Chapter 5
Importance of Model Instructions & Respondent Privacy
Chapter 5
IMPORTANCE OF MODEL INSTRUCTIONS &
RESPONDENT PRIVACY

5.1 Introduction

For ages, surveys have been conducted to obtain reliable information about population characteristics. But in some surveys, people may refuse/hesitate to answer or give incorrect responses when the question being asked is perceived by them as sensitive and private. The respondent does so to project himself/herself in a positive light and may be even to save himself/herself from punitive action. RRT based survey is always conducted under the assumption that the respondents will follow the instructions of the model truthfully. However, it is natural for a section of respondents to have misgivings about the randomization technique and be hesitant to use their true response while using the randomization device, believing that the interviewer will be able to determine their true response somehow. The respondent’s disbelief in randomized response procedure hampers the outcome of the models. The RRT must be designed with the aim of maintaining anonymity/privacy of the respondent’s response and efficiency of the model. These are two different issues where one aims to protect respondent’s privacy and second aims to protect the statistician’s interest.

Also, randomized response techniques were introduced with the objective that the respondents will be able to mask their true response in an interview which the interviewer will not be able to determine and thus the respondents’ privacy will be protected. It has been observed very often that when the RRT models are compared, one model is said to perform better than the other when the variance of the estimator of the first model is less than the corresponding variance of the second model. This is seen from a statistician’s view point. The aspect of respondent privacy protection is often neglected.
The above two problems are dealt with in this chapter. We first study the impact of defying instructions by the respondents when asked to respond to the sensitive question using the Gupta et al. (2010) model. The case of defying instructions is discussed for two scenarios in Section 5.2. In the first scenario, a known proportion of only those respondents who are asked to tell the truth, defy instructions and in the second scenario, a known proportion of all respondents defy instructions. In both cases, a proportion $F$ of the respondents pretends that they do not possess the sensitive characteristic when in fact they do. In Section 5.2, we describe the above scenarios and examine the impact of the defiance on model efficiency. The focus is on studying the impact on the accuracy of the two-stage optional RRT model of Gupta et al. (2010) when a proportion of respondents defies instructions and responds assuming that she/he does not have any trace of the sensitive characteristic. We study two different scenarios under which the instructions are defied and we examine the corresponding impact in terms of the increase in the variance of the mean estimator. Relevant results are in Mehta (2012).

In the second half of this chapter, we discuss the important aspect of respondent’s privacy protection in RRT. Several RRT’s exist in the literature and they are used to obtain the mean estimator (in quantitative response models) or estimator of the true proportion of respondents with sensitive characteristic in the population (in binary response models). Various models aim to reduce the $\text{Var} (\hat{\mu}_X)$ (in quantitative response models where $X$ denotes the sensitive variable whose mean is to be estimated and $\hat{\mu}_X$ denotes an estimator of the mean of $X$) or $\text{Var} (\hat{\pi})$ (in binary response models where $\pi$ denotes the true proportion of the respondents in the population with sensitive characteristic and $\hat{\pi}$ is its estimator). The efficiency of a model is often judged by comparing $\text{Var} (\hat{\mu}_X)$ (in quantitative response models) or $\text{Var} (\hat{\pi})$ (in binary response models). While comparing two RRT models, the model with smaller $\text{Var} (\hat{\mu}_X)$ or $\text{Var} (\hat{\pi})$ (as the case may be) is said to perform better than the model with larger $\text{Var} (\hat{\mu}_X)$ or $\text{Var} (\hat{\pi})$. This method of comparing models has been
used extensively by Greenberg et al. (1969), Eichhorn and Hayre (1983), and Gupta et al. (2011, 2012), among others.

In an effort to introduce models for which the variance of the estimator is small, one should also take into account respondent’s privacy. The issue of respondent privacy protection has been studied for binary randomized response models by several authors including Warner (1965) for the binary model introduced therein and Chang and Liang (1996) for a binary two-stage unrelated question model. In this chapter, we also study this issue in the context of additive optional RRT model. Diana and Perri (2011) have discussed it when there is an auxiliary variable that helps to estimate the mean prevalence of the sensitive characteristic. For binary response models, several authors like Lanke (1976), Nayak and Adeshiyan (2009), and Yan et al. (2009) have proposed several measures of respondents’ privacy protection and compared models taking the same into account. For quantitative response models also, the issue of privacy protection has been dealt with. Among the recent work is the contribution by Yan et al. (2009) where the measure of privacy protection is defined as $E[(Z - X)^2]$ where $X$ is the true response of the sensitive variable and $Z$ is the reported response. In this chapter, we modify this measure (which works for models with one sample) slightly so that it can work with models with two sub-samples. We then compare Gupta et al. (2006) model and Gupta et al. (2010) model with the Two-Stage $F$-model given in Mehta et al. (2012), in light of respondent’s privacy protection. The Three-Stage model of Mehta et al. (2012) is also compared with the Gupta et al. (2006) model, Gupta et al. (2010) model and with the Two-Stage $F$-model with regard to the same. These comparisons are given in Section 5.3.

