) = D_{\tilde{u}^r_{i, r}} + f^r_r, \]
\[ \tilde{G}'_{i, u} = g(r_{i, u}, t_{j, u}, \tilde{u}^l_{i, u}, \tilde{u}^r_{j, u}, \tilde{u}^u_{i, u}) = D_{\tilde{u}^u_{i, u}} + f^u_u, \]
\[ \hat{G}'_{i} = g(r_i, t_j, \hat{u}^l_i, \hat{u}^r_i, \hat{u}^u_i) = D_{\hat{u}^l_i} + f^l_i, \]
\[ \hat{u}^l_i = \bar{u}^l_i + \frac{h}{4} (\tilde{G}'_{i, l} - \tilde{G}'_{i, r}) - \frac{h}{8} (u_{i, l+1} - u_{i, l-1}) \]
\[ = \bar{u}^l_i + \frac{h}{4} \left( D_{\tilde{u}^l_{i, l}} + f^l_{i, l} - D_{\tilde{u}^r_{i, l}} - f^r_{i, l} \right) - \frac{h}{8} (u_{i, l+1} - u_{i, l-1}) \]
\[ = \bar{u}^l_i + \frac{h}{4} \left( D_{\tilde{u}^r_{i, r}} - D_{\tilde{u}^l_{i, r}} + f^r_{i, r} - f^l_{i, r} \right) - \frac{h}{8k^2} (\delta^2 (2\mu, \delta))\tilde{u}^r_i \]
\[ = \bar{u}^l_i + \frac{h}{4} \left( D_{\tilde{u}^r_{i, u}} - D_{\tilde{u}^l_{i, u}} + f^r_{i, u} - f^l_{i, u} \right) + \frac{h}{4} (f^r_{i, u} - f^l_{i, u}) - \frac{h}{8k^2} (\delta^2 (2\mu, \delta))\tilde{u}^r_i, \]
\[ \therefore \tilde{G}'_{i, i} + \tilde{G}'_{i, r} + \tilde{G}'_{i, u} = D_{\tilde{u}^l_{i, i}} + f^l_i + D_{\tilde{u}^r_{i, r}} + f^r_i + D_{\tilde{u}^u_{i, u}} + f^u_i + D_{\hat{u}^l_i} + f^l_i \]
\[ = D_{\tilde{u}^l_{i, i}} + D_{\tilde{u}^r_{i, r}} + \left( f^l_i + f^r_i + f^u_i \right) \]
\[ + D_i \left( \tilde{u}^l_i + \frac{h}{4} \left( D_{\tilde{u}^r_{i, l}} - D_{\tilde{u}^l_{i, r}} + f^r_{i, l} - f^l_{i, r} \right) + \frac{h}{4} (f^r_{i, l} - f^l_{i, r}) - \frac{h}{8k^2} (\delta^2 (2\mu, \delta))\tilde{u}^r_i \right). \]

Also,
\[ \left[ \tilde{u}_{i, l+1} + \tilde{u}_{i, l+1} + 10\tilde{u}_{i, l} \right] = \left[ \tilde{u}_{i, l+1} - 2\tilde{u}_{i, l} + \tilde{u}_{i, l-1} + 12\tilde{u}_{i, l} \right] \]
\[ = \left[ \delta^2 + 12 \right] \frac{\delta^2}{k^2} \tilde{u}^l_i \]
\[ = 12 \left[ 1 + \frac{\delta^2}{12} \right] \delta^2 \tilde{u}^l_i \]

where \( \mu u^l_i = \frac{1}{2} (u^l_{i+1} + u^l_{i-1}) \) and \( \delta u^l_i = (u^l_{i+1} - u^l_{i-1}) \) are averaging and central difference operators with respect to \( r \)-direction etc. This implies \( 2\mu, \delta \) \( u^l_i = u^l_{i+1} - u^l_{i-1}, \)
\( \delta^2 u^l_i = u^l_{i+1} - 2u^l_i + u^l_{i-1}, \delta^2 u^l_i = u^l_{i+1} - 2u^l_i + u^l_{i-1}, \) etc.
Hence, the scheme (2.4.3) becomes

\[ \lambda^2 \delta^2 u_i' = \left(1 + \frac{\delta^2}{12}\right) \delta^2 u_i' \]

\[ - \frac{k^2}{3} \left[ \frac{hD}{4} \left( f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}} \right) \right] - \frac{k^2}{3} \left[ f_{i+\frac{1}{2}}^f + f_{i-\frac{1}{2}}^f + f_i^f \right] \]

\[ \Rightarrow \lambda^2 \delta^2 u_i' = \left(1 + \frac{\delta^2}{12}\right) \delta^2 u_i' \]

\[ - \frac{k^2}{3} \left[ \frac{hD}{4} \left( f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}} \right) \right] - \frac{k^2}{3} \left[ f_{i+\frac{1}{2}}^f + f_{i-\frac{1}{2}}^f + f_i^f \right]. \]  

(2.4.4)

Note that, the scheme (2.4.3) is of \( O(k^2 + k^2 h^2 + h^4) \) for the solution of the differential equation (2.4.1) and is free from the terms \( \frac{1}{(\mu \delta)} \), thus very easily solved for \( l=1(1)N; \)

\( j=0,1,2,\ldots \) in the solution region without any modification. We do not require any fictitious point to solve the singular problem. But, in order to obtain the stability region of the scheme (2.4.4), we need the following approximations:

\[ D_{i\pm\frac{1}{2}} = D_i \pm \frac{h}{2} D_n + \frac{h^2}{8} D_n \pm O(h^3), \]  

(2.4.5a)

\[ f_{i\pm\frac{1}{2}}^f = f_i^f \pm \frac{h}{2} f_n^f + \frac{h^2}{8} f_n^f \pm O(h^3). \]  

(2.4.5b)

Using (2.4.5b), we have

\[ f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}} = hf_n^f + O(h^3), \]  

(2.4.6a)

\[ f_{i+\frac{1}{2}}^f + f_{i-\frac{1}{2}}^f + f_i^f = 3f_i^f + \frac{h^2}{4} f_n^f + O(h^4), \]  

(2.4.6b)
Now, with the help of the approximations defined in section 2.3 (i.e (2.3.5b) and (2.3.5c)) and (2.4.5a), we have

\[
D_{h+\frac{1}{2},n+\frac{1}{2}}\bar{\pi}^l_{i+\frac{1}{2}} + D_{n+\frac{1}{2},n+\frac{1}{2}}\bar{\pi}^l_{i+\frac{1}{2}} = \left(D_i + \frac{h^2}{8} D_{\eta i}\right)\left(\bar{\pi}^l_{i+\frac{1}{2}} + \bar{\pi}^l_{i+\frac{1}{2}}\right) + \frac{h}{2} D_{\eta i} \left(\bar{\pi}^l_{i+\frac{1}{2}} + \bar{\pi}^l_{i+\frac{1}{2}}\right)
\]

\[
= \left(D_i + \frac{h^2}{8} D_{\eta i}\right)\left(\frac{1}{h} (2\mu, \delta_i)u^l_j + \frac{1}{2} D_{\eta i} (\delta^2_i)u^l_j\right)
\]

\[
= \left(D_i + \frac{h^2}{8} D_{\eta i}\right)\left(\frac{1}{h} (2\mu, \delta_i)u^l_j + \frac{1}{2} D_{\eta i} (\delta^2_i)u^l_j\right),
\]  

(2.4.6c)

\[
D_{n+\frac{1}{2},n+\frac{1}{2}}\bar{\pi}^l_{i+\frac{1}{2}} - D_{n+\frac{1}{2},n+\frac{1}{2}}\bar{\pi}^l_{i+\frac{1}{2}} = \left(D_i + \frac{h^2}{8} D_{\eta i}\right)\left(\bar{\pi}^l_{i+\frac{1}{2}} - \bar{\pi}^l_{i+\frac{1}{2}}\right) + \frac{h}{2} D_{\eta i} \left(\bar{\pi}^l_{i+\frac{1}{2}} + \bar{\pi}^l_{i+\frac{1}{2}}\right)
\]

\[
= \left(D_i + \frac{h^2}{8} D_{\eta i}\right)\left(\frac{1}{h} (\delta^2_i)u^l_j + \frac{1}{2} D_{\eta i} (\delta^2_i)u^l_j\right)
\]

\[
= \left(D_i + \frac{h^2}{8} D_{\eta i}\right)\left(\frac{1}{h} (\delta^2_i)u^l_j + \frac{1}{2} D_{\eta i} (\delta^2_i)u^l_j\right).
\]  

(2.4.6d)

Hence, using (2.4.6a – 2.4.6d), the scheme (2.4.4) may be written as

\[
\lambda^2 \delta^2_i u^l_j = \left(1 + \frac{\delta^2_i}{12}\right) \delta^2_i u^l_j
\]

\[
- \frac{k^2}{3} \left[ \left(D + \frac{h^2}{8} D_{\eta i}\right)\left(\frac{1}{h} (2\mu, \delta_i)u^l_j + \frac{1}{2} D_{\eta i} (\delta^2_i)u^l_j\right) + D_{\eta i} \bar{u}^l_j\right]
\]

\[
+ \frac{hD_{\eta i}}{4} \left[ \left(D + \frac{h^2}{8} D_{\eta i}\right)\left(\frac{1}{h} (\delta^2_i)u^l_j + \frac{1}{2} D_{\eta i} (2\mu, \delta_i)u^l_j\right) - \frac{hD_{\eta i}}{8} (\delta^2_i (2\mu, \delta_i))\bar{\pi}^l_j\right]
\]

\[
- \frac{k^2}{3} \left[ \frac{h^2 D_{\eta i}}{4} f_{\eta i}\right] - \frac{k^2}{3} \left[3 f^l_j + \frac{h^2}{4} f_{\eta i}\right]
\]
\[
\Rightarrow \lambda^2\delta^2 u_i' = \left(1 + \frac{\delta^2}{12}\right)\delta_i^2 u_i'
\]

\[
- \frac{\lambda^2}{3} \left[ \frac{h^2 D_1}{8} (2\mu, \delta) \right] u_i' + \frac{h^2 D_2}{2} (\delta^2) u_i' + \frac{hD_0}{2} (2\mu, \delta, \delta) u_i'
\]

\[
+ \frac{h^2 D_1}{4} \left( \frac{h^2}{8} D_n \right) (\delta^2) u_i' + \frac{h^2 D_2}{8} (2\mu, \delta) u_i'
\]

\[
+ \frac{hD_0}{24} (\delta^2 (2\mu, \delta, \delta)) u_i'
\]

Neglecting the higher order term, we have

\[
\left(1 + \frac{\delta^2}{12} + \frac{hD_1}{24} (2\mu, \delta) \right) \delta^2 u_i' = \frac{\lambda^2}{3} \left[ \left( 1 + \frac{h^2}{4} (2D_n + D_1^2) \right) \delta^2 u_i' + \frac{3hD_1}{2} + \frac{h}{8} (D_n + D_1) \right] (2\mu, \delta) u_i'
\]

\[
+ \frac{k^2}{3} \left[ 3f_i' + \frac{h^2}{4} f_i' + \frac{h^2 D_1}{4} f_i' \right]
\]

\[
\Rightarrow \left(1 + \frac{1}{12} \left( \delta^2 + \frac{hD_1}{2} (2\mu, \delta) \right) \right) \delta^2 u_i' = \frac{\lambda^2}{3} \left[ \left( 1 + \frac{h^2}{12} (2D_n + D_1^2) \right) \delta^2 u_i' + \frac{hD_0}{2} + \frac{h^3}{24} (D_n + D_1) \right] (2\mu, \delta) u_i'
\]

\[
+ k^2 \left[ f_i' + \frac{h^2}{4} \left( f_{i'} + D_1 f_i' \right) \right].
\]  

(2.4.7)

We can re-write the scheme (2.4.7) in three-level operator compact implicit form

\[
[1 + \frac{1}{12} (\delta^2 + R_i (2\mu, \delta)) | \delta^2 u_i' = \lambda^2 \left( R_i \delta^2 + R_j (2\mu, \delta) \right) u_i' + \sum f; \quad l = 1(1)N, j = 1(1)J \ ] \]  

(2.4.8)

where
\[ R_1 = \frac{hD_1}{2}, \]
\[ R_2 = 1 + \frac{h^2}{12} \left( 2D_{11} + D_{1} \right), \]
\[ R_3 = R_1 + \frac{h^3}{24} \left( D_{111} + D_{1}D_{11} \right) \]
and
\[ \sum f = k^2 \left[ f_{i}^{j} + \frac{h^2}{4} \left( f_{i+1}^{j+1} + D_{i}f_{i}^{j+1} \right) \right]. \]

The scheme (2.4.8) has a local truncation error of \( O(k^2 + k^2 h^2 + h^4) \) and is free from the terms \( \frac{1}{12} \) and hence, it can also be solved for \( l=1(1)N, j=1(1)J \) in the region \( 0<r<1, t>0 \).

For stability of the method (2.4.8), we follow the technique used by Mohanty [122]. We may re-write (2.4.8) as

\[ [1 + \frac{1}{12} (R_2\delta_{1}^2 + R_3(2\mu, \delta_j))]\delta_{1}^2 u_i^j = \lambda^2 [R_2\delta_{1}^2 + R_3(2\mu, \delta_j)]u_i^j + \sum f. \quad (2.4.9) \]

The additional terms are of high orders and do not affect the accuracy of the scheme. The exact value \( U_i^j = u(r_i, t_j) \) satisfies

\[ [1 + \frac{1}{12} (R_2\delta_{1}^2 + R_3(2\mu, \delta_j))]\delta_{1}^2 U_i^j = \lambda^2 [R_2\delta_{1}^2 + R_3(2\mu, \delta_j)]U_i^j + \sum f + O(k^4 + k^2 h^2 + k^2 h^4). \]

\[ (2.4.10) \]

We assume that there exists an error \( \varepsilon_i^j = U_i^j - u_i^j \) at the grid point \((x_i, t_j)\). Subtracting (2.4.9) from (2.4.10), we obtain the error equation

\[ [1 + \frac{1}{12} (R_2\delta_{1}^2 + R_3(2\mu, \delta_j))]\delta_{1}^2 \varepsilon_i^j = \lambda^2 [R_2\delta_{1}^2 + R_3(2\mu, \delta_j)]\varepsilon_i^j + O(k^4 + k^2 h^2 + k^2 h^4). \]

\[ (2.4.11) \]

For stability of the modified scheme (2.4.9), we assume that \( \varepsilon_i^j = p|e^{i\theta}|e^{i\theta} \) (where \( \xi = e^{i\theta} \) such that \( |\xi| = 1 \)) at the grid point \((r_i, t_j)\), where \( \xi \) is in general complex, \( \theta \) is an arbitrary real number and \( p \) is a non-zero real parameter to be determined. Hence,
\[ \delta_i^2 \xi_j = \xi_{i+1}^{j-2} + \xi_i^{-1} = (\xi - 2 + \xi^{-1}) p^i \xi_j e^{i\theta}, \]
\[ \delta_i^2 \xi_j = \xi_{i+1}^{j-2} + \xi_i^{-1} = p^i \xi_j^{-1} \left( p e^{i\theta} - 2 + p^{-1} e^{-i\theta} \right) e^{i\theta}, \]
\[ (2 \mu, \delta) \xi_j = \xi_{i+1} - \xi_i^{-1} = (p e^{i\theta} - p^{-1} e^{-i\theta}) p^i \xi_j^{-1} e^{i\theta}, \]
\[ \delta_i^2 (2 \mu, \delta) \xi_j = \delta_i^2 (\xi_{i+1}^{j-2} - \xi_i^{-1}) \]
\[ = (\xi_{i+1}^{j-2} - 2 \xi_{i+1}^{j-1} + \xi_{i+1}^{-1}) - (\xi_{i+1}^{j-2} - 2 \xi_{i+1}^{-1} + \xi_{i+1}^{-1}) \]
\[ = (\xi - 2 + \xi^{-1}) p^i \xi_j^{-1} e^{i\theta} - (\xi - 2 + \xi^{-1}) p^i \xi_j^{-1} e^{i\theta} \]
\[ = (p e^{i\theta} - p^{-1} e^{-i\theta})(\xi - 2 + \xi^{-1}) p^i \xi_j^{-1} e^{i\theta}, \]
\[ \delta_i^2 \xi_j = \delta_i^2 (\xi_{i+1}^{j-2} - 2 \xi_{i+1}^{j-1} + \xi_{i+1}^{-1}) \]
\[ = (\xi_{i+1}^{j-1} - 2 \xi_{i+1}^{j-1} + \xi_{i+1}^{-1}) - 2(\xi_{i+1}^{j-1} - 2 \xi_{i+1}^{j-1} + \xi_{i+1}^{-1}) \]
\[ = (p e^{i\theta} - 2 + p^{-1} e^{-i\theta}) p^i \xi_j^{-1} e^{i\theta} - 2(p e^{i\theta} - 2 + p^{-1} e^{-i\theta}) p^i \xi_j^{-1} e^{i\theta} \]
\[ + (p e^{i\theta} - 2 + p^{-1} e^{-i\theta}) p^i \xi_j^{-1} e^{i\theta} \]
\[ = (p e^{i\theta} - 2 + p^{-1} e^{-i\theta}) p^i \xi_j^{-1} e^{i\theta} (\xi - 2 + \xi^{-1}). \]