In Section 5.4, a new measure of respondents’ privacy protection is proposed by modifying the Lanke (1976)’s approach for binary RRT models. We adapt this approach to make it work for quantitative response models. The underlying idea behind the second proposed measure is that when a respondent reports a non-zero response using a randomization technique, then he/she may or may not actually possess the sensitive characteristic. We examine the cases when the reported response
of the respondent is zero or non-zero. We try to find answer to the following questions:

1) What is the probability that the respondent possesses the sensitive characteristic given that his/her reported response is non-zero, and

2) What is the probability that the respondent possesses the sensitive characteristic given that his/her reported response is zero?

These probabilities are important because if these probabilities are too high, then it will affect respondent’s cooperation. We examine these probabilities with regard to the Gupta et al. (2006) model, Gupta et al. (2010) model, and Two-Stage $F$-model and Three-Stage model (introduced in Mehta et al. (2010)) after briefly discussing these models in Section 5.4 of this chapter. We conclude the chapter by summarizing the findings in Section 5.5.

### 5.2 The impact of defying instructions

In this section, we study the impact of defying instructions by a known proportion of respondents in the two-stage Gupta et al. (2010) model. In this model, a known proportion ($T$) of the respondents are asked to tell the truth, and the remaining proportion of respondents are asked to provide a true response (or additively scrambled response) if the question is considered non-sensitive (or sensitive) by the respondent. This is done using colour-coded cards and the researcher will not know which type of response has been provided, thus anonymity is maintained.

We present two scenarios in this section. In the first scenario, a fraction ($F$) of the respondents who are asked to tell the truth hesitate to answer truthfully and in the second scenario, a fraction ($F$) of all respondents whether they are asked to tell the truth or respond using the optional scrambling, hesitate to use their true response. In
either scenario, all the respondents who are hesitant to respond to the question respond assuming their true response to be zero.

In both the scenarios, the sample of size \( n \) is split into two sub-samples of sizes \( n_1 \) and \( n_2 \) \((n_1 + n_2 = n)\) with each sub-sample using a different scrambling device. Let \( X \) denote the true response of the respondent to the sensitive question with unknown mean \( \mu_X \) and unknown variance \( \sigma_X^2 \). Let \( S_i \) \((i = 1, 2)\) be the scrambling variable associated with the \( i^{th} \) sub-sample. Let the known mean of \( S_i \) be \( \theta_i \) and known variance of \( S_i \) be \( \sigma_{S_i}^2 \). Let \( W \) denote the sensitivity level of the underlying question. Assume that \( X, S_1 \) and \( S_2 \) are mutually independent. Let \( Z_i \) \((i = 1, 2)\) be the reported response in the \( i^{th} \) sub-sample \((i = 1, 2)\). As mentioned earlier, in Gupta et al. (2010) model, in the \( i^{th} \) sub-sample \((i = 1, 2)\), a known proportion \((T)\) of the respondents are asked to tell the truth, and the remaining \((1 - T)\) proportion of respondents provide the true response \( X \) if the question is considered non-sensitive by the respondent and the response \( X + S_i \) if the question is considered sensitive.

**Scenario 1:**

In the first scenario, we assume that among the proportion of respondents who are asked to tell the truth, a known proportion \((F)\) of respondents will hesitate to respond truthfully to the question being asked and defy the interviewer’s instructions. These respondents give their response as 0, that is, the proportion \(TF\) of respondents gives their response as 0. The remaining respondents in this group viz., \(T(1 - F)\), when asked to tell the truth, oblige and provide their true response \( X \). The respondents who were not asked to tell the truth (a proportion \((1 - T))\) respond using the usual optional additive scrambling strategy and their response depends upon how they perceive the question. The proportion \(W(1 - T)\) of these respondents considers the question sensitive, and their response is \( X + S_i \) and the remaining respondents (proportion
do not consider the question sensitive and their reported response is the true response $X$.

The reported response in the $i^{th}$ ($i = 1, 2$) sub-sample is given by

$$ Z_i = \begin{cases} 
0 & \text{with probability } TF \\
X & \text{with probability } (1 - F)T + (1 - T)(1 - W) \\
X + S_i & \text{with probability } (1 - T)W 
\end{cases} \quad (5.1) $$

Thus, for $i = 1, 2$,

$$ E(Z_i) = E(X)[(1 - F)T + (1 - T)(1 - W)] + E(X + S_i)[W (1 - T)] 
= E(X)[(1 - F)T + (1 - T)(1 - W)] + E(X)[W (1 - T)] + E(S_i)[W (1 - T)] 
= E(X)[(1 - F)T + (1 - W)(1 - T) + W (1 - T)] + E(S_i)[W (1 - T)] 
= E(X)(1 - TF) + \theta_i[W (1 - T)] \quad (5.2) $$