Now, the homogeneous part of the error equation (2.4.11), becomes

\[ ((\xi - 2 + \xi^{-1}) p^i \xi_j e^{i\theta} + \frac{1}{12} (R_2(\xi - 2 + \xi^{-1}) p^i \xi_j^{-1} (p e^{i\theta} - 2 + p^{-1} e^{-i\theta}) e^{i\theta}) \]
\[ + R_1(p e^{i\theta} - p^{-1} e^{-i\theta})(\xi - 2 + \xi^{-1}) p^i \xi_j^{-1} e^{i\theta} \]
\[ = \lambda^2 [R_2(p e^{i\theta} - 2 + p^{-1} e^{-i\theta}) e^{i\theta}] \]
\[ \Rightarrow (\xi - 2 + \xi^{-1}) p^i \xi_j e^{i\theta} \left[ 1 + \frac{1}{12} (R_2(p e^{i\theta} - 2 + p^{-1} e^{-i\theta}) + R_1(p e^{i\theta} - p^{-1} e^{-i\theta}) \right] \]
\[ = \lambda^2 p^i \xi_j^{-1} e^{i\theta} \left[ R_2(\xi - 2 + p^{-1} e^{-i\theta}) + R_1(p e^{i\theta} - p^{-1} e^{-i\theta}) \right] \]
\[ \Rightarrow (\xi - 2 + \xi^{-1}) = \lambda^2 \left[ R_2(p e^{i\theta} - 2 + p^{-1} e^{-i\theta}) + R_1(p e^{i\theta} - p^{-1} e^{-i\theta}) \right] \]
\[ \left[ 1 + \frac{1}{12} (R_2(p e^{i\theta} - 2 + p^{-1} e^{-i\theta}) + R_1(p e^{i\theta} - p^{-1} e^{-i\theta}) \right]. \]
Hence, we obtain the amplification factor

\[
-4\sin^2\left(\frac{\lambda}{2}\right) = \lambda^2 \left[ R_1 \left\{ (p + p^{-1})\cos \theta + 2 + i(p - p^{-1})\sin \theta \right\} + R_1 \left\{ (p - p^{-1})\cos \theta + i(p + p^{-1})\sin \theta \right\} \right] + \frac{R_1}{1 + \frac{R_2}{R_3}} \left[ (p + p^{-1})\cos \theta + 2 + i(p - p^{-1})\sin \theta \right].
\]  \hfill (2.4.12)

Since left-hand side of (2.4.12) is a real quantity, hence the imaginary part of right-hand side of (2.4.12) must be zero, from which we obtain

\[
R_1 (p - p^{-1}) + R_1 (p + p^{-1}) = 0
\]

or,

\[
p = \frac{R_1 - R_1}{\sqrt{R_2 + R_3}}
\]  \hfill (2.4.13)

where \( R_2 \pm R_1 > 0 \). Substituting the values of \( p \) and \( p^{-1} \) in (2.4.12), we get

\[
\sin^2\left(\frac{\phi}{2}\right) = \frac{\lambda^2 \left[ R_2 + \left( R_2^2 - R_3^2 \right) \left( 2\sin^2\left(\frac{\phi}{2}\right) - 1 \right) \right]}{2 - \frac{1}{2} \left[ R_2 + \left( R_2^2 - R_3^2 \right) \left( 2\sin^2\left(\frac{\phi}{2}\right) - 1 \right) \right]}. \]  \hfill (2.4.14)

Since \( 0 \leq \sin^2\left(\frac{\phi}{2}\right) \leq 1 \), \( \max \sin^2\left(\frac{\phi}{2}\right) = 1 \), \( \min \sin^2\left(\frac{\phi}{2}\right) = 0 \), it follows from (2.4.14) that the cubic spline finite difference scheme (2.4.9) is stable if

\[
0 < \lambda^2 \leq \frac{2 - \frac{1}{2} \left[ R_2 - \sqrt{R_2^2 - R_3^2} \right]}{R_2 + \sqrt{R_2^2 - R_3^2}} \]  \hfill (2.4.15)

leading to \( |\xi| = 1 \). It is easy to verify that as \( l \to \infty \), \( 0 < \lambda^2 \leq 1 \).
2.5 Application to the telegraph equation and stability analysis

In this section, we discuss the application of the method (2.2.10) to the telegraphic equation with forcing function

\[
\frac{\partial^2 u}{\partial t^2} + 2\alpha \frac{\partial u}{\partial t} + \beta^2 u = \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad 0 < x < 1, \quad t > 0
\]  

(2.5.1)

subject to the initial conditions

\[
u(x,0) = \varphi(x), \quad \frac{\partial \nu(x,0)}{\partial t} = \psi(x), \quad 0 \leq x \leq 1
\]

and the boundary conditions

\[
u(0,t) = a(t), \quad \nu(1,t) = b(t), \quad t \geq 0
\]

where \(\alpha > 0, \quad \beta \geq 0\) are real parameters. For \(\beta = 0\), the equation above represents damped wave equation.

Equation (2.5.1) can be written as

\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + g(x,t,u,\frac{\partial u}{\partial x}, \frac{\partial u}{\partial t})
\]  

(2.5.2)

where \(g(x,t,u,u_x,u_t) = -2\alpha u_x - \beta^2 u + f(x,t) \equiv G\).

Applying method (2.2.10) to equation (2.5.2), we have

\[
\hat{\lambda}^2 \hat{\delta}_x^2 U_j^{(l)} = \frac{k^2}{12} \left[ \bar{u}_{i+1,j} + \bar{u}_{i-1,j} + 10\bar{u}_{i,j} \right] - \frac{k^2}{3} \left[ G_{i+1,j}^{(l)} + G_{i-1,j}^{(l)} + G_{i,j}^{(l)} \right].
\]  

(2.5.3)

Now,
\[
\begin{bmatrix}
\bar{u}_{n+1} + \bar{u}_{n-1} + 10\bar{u}_{n}
\end{bmatrix} = \begin{bmatrix}
\bar{u}_{n+1} - 2\bar{u}_{n} + \bar{u}_{n-1} + 12\bar{u}_{n}
\end{bmatrix} = \begin{bmatrix}
\delta_t^2 + 12\frac{\delta_t^2}{k^2} \bar{u}'_i \\
= \frac{12}{k^2} \left[ 1 + \frac{\delta_t^2}{12} \right] \delta_t^2 \bar{u}'_i
\end{bmatrix}
\]

where \( \mu, u'_i = \frac{1}{2}(u_{i+1}' + u_{i-1}') \) and \( \delta, u'_i = (u_{i+1}' - u_{i-1}') \) are averaging and central difference operators with respect to \( x \)-direction etc. This implies \( (2\mu, \delta) u'_i = u'_{i+1} - u'_{i-1} \), \( \delta_t^2 u'_i = u'_{i+1} - 2u'_{i} + u'_{i-1} \), \( \delta_t^2 u'_i = u_{i+1}' - 2u'_{i} + u'_{i-1} \), etc.

Also,
\[
\bar{G}_{i+\frac{1}{2}} = -2\alpha \bar{u}_{i+\frac{1}{2}} - \beta^2 \bar{u}_{i+\frac{1}{2}} + f_{i+\frac{1}{2}} = \frac{-2\alpha}{k^2} \left( u_{i+1}^{r+1} + u_{i+1}^{r-1} - u_{i-1}^{r-1} - u_{i-1}^{r+1} \right) - \frac{\beta^2}{k^2} \left( u_{i+1}^{r+1} + u_{i+1}^{r-1} \right) + f_{i+\frac{1}{2}},
\]

\[
\bar{G}_{i-\frac{1}{2}} = -2\alpha \bar{u}_{i-\frac{1}{2}} - \beta^2 \bar{u}_{i-\frac{1}{2}} + f_{i-\frac{1}{2}} = \frac{-2\alpha}{k^2} \left( u_{i-1}^{r+1} + u_{i-1}^{r-1} - u_{i+1}^{r-1} - u_{i+1}^{r+1} \right) - \frac{\beta^2}{k^2} \left( u_{i-1}^{r+1} + u_{i-1}^{r-1} \right) + f_{i-\frac{1}{2}},
\]

\[
\bar{G}_{i+\frac{1}{2}} + \bar{G}_{i-\frac{1}{2}} = \frac{-2\alpha}{k^2} \left( (4 + \delta_t^2) u_{i+1}^{r+1} - (4 + \delta_t^2) u_{i-1}^{r-1} \right) - \frac{\beta^2}{k^2} \left( (4 + \delta_t^2) u_{i+1}^{r+1} - (4 + \delta_t^2) u_{i-1}^{r-1} \right) + f_{i+\frac{1}{2}} + f_{i-\frac{1}{2}}
\]

Using (2.2.8a) and (2.2.8c),
\[
\hat{G}_i = -2\alpha \hat{u}_i - \beta^2 \hat{u}_i + f_i
\]

\[
= -2\alpha \left( \bar{u}_i - \frac{1}{4}(\bar{u}_{i+1} - 2\bar{u}_i + \bar{u}_{i-1}) \right) - \beta^2 \left( u_{i+1}' - \frac{\delta_t^2}{4} u_{i+1}' \right) + f_i
\]

\[
= \frac{-2\alpha}{k^2} \left[ (2\mu, \delta) u'_i - \frac{1}{4} (\delta_t^2 2\mu, \delta) u_i' \right] - \beta^2 u_i' + \frac{\beta^2}{k^2} \delta_t^2 u_i' + f_i.
\]
Hence, the scheme (2.5.3) becomes

\[
\left(1 + \frac{\delta^2}{12}\right) \delta^2 u_i = \lambda^2 \delta^2 u_i + \frac{k^2}{3} \left[ \frac{\alpha}{2\varepsilon} \left(2(4 + \delta^2)u_i - 4\frac{\beta^2}{2}(4 + \delta^2)u_i\right) \right] \\
- \frac{\alpha}{2\varepsilon} \left[ \left(2\mu, \delta_i\right)u_i - \frac{1}{2}(\delta^2 2\mu, \delta_i)u_i \right] \\
- \frac{\beta^2}{4} \delta^2 u_i' \\
+ \frac{k^2}{3} \left[ f_{i\pm\frac{1}{2}} + f_{i-\frac{1}{2}} + f_i' \right]
\]

\[
\Rightarrow \left(1 + \frac{\delta^2}{12}\right) \delta^2 u_i = \lambda^2 \delta^2 u_i + \frac{k^2}{3} \left[ \frac{\alpha}{2\varepsilon} (2\mu, \delta_i)u_i - \frac{\alpha}{4\varepsilon} (\delta^2 2\mu, \delta_i)u_i \right] \\
- 3\beta^2 u_i' - \frac{\beta^2}{4} \delta^2 u_i' \\
+ \frac{k^2}{3} \left[ f_{i\pm\frac{1}{2}} + f_{i-\frac{1}{2}} + f_i' \right]
\]

\[
\Rightarrow \delta^2 u_i' + \alpha k(2\mu, \delta_i)u_i' + \frac{\alpha}{12} (\delta^2 2\mu, \delta_i)u_i' + \left(\frac{\beta^2 k^2}{12} - \lambda^2\right) \delta^2 u_i' + \beta^2 k u_i' + \frac{\beta^2}{4} \delta^2 u_i' \\
= \frac{k^2}{3} \left[ f_{i\pm\frac{1}{2}} + f_{i-\frac{1}{2}} + f_i' \right].
\]

In this section, we denote \( a = \alpha^2 k^2, b = \beta^2 k^2 \).

Hence, we obtain a numerical approximation of \( O(k^2 + k^2 h^2 + h^4) \) as

\[
\delta^2 u_i' + \sqrt{a} (2\mu, \delta_i)u_i' + \frac{\sqrt{a}}{12} \left( \delta^2 2\mu, \delta_i \right)u_i' + \left(\frac{b}{12} - \lambda^2\right) \delta^2 u_i' \\
+ bu_i' + \frac{\delta^2}{12} u_i' = \frac{k^2}{3} \left[ f_{i\pm\frac{1}{2}} + f_{i-\frac{1}{2}} + f_i' \right].
\]

(2.5.4)

The exact value \( U_i' = u(x_i, t_i) \) satisfies

\[
\delta^2 U_i' + \sqrt{a} (2\mu, \delta_i)U_i' + \frac{\sqrt{a}}{12} \left( \delta^2 2\mu, \delta_i \right)U_i' + \left(\frac{b}{12} - \lambda^2\right) \delta^2 U_i'
\]
+bU_i + \frac{\delta^2}{12} U_i = \frac{k^2}{3} \left[ f_i^{j+1} + f_i^{j-1} + f_i^j \right] + O(k^4 + k^2h^2 + k^4h^4). \tag{2.5.5}

We assume that there exists an error \( e_i^j = U_i^j - u_i^j \) at the grid point \((x_l, t_j)\). Subtracting \(2.5.4\) from \(2.5.5\), we obtain the error equation

\[
\begin{align*}
\delta_i e_i^j + \sqrt{a} (2\mu, \delta_i) e_i^j + \frac{\sqrt{a}}{12} \left( \delta_i 2\mu, \delta_i \right) e_i^j + \left( \frac{b}{12} - \lambda^2 \right) \delta_i e_i^j & = -\frac{k^2}{3} \left[ f_i^{j+1} + f_i^{j-1} + f_i^j \right] + O(k^4 + k^2h^2 + k^4h^4). \tag{2.5.6}
\end{align*}
\]

Now, let us assume \( e_i^j = \xi^j e^{i\theta} \) (where \( \xi = e^{i\theta} \) such that \(|\xi| = 1\)) at the grid point \((x_l, t_j)\), where \( \xi \) is in general complex, \( \theta \) is an arbitrary real number, then

\[
\begin{align*}
\delta_i^2 e_i^j & = e_i^{j+1} - 2 e_i^j + e_i^{j-1} = (\xi - 2 + \xi^{-1}) e_i^j e^{i\theta}, \tag{2.5.7a}
\delta_i e_i^j & = e_i^{j+1} - 2 e_i^j + e_i^{j-1} = \xi^j (e^{i\theta} - 2 + e^{-i\theta}) e^{i\theta}, \tag{2.5.7b}
(2\mu, \delta_i) e_i^j & = e_i^{j+1} - e_i^{j-1} = (\xi - \xi^{-1}) \xi^j e^{i\theta}, \tag{2.5.7c}
\delta_i^2 (2\mu, \delta_i) e_i^j & = \delta_i^2 (e_i^{j+1} - e_i^{j-1}) \\
& = (e_i^{j+1} - 2e_i^j + e_i^{j-1}) - (e_i^{j+1} - 2e_i^j + e_i^{j-1}) \\
& = (\xi - \xi^{-1}) \xi^j (e^{i\theta} - 2 + e^{-i\theta}) e^{i\theta}, \tag{2.5.7d}
\delta_i^2 \delta_i e_i^j & = \delta_i^2 (e_i^{j+1} - 2e_i^j + e_i^{j-1}) \\
& = (e_i^{j+1} - 2e_i^j + e_i^{j+1}) - 2(e_i^{j+1} - 2e_i^j + e_i^{j-1}) + (e_i^{j-1} - 2e_i^j + e_i^{j+1}) \\
& = (\xi - 2 + \xi^{-1}) \xi^j (e^{i\theta} - 2 + e^{-i\theta}) e^{i\theta}. \tag{2.5.7e}
\end{align*}
\]

For stability, we put \( e_i^j = \xi^j e^{i\theta} \) in the homogeneous part of the error equation \(2.5.6\) and using the above approximations \(2.5.7a)-(2.5.7e)\), we get
\[
\begin{align*}
&\left[ (\xi - 2 + \xi^{-1}) + \sqrt{a}(\xi - \xi^{-1}) \\
&+ \frac{\sqrt{a}}{12}(\xi - \xi^{-1})(e^{i\theta} - 2 + e^{-i\theta}) + \left( \frac{b}{12} - \lambda^2 \right) (e^{i\theta} - 2 + e^{-i\theta}) + b \right] \xi' e^{i\omega} = 0 \\
\end{align*}
\]

\[
\begin{align*}
&\Rightarrow \\
&\left[ (\xi - 2 + \xi^{-1}) + \sqrt{a}(\xi - \xi^{-1}) \\
&+ \frac{\sqrt{a}}{12}(\xi - \xi^{-1})(-4 \sin^2 \frac{\theta}{2}) + \left( \frac{b}{12} - \lambda^2 \right)(-4 \sin^2 \frac{\theta}{2}) + b \\
&+ \frac{(\xi - 2 + \xi^{-1})(-4 \sin^2 \frac{\theta}{2})}{12} \right] = 0 \\
\end{align*}
\]

\[
\begin{align*}
&\Rightarrow \\
&\left[ (\xi^2 - 2\xi + 1) + \sqrt{a}(\xi^2 - 1) \\
&+ \frac{\sqrt{a}}{12}(\xi^2 - 1)(-4 \sin^2 \frac{\theta}{2}) + \left( \frac{b}{12} - \lambda^2 \right)(-4 \sin^2 \frac{\theta}{2})\xi + b\xi \\
&+ \frac{(\xi^2 - 2\xi + 1)(-4 \sin^2 \frac{\theta}{2})}{12} \right] = 0 \\
\end{align*}
\]