For $i = 1, 2$,

$$ E(Z_i^2) = E(X^2)[(1 - F)T + (1 - T)(1 - W)] + E([X + S_i]^2)[W (1 - T)] 
= E(X^2)[(1 - F)T + (1 - T)(1 - W)] + E[X^2 + S_i^2 + 2XS_i][W (1 - T)] 
= E(X^2)[(1 - F)T + (1 - T)(1 - W) + W (1 - T)] + E[S_i^2]W (1 - T) + 2E(X)E(S_i)(W (1 - T)) 
= E(X^2)[1 - TF] + E[S_i^2]W (1 - T) + 2E(X)E(S_i)(W (1 - T)) 
= (\sigma_x^2 + \mu_x^2)(1 - TF) + (\sigma_{S_i}^2 + \theta_i^2)W (1 - T) + 2\mu_x \theta_i W (1 - T) $$. 


\[ \text{Var} (Z_i) = E(Z_i^2) - [E(Z_i)]^2 \]

\[ = (\sigma_x^2 + \mu_x^2)(1 - TF) + (\sigma_{\theta_i}^2 + \theta_i^2)[W (1 - T)] + 2\mu_x\theta_i[W (1 - T)] - (\mu_x (1 - TF) + \theta_i(1 - T)W)^2 \]

\[ = (\sigma_x^2 + \mu_x^2)(1 - TF) + (\sigma_{\theta_i}^2 + \theta_i^2)[W (1 - T)] + 2\mu_x\theta_i[W (1 - T)] - \mu_x^2 (1 - TF)^2 - \theta_i^2[(1 - T)W]^2 - 2\mu_x\theta_i[W (1 - T)(1 - TF)] \]

\[ \sigma_{Z_i(TF)}^2 = [\text{Var} (X) + \mu_x^2 (TF)][1 - TF] + [\sigma_{\theta_i}^2 + \theta_i^2 + 2\mu_x\theta_i(TF)]W (1 - T) - \theta_i^2W^2(1 - T)^2 \]

On solving the Equations 5.2 (for \( i = 1, 2 \)) for \( \mu_x \) and \( W \), we get

\[ \mu_x = \frac{\theta_2 E(Z_1) - \theta_1 E(Z_2)}{(\theta_2 - \theta_1)(1 - TF)}, TF \neq 1, \theta_1 \neq \theta_2 \] (5.3)

\[ W = \frac{E(Z_1) - E(Z_2)}{(\theta_1 - \theta_2)(1 - T)}, T \neq 1, \theta_1 \neq \theta_2 \] (5.4)

Unbiased estimators for \( \mu_x \) and \( W \) respectively can be obtained by estimating \( E(Z_i) \) by \( \bar{Z}_i (i = 1, 2) \). The estimators \( \hat{\mu}_i \) and \( \hat{W}_i \) for \( \mu_x \) and \( W \) respectively are given below:

\[ \hat{\mu}_i = \frac{\theta_2 \bar{Z}_2 - \theta_1 \bar{Z}_1}{(\theta_1 - \theta_2)(1 - TF)}, TF \neq 1, \theta_1 \neq \theta_2 \] (5.5)

\[ \hat{W}_i = \frac{\bar{Z}_1 - \bar{Z}_2}{(\theta_1 - \theta_2)(1 - T)}, T \neq 1, \theta_1 \neq \theta_2 \] (5.6)

Note that for \( \theta_1 \neq \theta_2, TF \neq 1, E(\hat{\mu}_i) = E \left[ \frac{\theta_2 \bar{Z}_1 - \theta_1 \bar{Z}_2}{(\theta_2 - \theta_1)(1 - TF)} \right] \]

\[ = \frac{\theta_2 E(\bar{Z}_1) - \theta_1 E(\bar{Z}_2)}{(\theta_2 - \theta_1)(1 - TF)} \]
\[
\begin{align*}
= & \ \frac{\theta_2 E(Z_1) - \theta_1 E(Z_2)}{(\theta_2 - \theta_1)(1 - TF)} \\
= & \ \mu_X
\end{align*}
\]

Also, for \( \theta_1 \neq \theta_2 \), \( TF \neq 1 \), \( \text{Var}(\hat{\mu}_x) = \text{Var} \left[ \frac{\theta_2 \bar{Z}_1 - \theta_1 \bar{Z}_2}{(\theta_2 - \theta_1)(1 - TF)} \right] \)

\[
\begin{align*}
= & \ \frac{1}{(\theta_2 - \theta_1)^2(1 - TF)^2} \left[ \theta_2^2 \text{Var}(\bar{Z}_1) + \theta_1^2 \text{Var}(\bar{Z}_2) \right] \\
= & \ \frac{1}{(\theta_2 - \theta_1)^2(1 - TF)^2} \left[ \theta_2^2 \sigma_\text{Z_1(F)}^2 \frac{1}{n_1} + \theta_1^2 \sigma_\text{Z_2(F)}^2 \frac{1}{n_2} \right]
\end{align*}
\]