\[
\begin{align*}
&\Rightarrow \\
&\left[ \left( 1 + \sqrt{a} - \frac{\sqrt{a}}{3} \sin^2 \frac{\theta}{2} - \frac{1}{3} \sin^2 \frac{\theta}{2} \right)\xi^2 \\
&+ \left( -2 + 4 \left( \lambda^2 - \frac{b}{12} \right) \sin^2 \frac{\theta}{2} + b + \frac{2}{3} \sin^2 \frac{\theta}{2} \right) \xi \\
&+ 1 - \sqrt{a} + \frac{\sqrt{a}}{3} \sin^2 \frac{\theta}{2} - \frac{1}{3} \sin^2 \frac{\theta}{2} \right] = 0 \\
\end{align*}
\]
Hence, the characteristic equation is given by

\[ p_1 \xi^2 + p_2 \xi + p_3 = 0 \] (2.5.8)

where

\[ p_1 = (1 + \sqrt{a}) \left( 1 - \frac{1}{3} \sin^2 \frac{\theta}{2} \right), \]
\[ p_2 = \left( b - 2 \left( 1 - \frac{1}{3} \sin^2 \frac{\theta}{2} \right) + 4 \left( \lambda^2 - \frac{b}{12} \right) \sin^2 \frac{\theta}{2} \right), \]
\[ p_3 = (1 - \sqrt{a}) \left( 1 - \frac{1}{3} \sin^2 \frac{\theta}{2} \right). \]

Using the transformation \( \xi = \frac{1 + z}{1 - z} \), the characteristic equation (2.5.8) reduces to

\[ (p_1 - p_2 + p_3) z^2 + 2(p_1 - p_3) z + (p_1 + p_2 + p_3) = 0. \] (2.5.9)

The necessary and the sufficient condition for \( |\xi| < 1 \) is that \( p_1 + p_2 + p_3 > 0 \), \( p_1 - p_3 > 0 \) and \( p_1 - p_2 + p_3 > 0 \).

We have

\[ p_1 + p_2 + p_3 = (1 + \sqrt{a}) \left( 1 - \frac{1}{3} \sin^2 \frac{\theta}{2} \right) + \left( b - 2 \left( 1 - \frac{1}{3} \sin^2 \frac{\theta}{2} \right) + 4 \left( \lambda^2 - \frac{b}{12} \right) \sin^2 \frac{\theta}{2} \right) + (1 - \sqrt{a}) \left( 1 - \frac{1}{3} \sin^2 \frac{\theta}{2} \right) \]
\[ b + 4 \left( \lambda^2 - \frac{b}{12} \right) \sin^2 \frac{\theta}{2} = 4\lambda^2 \sin^2 \frac{\theta}{2} + b \left( 1 - \frac{1}{3} \sin^2 \frac{\theta}{2} \right), \]

\[ p_1 - p_3 = \left( 1 + \sqrt{a} \right) \left( 1 - \frac{1}{3} \sin^2 \frac{\theta}{2} \right) - \left( 1 - \sqrt{a} \right) \left( 1 - \frac{1}{3} \sin^2 \frac{\theta}{2} \right) = 2\sqrt{a} \left( 1 - \frac{1}{3} \sin^2 \frac{\theta}{2} \right), \]

\[ p_1 - p_2 + p_3 = \left( 1 + \sqrt{a} \right) \left( 1 - \frac{1}{3} \sin^2 \frac{\theta}{2} \right) - \left( b - 2 \left( 1 - \frac{1}{3} \sin^2 \frac{\theta}{2} \right) + 4 \left( \lambda^2 - \frac{b}{12} \right) \sin^2 \frac{\theta}{2} \right) \]

\[ + \left( 1 - \sqrt{a} \right) \left( 1 - \frac{1}{3} \sin^2 \frac{\theta}{2} \right) = 2 \left( 1 - \frac{1}{3} \sin^2 \frac{\theta}{2} \right) - b + 2 \left( 1 - \frac{1}{3} \sin^2 \frac{\theta}{2} \right) - 4 \left( \lambda^2 - \frac{b}{12} \right) \sin^2 \frac{\theta}{2} \]

\[ = 4 \left( 1 - \frac{1}{3} \sin^2 \frac{\theta}{2} \right) - 4\lambda^2 \sin^2 \frac{\theta}{2} + \frac{b}{3} \sin^2 \frac{\theta}{2} \]

\[ = 4 \left( 1 - \frac{1}{3} \sin^2 \frac{\theta}{2} \right) - 4\lambda^2 \sin^2 \frac{\theta}{2} - b \left( 1 - \frac{1}{3} \sin^2 \frac{\theta}{2} \right) \]

\[ = (4 - b) \left( 1 - \frac{1}{3} \sin^2 \frac{\theta}{2} \right) - 4\lambda^2 \sin^2 \frac{\theta}{2}. \]

The condition \( p_1 + p_2 + p_3 > 0 \) and \( p_1 - p_3 > 0 \) are satisfied for \( \alpha > 0, \beta \geq 0 \) and for all \( \theta \) except \( (\theta, \beta) = (0, 0) \) or \( (2\pi, 0) \). We can treat this case separately.

The condition \( p_1 - p_2 + p_3 > 0 \) is satisfied if

\[ 0 < \lambda^2 \leq \frac{4 - b}{6}, \text{ provided, } 0 \leq b < 4. \quad (2.5.10) \]

This scheme is conditionally stable [100].
In order to obtain an unconditionally stable scheme, we may re-write the scheme (2.5.4) as

\[
(1 + \eta b^2) \delta_t^2 u^j_l + \sqrt{a} (2 \mu, \delta_t) u^j_l + \frac{\sqrt{a}}{12} \left( \delta_t^2 2 \mu, \delta_t \right) u^j_l + \left( \frac{b}{12} - \lambda^2 \right) \delta_t^2 u^j_l + b u^j_l \\
+ \frac{\delta_t^2 \delta_x^2}{12} u^j_l - \gamma \lambda^2 \delta_t^2 \delta_x^2 u^j_l = \frac{k^2}{3} \left[ f_{i+1}^l + f_{i-1}^l + f_i^l \right].
\] (2.5.11)

where \( \eta \) and \( \gamma \) are free parameters to be determined. The additional terms are of high-orders and do not affect the accuracy of the scheme.

The exact value \( U^j_l = u(x_l, t_j) \) satisfies

\[
(1 + \eta b^2) \delta_t^2 U^j_l + \sqrt{a} (2 \mu, \delta_t) U^j_l + \frac{\sqrt{a}}{12} \left( \delta_t^2 2 \mu, \delta_t \right) U^j_l + \left( \frac{b}{12} - \lambda^2 \right) \delta_t^2 U^j_l + b U^j_l \\
+ \frac{\delta_t^2 \delta_x^2}{12} U^j_l - \gamma \lambda^2 \delta_t^2 \delta_x^2 U^j_l = \frac{k^2}{3} \left[ f_{i+1}^l + f_{i-1}^l + f_i^l \right] + O(k^4 + k^4 h^2 + k^2 h^4). \] (2.5.12)

We assume that there exists an error \( \epsilon^j_l = U^j_l - u^j_l \) at the grid point \((x_l, t_j)\). Subtracting (2.5.11) from (2.5.12), we obtain the error equation

\[
(1 + \eta b^2) \delta_t^2 \epsilon^j_l + \sqrt{a} (2 \mu, \delta_t) \epsilon^j_l + \frac{\sqrt{a}}{12} \left( \delta_t^2 2 \mu, \delta_t \right) \epsilon^j_l + \left( \frac{b}{12} - \lambda^2 \right) \delta_t^2 \epsilon^j_l \\
+ b \epsilon^j_l + \frac{\delta_t^2 \delta_x^2}{12} \epsilon^j_l - \gamma \lambda^2 \delta_t^2 \delta_x^2 \epsilon^j_l = O(k^4 + k^4 h^2 + k^2 h^4). \] (2.5.13)

For stability, we put \( \epsilon^j_l = \xi^j_l e^{i \theta} \) in the homogeneous part of the error equation (2.5.13) and using the above approximations (2.5.7a)-(2.5.7e), we get

\[
\begin{bmatrix}
(1 + \eta b^2)(\xi - 2 + \xi^{-1}) + \sqrt{a}(\xi - \xi^{-1}) \\
+ \frac{\sqrt{a}}{12}(\xi - \xi^{-1})(e^{i \theta} - 2 + e^{-i \theta}) + \left( \frac{b}{12} - \lambda^2 \right)(e^{i \theta} - 2 + e^{-i \theta}) + b \\
(\xi - 2 + \xi^{-1})(e^{i \theta} - 2 + e^{-i \theta})
\end{bmatrix}
\begin{bmatrix}
\xi^j_l e^{i \theta}
\end{bmatrix} = 0
\]
Hence, the characteristic equation is given by

\[ P_1 \xi^2 + P_2 \xi + P_3 = 0 \]  

(2.5.14)

where

\[ P_1 = \left( 1 + \eta b^2 + \sqrt{a} - \sqrt{a} \frac{\sin^2 \left( \frac{\theta}{2} \right)}{3} - \frac{1}{3} \sin^2 \left( \frac{\theta}{2} \right) + 4 \gamma \xi^2 \sin^2 \left( \frac{\theta}{2} \right) \right) \],

\[ P_2 = \left( -2 - 2 \eta b^2 + 4 \left( \lambda^2 - \frac{b}{12} \right) \sin^2 \left( \frac{\theta}{2} \right) + b + \frac{2}{3} \sin^2 \left( \frac{\theta}{2} \right) - 8 \gamma \lambda^2 \sin^2 \left( \frac{\theta}{2} \right) \right) \],

\[ P_3 = \left( 1 + \eta b^2 - \sqrt{a} + \sqrt{a} \frac{\sin^2 \left( \frac{\theta}{2} \right)}{3} - \frac{1}{3} \sin^2 \left( \frac{\theta}{2} \right) + 4 \gamma \lambda^2 \sin^2 \left( \frac{\theta}{2} \right) \right) \].
Using the transformation \( \xi = \frac{1 + z}{1 - z} \), the characteristic equation (2.5.14) reduces to

\[
(P_1 - P_2 + P_3) z^2 + 2(P_1 - P_3) z + (P_1 + P_2 + P_3) = 0.
\]

(2.5.15)

The necessary and sufficient condition for \( |\xi| < 1 \) is that

\[
P_1 + P_2 + P_3 > 0, P_1 - P_3 > 0, P_1 - P_2 + P_3 > 0.
\]

(2.5.16)

Thus for stability, we must have the conditions

\[
P_1 + P_2 + P_3 = b \cos^2 \left( \frac{\phi}{2} \right) + 4 \lambda^2 \sin^2 \left( \frac{\phi}{2} \right) + \frac{2b}{3} \sin^2 \left( \frac{\phi}{2} \right) > 0,
\]

\[
P_1 - P_3 = 2 \sqrt{a} \left( \cos^2 \left( \frac{\phi}{2} \right) + \frac{2}{3} \sin^2 \left( \frac{\phi}{2} \right) \right) > 0,
\]

\[
P_1 - P_2 + P_3 = 4 + 4\eta b^2 - b + \frac{b}{3} \sin^2 \left( \frac{\phi}{2} \right) + 4 \left( 4\gamma - 1 \right) \lambda^2 - \frac{1}{3} \sin^2 \left( \frac{\phi}{2} \right) > 0.
\]

First two conditions are satisfied for all choices of variable angle \( \theta \). Multiplying third condition by \( 16\eta \), we get

\[
(64\eta -1) + (8\eta b -1)^2 + \frac{16b\eta}{3} \sin^2 \frac{\theta}{2} + 64\eta \left[ (4\gamma -1) \lambda^2 - \frac{1}{3} \right] \sin^2 \frac{\theta}{2} > 0.
\]

(2.5.17)

Thus the scheme is stable if, \( \eta \geq \frac{1}{64} \), \( \gamma \geq \frac{1+3\lambda^2}{12\lambda^2} \), \( \alpha > 0 \) and \( \beta \geq 0 \) for all \( \theta \) except \( \theta = 0 \) and \( 2\pi \) (when \( b=0 \)). We treat this case separately.

For \( \theta = 0 \) or \( 2\pi \) and \( b = 0 \), we have the characteristic equation

\[
\left( 1 + \sqrt{a} \right) \xi^2 + 2\xi + \left( 1 - \sqrt{a} \right) = 0
\]

(2.5.18)

where \( P_1 = 1+\sqrt{a} \), \( P_2 = -2 \) and \( P_3 = 1-\sqrt{a} \).

The roots of (2.5.18) are \( \xi_{1,2} = 1, \frac{1-\sqrt{a}}{1+\sqrt{a}} \). In this case also \( |\xi| \leq 1 \).
Hence for $\alpha > 0, \beta \geq 0, \eta \geq \frac{1}{64}, \gamma \geq \frac{1 + 3\lambda^2}{12\lambda^2}$, the scheme (2.5.11) is unconditionally stable.

2.6. Computational results

In this section, we have solved some benchmark problems using the method described by equation (2.2.10) and compared our results with the results of the fourth order numerical method discussed by Mohanty et al [101] for the solution of 1-D non-linear wave equations. The exact solutions are provided in each case. The right hand side homogeneous functions, initial and boundary conditions may be obtained using the exact solution as a test procedure. The linear difference equations have been solved using a tridiagonal solver, whereas non-linear difference equations have been solved using the Newton-Raphson method. While using the Newton-Raphson method, the iterations were stopped when absolute error tolerance $\leq 10^{-12}$ was achieved. In order to demonstrate the fourth order convergence of the proposed method, throughout the computation we have chosen the fixed value of the parameter $\sigma = \frac{k}{h} = 3.2$. For this choice, our method behaves like a fourth order method in space. All computations were carried out using double precision arithmetic.

Note that, the proposed method (2.2.10) for second order hyperbolic equations is a three-level scheme. The value of $u$ at $t=0$ is known from the initial condition. To start any computation, it is necessary to know the numerical value of $u$ of required accuracy at $t=k$. In this section, we discuss an explicit scheme of $O(k^2)$ for $u$ at first time level, i.e., at $t=k$ in order to solve the differential equation (2.1.1) using the method (2.2.10), which is applicable to problems in cartesian and polar coordinates.

Since the values of $u$ and $u_t$ are known explicitly at $t=0$, this implies all their successive tangential derivatives are known at $t=0$, i.e. the values of $u, u_t, u_{tx}, \ldots, u, u_{tx}, \ldots, etc.$ are known at $t=0$.

An approximation for $u$ of $O(k^2)$ at $t=k$ may be written as

$$U_i^1 = U_i^0 + kU_{it}^0 + \frac{k^2}{2}(U_{tt}^0) + O(k^3)$$  \hspace{1cm} (2.6.1)

From equation (2.1.1), we have
Thus, using the initial values and their successive tangential derivative values, from (2.6.2), we can obtain the value of $(U_0)_0$, and then ultimately, from (2.6.1) we can compute the value of $u$ at first time level, i.e. at $t=k$. Replacing the variable $x$ by $r$ in (2.6.1), we can also obtain an approximation of $O(k^2)$ for $u$ at $t=k$ in polar coordinates.
Example 2.6.1. (Wave equation in polar coordinates)

\[ \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \alpha \frac{\partial u}{\partial r} - \left( 2 \cosh \frac{r}{\alpha} + \frac{\sinh r}{r} \right), \quad 0 < r < 1, \ t > 0. \]

The initial and boundary conditions, respectively, are given by

\[ u(r, 0) = 0, \quad \frac{\partial u(r, 0)}{\partial t} = \cosh r, \quad 0 \leq r \leq 1, \]
\[ u(0, t) = \sin t, \quad u(1, t) = \cosh(1) \sin t, \quad t \geq 0. \]

For \( \alpha = 1 \) and 2, the equation above represents the one-space dimensional wave equation in cylindrical and spherical coordinates, respectively. The analytical solution is \( u = \cosh r \sin t \). The maximum absolute errors and the CPU time (in second) are tabulated in Table-2.6.1 at \( t = 1 \) and \( t = 2 \) for \( \alpha = 1 \) and 2, respectively. Also, the 3-D plot of the analytical and the numerical solution are given in Fig. 2.6.1a and Fig. 2.6.1b, respectively, at \( \alpha = 1 \) for \( t = 1 \).