Also, for \( \theta_1 \neq \theta_2 \), \( \hat{\mu}_x \sim AN(\mu_X, V_{s1}) \) where

\[
\begin{align*}
V_{s1} = & \ \frac{1}{(\theta_1 - \theta_2)^2(1 - TF)^2} \left[ \theta_1^2 \sigma_\text{Z_1(F)}^2 \frac{1}{n_2} + \theta_2^2 \sigma_\text{Z_2(F)}^2 \frac{1}{n_1} \right]_{TF \neq 1} \\
\end{align*}
\]

\[
\begin{align*}
\sigma_\text{Z_1(F)}^2 = & \ [\sigma_X^2 + \mu_X^2 (TF)](1 - TF) + [\sigma_X^2 + \theta_1^2 + 2 \mu_X \theta_1 (TF)]W (1 - T) - \theta_1^2 W^2 (1 - T)^2 \\
\sigma_\text{Z_2(F)}^2 = & \ [\sigma_X^2 + \mu_X^2 (TF)](1 - TF) + [\sigma_X^2 + \theta_2^2 + 2 \mu_X \theta_2 (TF)]W (1 - T) - \theta_2^2 W^2 (1 - T)^2
\end{align*}
\]

As before, we may note that for \( \theta_1 \neq \theta_2 \), \( T \neq 1 \), \( E(\hat{W}) = E \left[ \frac{\bar{Z}_1 - \bar{Z}_2}{(\theta_1 - \theta_2)(1 - T)} \right] \)

\[
\begin{align*}
= & \ \frac{E(\bar{Z}_1) - E(\bar{Z}_2)}{(\theta_1 - \theta_2)(1 - T)} \\
= & \ \frac{E(Z_1) - E(Z_2)}{(\theta_1 - \theta_2)(1 - T)} \\
= & \ W
\end{align*}
\]
Also, for \( \theta_i \neq \theta_2, T \neq 1 \), \( \text{Var} (\hat{W}) = \text{Var} \left[ \frac{\bar{Z}_1 - \bar{Z}_2}{(\theta_i - \theta_2)(1 - T)} \right] \)

\[
= \frac{1}{(\theta_i - \theta_2)^2 (1 - T)^2} \left[ \text{Var}(\bar{Z}_1) + \text{Var}(\bar{Z}_2) \right]
\]

\[
= \frac{1}{(\theta_i - \theta_2)^2 (1 - T)^2} \left[ \frac{\sigma_{Z_1(F)}^2}{n_1} + \frac{\sigma_{Z_2(F)}^2}{n_2} \right]
\]

where \( \sigma_{Z_1}^2 \) and \( \sigma_{Z_2}^2 \) are as given in (5.8) and (5.9) respectively.

Thus, we can conclude that for \( \theta_i \neq \theta_2, T \neq 1 \), \( \hat{W} \sim AN (W, V_{s_2}) \), where

\[
V_{s_2} = \frac{1}{(\theta_i - \theta_2)^2 (1 - T)^2} \left\{ \frac{\sigma_{Z_1(F)}^2}{n_1} + \frac{\sigma_{Z_2(F)}^2}{n_2} \right\}
\]

(5.10)

where \( \sigma_{Z_1}^2 \) and \( \sigma_{Z_2}^2 \) are as given in (5.8) and (5.9) respectively.

Under Scenario 1, with \( TF \neq 1, \theta_i \neq \theta_2 \), the burden \( B_1 \) on variance of mean estimator because of defying instructions is given by

\[
B_1 = \frac{TF}{(\theta_i - \theta_2)^2 (1 - T)^2} \left[ \frac{\theta_i^2}{n_2} \left[ (\text{Var}(X) + \mu_x^2)[1 - TF] + 2 \mu_x \theta_2 W (1 - T) \right] + (2 - TF) W (1 - T) \left[ (\sigma_{Z_2}^2 + \theta_2^2 - \theta_2^2 W (1 - T) \right] \right] + \frac{\theta_i^2}{n_1} \left[ (\text{Var}(X) + \mu_x^2)[1 - TF] + 2 \mu_x \theta_1 W (1 - T) \right] + (2 - TF) W (1 - T) \left[ (\sigma_{Z_1}^2 + \theta_1^2) - \theta_1^2 W (1 - T) \right] \]
\]

where the burden on variance of mean estimator is defined as \( V_{s_1} - \text{Var}(\hat{\mu}_T) \), and \( \text{Var}(\hat{\mu}_T) \) is the mean estimator variance for the Gupta et al. (2010) model. It may be observed that the burden on the variance of the mean estimator in the first scenario is a positive quantity, as expected.
**Scenario 2:**

In the second scenario, we assume that in the population, a known proportion \( F \) of respondents defy the interviewer’s instructions, and consider their true response to be zero. In \( i^{th} \) \((i = 1, 2)\) sub-sample, among the proportion \( T \) of respondents who are asked to tell the truth, \( T(1 - F) \) proportion of respondents oblige and give true response \( (X) \) to the question and the proportion \( TF \) of respondents hesitate to respond truthfully and give their response as 0. The remaining proportion \((1 - T)\) responds using the usual optional additive scrambling strategy. The proportion \((1 - T)W \) \((1 - F)\) of respondents considers the question sensitive, but follows the instructions and their additively scrambled response is \( X + S_i \) and the proportion \((1 - T)WF \) of these respondents considers the question sensitive but gives its response as \( 0 + S_i \), that is \( S_i \). The proportion \((1 - T)(1 - W)F \) of respondents are not asked to tell the truth and do not consider the question as sensitive but hesitate to answer and give their response as 0 and the remaining proportion \((1 - T)(1 - W)(1 - F)\) of respondents neither consider the question as sensitive nor hesitate to answer, and their response is \( X \).

The reported response in the \( i^{th} \) \((i = 1, 2)\) sub-sample is given by

\[
Z_i = \begin{cases} 
X & \text{with probability } y (1 - F) + (1 - T)(1 - W)(1 - F) \\
X + S_i & \text{with probability } y (1 - T)(1 - F)W \\
0 & \text{with probability } y TF + (1 - T)(1 - W)F \\
S_i & \text{with probability } y (1 - T)WF 
\end{cases}
\]  

(5.11)

Thus, for \( i = 1, 2 \),

\[
\]

\[
\]

\[
= E(X)[1 - F ] + E(S_i)|(1 - T)W ]
\]

So, \( E(Z_i) = E(X),(1 - F) + \theta_i, [W(1 - T)] \)  

(5.12)
For \( i = 1, 2, \)

\[
E(Z_i^2) = E(X^2)[(1 - F) + (1 - T)(1 - W)(1 - F)] + E[(X + S_i)^2](W(1-F)(1-T)) + E[S_i^2](WF)(1-T)
\]

\[
= E(X^2)[1 - F] + E[S_i^2].[W (1-T)] + 2E(X)E(S_i)W (1-F)(1-T)
\]

\[
= (\sigma_X^2 + \mu_X^2)(1-F) + (\sigma_{S_i}^2 + \theta_i^2)W (1-T) + 2\mu_X\theta_iW (1-T)(1-F)
\]

Further, for \( i = 1, 2, \)

\[
Var(Z_i) = E(Z_i^2) - [E(Z_i)]^2
\]

\[
= (\sigma_X^2 + \mu_X^2)(1-F) + (\sigma_{S_i}^2 + \theta_i^2)[W (1-T)]
\]

\[
+ 2\mu_X\theta_i[W (1-T)(1-F)] - (\mu_X(1-F) + \theta_i(1-T)W)^2
\]

\[
= (\sigma_X^2 + \mu_X^2)(1-F) + (\sigma_{S_i}^2 + \theta_i^2)[W (1-T)] + 2\mu_X\theta_i[W (1-T)(1-F)]
\]

\[
- \mu_X^2 (1-F)^2 - \theta_i^2[(1-T)W]^2 - 2\mu_X\theta_i[W (1-T)(1-F)]
\]

\[
\sigma_{Z_i,(F)}^2 = Var(X)[1-F] + \mu_X^2(1-F)(F) + (\sigma_{S_i}^2 + \theta_i^2).W (1-T) - \theta_i^2.W^2(1-T)^2
\]

That is,

\[
\sigma_{Z_i,(F)}^2 = Var(X)[1-F] + \mu_X^2(1-F)(F) + (\sigma_{S_i}^2 + \theta_i^2).W (1-T) - \theta_i^2.W^2(1-T)^2
\]

\[
\sigma_{Z_i,(T)}^2 = Var(X)[1-F] + \mu_X^2(1-F)(F) + (\sigma_{S_i}^2 + \theta_i^2).W (1-T) - \theta_i^2.W^2(1-T)^2
\]

On solving the equations (5.12) (for \( i = 1, 2 \)) for \( \mu_X \) and \( W \), we get

\[
\mu_X = \frac{\theta_iE(Z_i) - \theta_1E(Z_1)}{(\theta_2 - \theta_1)(1-F)} \quad F \neq 1, \theta_i \neq \theta_2
\]

\[
W = \frac{E(Z_i) - E(Z_1)}{(\theta_1 - \theta_2)(1-T)} \quad T \neq 1, \theta_i \neq \theta_2
\]
Unbiased estimators $\hat{\mu}_n$ and $\hat{W}_n$ for $\mu_X$ and $W$ respectively can be obtained by estimating $E(Z_i)$ by $\overline{Z}_i (i = 1, 2)$. The estimators $\hat{\mu}_n$ and $\hat{W}_n$ for $\mu_X$ and $W$ respectively are given below:

\[
\hat{\mu}_n = \frac{\theta_1 \overline{Z}_2 - \theta_2 \overline{Z}_1}{(\theta_1 - \theta_2)(1 - F)}, \quad F \neq 1, \theta_1 \neq \theta_2 \tag{5.17}
\]

\[
\hat{W}_n = \frac{\overline{Z}_1 - \overline{Z}_2}{(\theta_1 - \theta_2)(1 - T)}, \quad T \neq 1, \theta_1 \neq \theta_2 \tag{5.18}
\]