<table>
<thead>
<tr>
<th>( h )</th>
<th>proposed ( O(k^2+k^2h^2+h^4) )-method</th>
<th>( O(k^4+k^2h^2+h^4) )-method discussed in [101]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha = 1 )</td>
<td>( \alpha = 2 )</td>
<td>( \alpha = 1 )</td>
</tr>
<tr>
<td>( t = 1 )</td>
<td>( t = 2 )</td>
<td>( t = 1 )</td>
</tr>
<tr>
<td>( \frac{1}{8} ) (CPU Time)</td>
<td>.1975(-03) (0.351)</td>
<td>.2148(-03) (0.363)</td>
</tr>
<tr>
<td>( \frac{1}{16} ) (CPU Time)</td>
<td>.1252(-04) (0.369)</td>
<td>.1284(-04) (0.374)</td>
</tr>
<tr>
<td>( \frac{1}{32} ) (CPU Time)</td>
<td>.7889(-06) (0.389)</td>
<td>.8483(-06) (0.419)</td>
</tr>
<tr>
<td>( \frac{1}{64} ) (CPU Time)</td>
<td>.5004(-07) (0.601)</td>
<td>.5304(-07) (0.879)</td>
</tr>
</tbody>
</table>
Fig 2.6.1a: Analytical Solution at $\alpha = 1$ for $t = 1$

Fig 2.6.1b: Numerical Solution at $\alpha = 1$ for $t = 1$
Example 2.6.2. ( Telegraphic equation with forcing function )

\[
\frac{\partial^2 u}{\partial t^2} + 2\alpha \frac{\partial u}{\partial t} + \beta^2 u = \frac{\partial^2 u}{\partial x^2} + (3 - 4\alpha + \beta^2)e^{-2t} \sinh x, \quad 0 < x < 1, \quad t > 0. 
\]

The initial and boundary conditions are, respectively, given by

\[
u(x, 0) = \sinh x, \quad u_t(x, 0) = -2\sinh x, \quad 0 \leq x \leq 1, \\
u(0, t) = 0, \quad u(1, t) = \frac{1}{2}(e^{-e^{-1}})e^{-2t}, \quad t \geq 0.
\]

where \( \alpha > 0, \beta \geq 0 \) are real parameters. The analytical solution is \( u = e^{-2t} \sinh x \). The maximum absolute errors and the CPU time (in second) are tabulated in Table-2.6.2 at \( t = 5 \) for \( \alpha > 0, \beta \geq 0 \). Also, the 3-D plot of the analytical and the numerical solution are given in Fig. 2.6.2a and Fig. 2.6.2b, respectively, at \( \alpha = 10, \beta = 5, \eta = 0.5 \) and \( \gamma = 1 \) for \( t = 5 \).

### Table-2.6.2

Example 2.6.2: The maximum absolute errors [using proposed \( O(k^2 + k^2h^2 + h^4) \)-method]

<table>
<thead>
<tr>
<th>( h )</th>
<th>( \alpha = 10, \beta = 5, \eta = 0.5, \gamma = 1 )</th>
<th>( \alpha = 20, \beta = 10, \eta = 1, \gamma = 1 )</th>
<th>( \alpha = 40, \beta = 4, \eta = 10, \gamma = 20 )</th>
<th>( \alpha = 50, \beta = 5, \eta = 0.25, \gamma = 0.75 )</th>
<th>( \alpha = 50, \beta = 2, \eta = 10, \gamma = 5 )</th>
<th>( \alpha = 10, \beta = 0, \eta = 5, \gamma = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{8} ) (CPU time)</td>
<td>0.7264(-06) (0.392)</td>
<td>0.1025(-06) (0.395)</td>
<td>0.3852(-03) (0.390)</td>
<td>0.2359(-03) (0.398)</td>
<td>0.6604(-03) (0.386)</td>
<td>0.2078(-03) (0.393)</td>
</tr>
<tr>
<td>*0.7451(-06) (0.588)</td>
<td>*0.1105(-06) (0.590)</td>
<td>*0.3855(-03) (0.586)</td>
<td>*0.2363(-03) (0.592)</td>
<td>*0.6685(-03) (0.576)</td>
<td>*0.2132(-03) (0.589)</td>
<td></td>
</tr>
<tr>
<td>( \frac{1}{16} ) (CPU time)</td>
<td>0.5093(-07) (0.616)</td>
<td>0.7455(-08) (0.621)</td>
<td>0.2910(-04) (0.616)</td>
<td>0.1542(-04) (0.619)</td>
<td>0.4399(-04) (0.607)</td>
<td>0.1335(-04) (0.611)</td>
</tr>
<tr>
<td>*0.9033(-07) (0.920)</td>
<td>*0.3980(-07) (0.922)</td>
<td>*0.6530(-04) (0.921)</td>
<td>*0.5137(-04) (0.921)</td>
<td>*0.9227(-04) (0.760)</td>
<td>*0.4103(-04) (0.901)</td>
<td></td>
</tr>
<tr>
<td>( \frac{1}{32} ) (CPU time)</td>
<td>0.3221(-08) (2.302)</td>
<td>0.5208(-09) (2.297)</td>
<td>0.1848(-05) (2.305)</td>
<td>0.9760(-06) (2.288)</td>
<td>0.2784(-05) (2.312)</td>
<td>0.8408(-06) (2.268)</td>
</tr>
<tr>
<td>*0.2508(-07) (3.451)</td>
<td>*0.6686(-08) (3.301)</td>
<td>*0.9455(-05) (3.322)</td>
<td>*0.7204(-05) (3.287)</td>
<td>*0.3625(-04) (3.567)</td>
<td>*0.6532(-05) (3.221)</td>
<td></td>
</tr>
<tr>
<td>( \frac{1}{64} ) (CPU time)</td>
<td>0.2014(-09) (15.263)</td>
<td>0.3276(-10) (15.738)</td>
<td>0.1158(-06) (15.641)</td>
<td>0.6112(-07) (15.597)</td>
<td>0.1744(-06) (15.956)</td>
<td>0.5264(-07) (215.959)</td>
</tr>
<tr>
<td>*0.5205(-08) (22.814)</td>
<td>*0.8882(-09) (24.027)</td>
<td>*0.1218(-05) (23.889)</td>
<td>*0.9366(-06) (22.999)</td>
<td>*0.6267(-05) (22.954)</td>
<td>*0.7643(-06) (23.097)</td>
<td></td>
</tr>
</tbody>
</table>

* Result obtained by using the method discussed by Mohanty [109].
Fig 2.6.2a: Analytical Solution at $\alpha=10, \beta=5, \eta=0.5$ and $\gamma=1$ for $t=5$

Fig 2.6.2b: Numerical Solution at $\alpha=10, \beta=5, \eta=0.5$ and $\gamma=1$ for $t=5$
Example 2.6.3. (Van der Pol type nonlinear wave equation)

\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \gamma(u^2 - 1)\frac{\partial u}{\partial t} + \left(\pi^2 + \gamma^2 e^{-\gamma^2} \sin^2(\pi x)\right)e^{\gamma^2} \sin(\pi x), \quad 0 < x < 1, \quad t > 0.
\]

The initial and boundary conditions are, respectively, given by

\[
u(x,0) = \sin(\pi x), \quad \frac{\partial u}{\partial t}(x,0) = -\gamma \sin(\pi x), \quad 0 \leq x \leq 1,
\]

\[
u(0,t) = 0, \quad \nu(1,t) = 0, \quad t \geq 0
\]

with the exact solution, \( u = e^{-\gamma^2} \sin(\pi x) \). The maximum absolute errors and CPU time (in second) are tabulated in Table-2.6.3 at \( t = 2 \) for \( \gamma = 1, 2 \) and 3. Also, the 3-D plot of the analytical and the numerical solution are given in Fig. 2.6.3a and Fig. 2.6.3b, respectively, at \( \gamma = 3 \) for \( t = 2 \).

**Table-2.6.3**

Example 2.6.3: The maximum absolute errors

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>( O(k^2 + k^2 h^2 + h^4) )-method discussed in [101]</th>
<th>( O(k^4 + k^2 h^2 + h^4) )-method discussed in [101]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( h = \frac{1}{8} ) (CPU time)</td>
<td>( h = \frac{1}{16} ) (CPU time)</td>
</tr>
<tr>
<td>1</td>
<td>0.2221(-04) (0.475)</td>
<td>0.2280(-04) (0.483)</td>
</tr>
<tr>
<td>2</td>
<td>0.1342(-05) (1.073)</td>
<td>0.1351(-05) (1.083)</td>
</tr>
<tr>
<td>3</td>
<td>0.8934(-07) (5.210)</td>
<td>0.9006(-07) (5.322)</td>
</tr>
<tr>
<td>4</td>
<td>0.3812(-08) (32.120)</td>
<td>0.3733(-08) (33.783)</td>
</tr>
</tbody>
</table>
Fig 2.6.3a: Analytical Solution at $\gamma = 3$ for $t = 2$

Fig 2.6.3b: Numerical Solution at $\gamma = 3$ for $t = 2$
Example 2.6.4. (Dissipative nonlinear wave equation)

\[
\frac{\partial^2 u}{\partial t^2} + 2u \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + (2\sin(\pi x) \cos t + \pi^2 - 1) \sin(\pi x) \sin t, \quad 0 < x < 1, \quad t > 0.
\]

The initial and boundary conditions are, respectively, given by

\[
u(x,0) = 0, \quad \frac{\partial u(x,0)}{\partial t} = \sin(\pi x), \quad 0 \leq x \leq 1, \quad u(0,t) = 0, \quad u(1,t) = 0, \quad t \geq 0.
\]

with exact solution \( u = \sin(\pi x) \sin t \). The maximum absolute errors and the CPU time (in second) are tabulated in Table-2.6.4 at \( t = 1 \) and \( t = 2 \). Also, the 3-D plot of the analytical and the numerical solution are given in Fig. 2.6.4a and Fig. 2.6.4b, respectively, at \( t = 1 \).