Note that for $\theta_1 \neq \theta_2$, $F \neq 1$, $E(\hat{\mu}_n) = E \left[ \frac{\theta_2 \overline{Z}_1 - \theta_1 \overline{Z}_2}{(\theta_2 - \theta_1)(1 - F)} \right]$

\[
= \frac{\theta_2 E(\overline{Z}_1) - \theta_1 E(\overline{Z}_2)}{(\theta_2 - \theta_1)(1 - F)}
\]

\[
= \frac{\theta_2 E(Z_1) - \theta_1 E(Z_2)}{(\theta_2 - \theta_1)(1 - F)}
\]

\[
= \mu_X
\]

Also, for $\theta_1 \neq \theta_2$, $F \neq 1$, $\text{Var}(\hat{\mu}_n) = \text{Var} \left[ \frac{\theta_2 \overline{Z}_1 - \theta_1 \overline{Z}_2}{(\theta_2 - \theta_1)(1 - F)} \right]$

\[
= \frac{1}{(\theta_2 - \theta_1)^2(1 - F)^2} \left[ \theta_2^2 \text{Var}(\overline{Z}_1) + \theta_1^2 \text{Var}(\overline{Z}_2) \right]
\]

\[
= \frac{1}{(\theta_2 - \theta_1)^2(1 - F)^2} \left[ \theta_2^2 \frac{\sigma^2_{Z_1(F)}}{n_1} + \theta_1^2 \frac{\sigma^2_{Z_2(F)}}{n_2} \right]
\]

where $\sigma^2_{Z_1(F)}$ and $\sigma^2_{Z_2(F)}$ are as given in (5.13) and (5.14) respectively.
Also, for \( \theta_1 \neq \theta_2 \), \( \hat{\mu}_n \sim AN (\mu, V_{s3}) \) where

\[
V_{s3} = \frac{1}{(\theta_1 - \theta_2)^2 (1 - T)^2} \left[ \frac{\theta_1^2 \sigma_{Z_1(F)}^2}{n_2} + \frac{\theta_2^2 \sigma_{Z_2(F)}^2}{n_1} \right], \quad F \neq 1
\]

(5.19)

where \( \sigma_{Z_1(F)}^2 \) and \( \sigma_{Z_2(F)}^2 \) are as given in (5.13) and (5.14) respectively.

As before, we may note that for \( \theta_1 \neq \theta_2 \), \( T \neq 1 \), \( E(\hat{W}_n) = E \left( \frac{\hat{Z}_1 - \hat{Z}_2}{(\theta_1 - \theta_2)(1 - T)} \right) \)

\[
= \frac{E(\hat{Z}_1) - E(\hat{Z}_2)}{(\theta_1 - \theta_2)(1 - T)}
\]

\[
= \frac{E(Z_1) - E(Z_2)}{(\theta_1 - \theta_2)(1 - T)}
\]

\[
= W
\]

Also, for \( \theta_1 \neq \theta_2 \), \( T \neq 1 \), \( \text{Var}(\hat{W}_n) = \text{Var} \left( \frac{\hat{Z}_1 - \hat{Z}_2}{(\theta_1 - \theta_2)(1 - T)} \right) \)

\[
= \frac{1}{(\theta_1 - \theta_2)^2 (1 - T)^2} \left[ \text{Var}(\hat{Z}_1) + \text{Var}(\hat{Z}_2) \right]
\]

\[
= \frac{1}{(\theta_1 - \theta_2)^2 (1 - T)^2} \left[ \frac{\sigma_{Z_1(F)}^2}{n_1} + \frac{\sigma_{Z_2(F)}^2}{n_2} \right]
\]

where \( \sigma_{Z_1(F)}^2 \) and \( \sigma_{Z_2(F)}^2 \) are as given in (5.13) and (5.14) respectively.

Concluding, we get that for \( \theta_1 \neq \theta_2 \), \( T \neq 1 \), \( \hat{w}_n \sim AN (W, V_{s4}) \), where

\[
V_{s4} = \frac{1}{(\theta_1 - \theta_2)^2 (1 - T)^2} \left[ \frac{\sigma_{Z_1(F)}^2}{n_1} + \frac{\sigma_{Z_2(F)}^2}{n_2} \right]
\]

(5.20)

where \( \sigma_{Z_1(F)}^2 \) and \( \sigma_{Z_2(F)}^2 \) are as given in (5.13) and (5.14) respectively.
It can be verified that for \( F \neq 1, \theta_1 \neq \theta_2 \), the burden \( B_2 \) on the mean estimator for defying instructions is as given below:

\[
\begin{align*}
\text{where the burden on the mean estimator because of defying instructions is defined as the difference } V_{s3} - \text{Var} (\hat{\mu}_T) . \text{ In the second scenario also, the burden is positive.}
\end{align*}
\]

Note that \( B_1 < B_2 \) will be true if and only if the following condition is satisfied:

\[
\begin{align*}
\left[ \frac{\theta_1^2}{n_2} [\sigma_{s_1}^2 + \sigma_{s_2}^2 W (1 - T)] + \frac{\theta_2^2}{n_1} [(\sigma_{s_1}^2 + \theta_1^2 - \theta_1^2 W (1 - T)] \right] W (1 - T) (2 - F - TF) + \\
[\text{Var} (X) + \mu_X^2 ] (1 - F) (1 - TF) \left[ \frac{\theta_1^2}{n_2} + \frac{\theta_2^2}{n_1} \right] > 2 \mu_X W \theta_1 \theta_2 \left[ \frac{\theta_1^2}{n_2} + \frac{\theta_2^2}{n_1} \right]
\end{align*}
\]

As an illustration, one may note that the above condition holds true for the data:

\( w = 0.8, \) sample size \( n = 1000, \) equal sub-sample size i.e., \( n_1 = n_2 = 500, \)

\( X \sim \text{Poisson} \) (4), \( S_1 \sim \text{Poisson} \) (2), \( S_2 \sim \text{Poisson} \) (5) , \( T = 0.1, 0.3, 0.5, 0.7, 0.9 \text{ and } F = 0.1, 0.3, 0.5, 0.7, 0.9. \)

Table 5.1 below provides a comparison of the variance of the mean estimator of the two-stage Gupta et al. (2010) model and the variance of the mean estimator in the two scenarios for various combinations of \( T \) and \( F \). In this table, we assume that the sensitivity of the underlying question is 0.8, i.e. \( W = 0.8 \). We denote the variance of
the mean estimator under Scenario 1, Scenario 2 and the Gupta et al. (2010) model
by $V(\hat{\mu}_T)$, $V(\hat{\mu}_H)$, and $V(\hat{\mu}_F)$ respectively.

Table 5.1:

$V(\hat{\mu}_T)$ (in bold), $V(\hat{\mu}_H)$, and $V(\hat{\mu}_F)$ (in brackets) for several combinations of $T$ and $F$, with $w = 0.8$, $n = 1000$, $n_1 = n_2 = 500$, $X \sim Poisson$ (4), $S_1 \sim Poisson$ (2) and $S_2 \sim Poisson$ (5)

<table>
<thead>
<tr>
<th>$F$</th>
<th>$T = 0.1$</th>
<th>$T = 0.3$</th>
<th>$T = 0.5$</th>
<th>$T = 0.7$</th>
<th>$T = 0.9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.04594</td>
<td>0.04544</td>
<td>0.04267</td>
<td>0.03762</td>
<td>0.03029</td>
</tr>
<tr>
<td></td>
<td>(0.04594)</td>
<td>(0.04544)</td>
<td>(0.04267)</td>
<td>(0.03762)</td>
<td>(0.03029)</td>
</tr>
<tr>
<td>0.1</td>
<td>0.04856</td>
<td>0.05288</td>
<td>0.04267</td>
<td>0.03762</td>
<td>0.03029</td>
</tr>
<tr>
<td></td>
<td>(0.06499)</td>
<td>(0.06437)</td>
<td>(0.06095)</td>
<td>(0.05472)</td>
<td>(0.04567)</td>
</tr>
<tr>
<td>0.3</td>
<td>0.05405</td>
<td>0.06984</td>
<td>0.08223</td>
<td>0.08906</td>
<td>0.08697</td>
</tr>
<tr>
<td></td>
<td>(0.12216)</td>
<td>(0.12114)</td>
<td>(0.11548)</td>
<td>(0.10518)</td>
<td>(0.09023)</td>
</tr>
<tr>
<td>0.5</td>
<td>0.05986</td>
<td>0.09021</td>
<td>0.12089</td>
<td>0.14794</td>
<td>0.16097</td>
</tr>
<tr>
<td></td>
<td>(0.23531)</td>
<td>(0.23332)</td>
<td>(0.22222)</td>
<td>(0.20203)</td>
<td>(0.17273)</td>
</tr>
<tr>
<td>0.7</td>
<td>0.06604</td>
<td>0.11499</td>
<td>0.17639</td>
<td>0.25140</td>
<td>0.32404</td>
</tr>
<tr>
<td></td>
<td>(0.55052)</td>
<td>(0.54499)</td>
<td>(0.51417)</td>
<td>(0.45807)</td>
<td>(0.37669)</td>
</tr>
<tr>
<td>0.9</td>
<td>0.07261</td>
<td>0.14565</td>
<td>0.26111</td>
<td>0.46917</td>
<td>0.92372</td>
</tr>
<tr>
<td></td>
<td>(3.20178)</td>
<td>(3.15200)</td>
<td>(2.87467)</td>
<td>(2.36978)</td>
<td>(1.63733)</td>
</tr>
</tbody>
</table>
It may be observed from Table 5.1 that when \( F = 0 \), then the value of the variances for all values of \( T \) coincide. This is so because in that case, no respondent defies instructions and both the scenarios are same as the Gupta et al. (2010) model. Note also that in both scenarios the variance of the mean estimator increases with an increase in \( F \). The increase is substantial when the value of \( F \) becomes large.

Further, it may be observed that for all combinations of \( T \) and \( F \), \( V(\hat{\mu}_T) \leq V(\hat{\mu}_1) \leq V(\hat{\mu}_2) \). An important message here is that extra caution should be exercised in educating the respondents regarding the importance of following the model instructions.