**Table-2.6.4**

Example 2.6.4: The maximum absolute errors

<table>
<thead>
<tr>
<th>( h )</th>
<th>( O(k^2 + k^2 h^2 + h^4) )-method</th>
<th>( O(k^4 + k^2 h^4 + h^4) )-method discussed in [101]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( t=1 )</td>
<td>( t=2 )</td>
</tr>
<tr>
<td></td>
<td>( t=1 )</td>
<td>( t=2 )</td>
</tr>
<tr>
<td>\frac{1}{8} (CPU time)</td>
<td>0.9889(-04) (0.537)</td>
<td>0.7040(-04) (0.523)</td>
</tr>
<tr>
<td>\frac{1}{16} (CPU time)</td>
<td>0.6112(-05) (1.083)</td>
<td>0.4336(-05) (0.958)</td>
</tr>
<tr>
<td>\frac{1}{32} (CPU time)</td>
<td>0.3777(-06) (5.527)</td>
<td>0.2672(-06) (5.479)</td>
</tr>
<tr>
<td>\frac{1}{64} (CPU time)</td>
<td>0.2584(-07) (34.17)</td>
<td>0.1180(-07) (34.512)</td>
</tr>
</tbody>
</table>
Fig 2.6.4a: Analytical Solution for $t = 1$

Fig 2.6.4b: Numerical Solution for $t = 1$
Example 2.6.5. (Nonlinear wave equation with variable coefficients)

\[
\frac{\partial^2 u}{\partial t^2} = \left(1 + x^2\right) \frac{\partial^2 u}{\partial x^2} + \gamma \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}\right) + f(x, t), \quad 0 < x < 1, \ t > 0.
\]

The initial and boundary conditions are, respectively, given by

\[
\begin{align*}
  u(x, 0) &= \cosh x, \quad \frac{\partial u}{\partial t}(x, 0) = -\cosh x, \quad 0 \leq x \leq 1, \\
  u(0, t) &= 0, \quad u(1, t) = \frac{1}{2}(e^{-\gamma t})e^{\gamma t}, \quad t \geq 0,
\end{align*}
\]

with the exact solution \( u = e^{-\gamma t} \cosh x \). \textbf{The maximum absolute errors and the CPU time (in second)} are tabulated in Table-2.6.5 at for \( \gamma = 2,5 \text{ and } 10 \) at \( t = 1 \). Also, \textbf{the 3-D plot} of the analytical and the numerical solution are given in Fig. 2.6.5a and Fig. 2.6.5b, respectively, at \( \gamma = 5 \) for \( t = 1 \).

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
\( h \) & \multicolumn{3}{c|}{proposed \( O(k^2 + k^2h^2 + h^4) \)-method} & \multicolumn{3}{c|}{\( O(k^4 + k^2h^2 + h^4) \)-method discussed in [101]} \\
\hline

\( \gamma \) & 2 & 5 & 10 & 2 & 5 & 10 \\
\hline
\( \frac{1}{8} \) & \\
(CPU time) & 0.1342(-03) & 0.7055(-03) & 0.1222(-01) & 0.8675(-03) & 0.2419(-02) & 0.8226(-01) \\
& (0.477) & (0.488) & (0.472) & (0.713) & (0.745) & (0.707) \\
\hline
\( \frac{1}{16} \) & \\
(CPU time) & 0.8615(-05) & 0.4204(-04) & 0.8677(-03) & 0.5219(-04) & 0.1214(-03) & 0.4848(-02) \\
& (0.928) & (1.163) & (0.876) & (1.407) & (1.893) & (1.303) \\
\hline
\( \frac{1}{32} \) & \\
(CPU time) & 0.5316(-06) & 0.2551(-05) & 0.5470(-04) & 0.3118(-05) & 0.8644(-05) & 0.2661(-03) \\
& (3.063) & (3.277) & (3.622) & (4.594) & (4.899) & (5.411) \\
\hline
\( \frac{1}{64} \) & \\
(CPU time) & 0.2744(-07) & 0.1484(-06) & 0.3361(-05) & 0.2033(-06) & 0.5558(-06) & 0.1442(-04) \\
& (18.507) & (20.552) & (22.935) & (28.092) & (31.967) & (34.456) \\
\hline
\end{tabular}
\end{table}
Fig. 2.6.5a: Analytical Solution at $\gamma = 5$ for $t = 1$

Fig. 2.6.5b: Numerical Solution at $\gamma = 5$ for $t = 1$
The order of convergence may be obtained by using the formula

\[
\frac{\log(e_{h_1}) - \log(e_{h_2})}{\log(h_1) - \log(h_2)}
\]

where \( e_{h_1} \) and \( e_{h_2} \) are maximum absolute errors for two uniform mesh widths \( h_1 \) and \( h_2 \), respectively. For computation of order of convergence of the proposed method, we have considered \( h_1 = \frac{1}{32} \) and \( h_2 = \frac{1}{64} \) for all cases and results are reported in Table-2.6.6.

**Table-2.6.6**

Order of convergence: \( h_1 = \frac{1}{32} \), \( h_2 = \frac{1}{64} \)

<table>
<thead>
<tr>
<th>Example</th>
<th>Parameters</th>
<th>Order of the method</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.6.1</td>
<td>( \gamma = 1 ) at ( t=1 )</td>
<td>3.978</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 1 ) at ( t=2 )</td>
<td>3.999</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 2 ) at ( t=1 )</td>
<td>3.930</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 2 ) at ( t=2 )</td>
<td>4.000</td>
</tr>
<tr>
<td>2.6.2</td>
<td>( \alpha = 10, \beta = 5, \eta = 0.5, \gamma = 1 ) at ( t=2 )</td>
<td>3.999</td>
</tr>
<tr>
<td></td>
<td>( \alpha = 20, \beta = 10, \eta = 1, \gamma = 1 ) at ( t=2 )</td>
<td>3.990</td>
</tr>
<tr>
<td></td>
<td>( \alpha = 40, \beta = 4, \eta = 10, \gamma = 20 ) at ( t=2 )</td>
<td>3.993</td>
</tr>
<tr>
<td></td>
<td>( \alpha = 50, \beta = 5, \eta = 0.25, \gamma = 0.75 ) at ( t=2 )</td>
<td>3.997</td>
</tr>
<tr>
<td></td>
<td>( \alpha = 50, \beta = 2, \eta = 10, \gamma = 5 ) at ( t=2 )</td>
<td>3.996</td>
</tr>
<tr>
<td></td>
<td>( \alpha = 10, \beta = 0, \eta = 5, \gamma = 5 ) at ( t=2 )</td>
<td>3.997</td>
</tr>
<tr>
<td>2.6.3</td>
<td>( \gamma = 1 ) at ( t=2 )</td>
<td>4.550</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 2 ) at ( t=2 )</td>
<td>4.592</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 3 ) at ( t=2 )</td>
<td>4.951</td>
</tr>
<tr>
<td>2.6.4</td>
<td>at ( t=1 )</td>
<td>3.869</td>
</tr>
<tr>
<td></td>
<td>at ( t=2 )</td>
<td>4.501</td>
</tr>
<tr>
<td>2.6.5</td>
<td>( \gamma = 2 ) at ( t=1 )</td>
<td>4.276</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 5 ) at ( t=1 )</td>
<td>4.103</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 10 ) at ( t=1 )</td>
<td>4.024</td>
</tr>
</tbody>
</table>
2.7. Conclusions

Available numerical methods for the numerical solution of second order non-linear wave equations are of order four, which require 9-grid points. In this chapter, using the same number of grid points and three evaluations of the function \( g \) (as compared to five evaluations of the function \( g \) discussed in [100]), we have derived a new stable method of \( O(k^2 + k^3 h^2 + h^4) \) accuracy for the solution of non-linear wave equation (2.1.1). Ultimately, we use less algebra for computation and for a fixed parameter value \( \sigma = \frac{k}{h^2} \), the proposed method behaves like a fourth order method, which is exhibited from the computed results. The proposed numerical method is directly applicable to wave equation in polar coordinates, and we do not require any fictitious points for computation, which is a main advantage of our work.