Table 5.2 gives the same information in terms of percentage increase in the variance of the Gupta et al. (2010) model when instructions are defied. The percentage increase in Scenario \( i (i = 1, 2) \) is given by \( \left( \frac{B}{V(\hat{\mu}_T)} \right) \times 100 \). The variance values are calculated for the following data: \( w = 0.8 \), \( n = 1000 \), \( n_1 = n_2 = 500 \), \( X \sim Poisson (4) \), \( S_1 \sim Poisson (2) \) and \( S_2 \sim Poisson (5) \).
Table 5.2:

Percentage increase in $V(\hat{\mu}_I)$ (in bold) and percentage increase in $V(\hat{\mu}_H)$ for several combinations of $T$ and $F$, with $w = 0.8$, $n = 1000$, $n_1 = n_2 = 500$.

$X \sim \text{Poisson} \ (4)$, $S_1 \sim \text{Poisson} \ (2)$ and $S_2 \sim \text{Poisson} \ (5)$

<table>
<thead>
<tr>
<th>$F$</th>
<th>$T = 0.1$</th>
<th>$T = 0.3$</th>
<th>$T = 0.5$</th>
<th>$T = 0.7$</th>
<th>$T = 0.9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>5.70</td>
<td>16.37</td>
<td>26.62</td>
<td>37.13</td>
<td>48.76</td>
</tr>
<tr>
<td></td>
<td>41.47</td>
<td>41.66</td>
<td>42.84</td>
<td>45.45</td>
<td>50.78</td>
</tr>
<tr>
<td>0.3</td>
<td>17.65</td>
<td>53.70</td>
<td>92.71</td>
<td>136.74</td>
<td>187.12</td>
</tr>
<tr>
<td></td>
<td>165.91</td>
<td>166.59</td>
<td>170.64</td>
<td>179.59</td>
<td>197.89</td>
</tr>
<tr>
<td>0.5</td>
<td>30.30</td>
<td>98.53</td>
<td>183.31</td>
<td>293.25</td>
<td>431.43</td>
</tr>
<tr>
<td></td>
<td>412.21</td>
<td>413.47</td>
<td>420.79</td>
<td>437.03</td>
<td>470.25</td>
</tr>
<tr>
<td>0.7</td>
<td>43.75</td>
<td>153.06</td>
<td>313.38</td>
<td>568.26</td>
<td>969.79</td>
</tr>
<tr>
<td></td>
<td>1098.35</td>
<td>1099.36</td>
<td>1104.99</td>
<td>1117.62</td>
<td>1143.61</td>
</tr>
<tr>
<td>0.9</td>
<td>58.05</td>
<td>220.53</td>
<td>511.93</td>
<td>1147.13</td>
<td>2949.59</td>
</tr>
<tr>
<td></td>
<td>6869.48</td>
<td>6836.62</td>
<td>6636.98</td>
<td>6199.26</td>
<td>5305.51</td>
</tr>
</tbody>
</table>

5.3 Privacy protection of respondents

The importance of privacy protection has been mentioned in the introductory section of this chapter. Several measures of privacy protection of respondents exist. Yan et al. (2009) introduced a measure of privacy protection given by

$\Delta = E[(Z - X)^2]$ where $X$ is the true response of the respondent to the sensitive question and $Z$ is the reported response of the respondent to the sensitive question. Using this definition, while comparing two models, a model may be regarded as better than the other model in two scenarios. Firstly, the model with smaller value of
Var (\( \hat{\mu}_x \)) is regarded to perform better than the one with larger value of Var (\( \hat{\mu}_x \)), provided their \( \Delta \) value is same. Secondly, the model with larger value of \( \Delta \) is regarded to perform better than the one with the smaller value of \( \Delta \), provided the value of Var (\( \hat{\mu}_x \)) of both the models is same.

Yan et al. (2009) introduced the above measure for respondent’s privacy protection and several models were compared in the same paper but all the models that were compared used only one sample. In order to work with models which use split sample approach, we modify Yan et al (2009)’s measure slightly. Let the total sample size \( n \) be split into two sub-samples of sizes \( n_1 \) and \( n_2 \) (\( n_1 + n_2 = n \)). Let \( \Delta_1 \) and \( \Delta_2 \) represent the value of Yan et al. (2009)’s measure for samples of size \( n_1 \) and \( n_2 \) respectively. Then define the weighted average of \( \Delta_1 \) and \( \Delta_2 \), given by

\[
\Delta = \frac{n_1 \Delta_1 + n_2 \Delta_2}{n}
\]

as the measure of respondent’s privacy protection when the model involves two sub-samples. This idea can be extended to models with even three or more sub-samples.

In this section, we compare the Gupta et al. (2006) model, the Gupta et al. (2010) model with the Two-Stage F-model and the Three-Stage model of Mehta et al. (2012) on the basis of privacy protection of respondents. All models except the last one are two-stage quantitative response models and the last one is a three-stage model as the name suggests. In all these models, let \( W \) denote the sensitivity level, that is, it is the unknown proportion of respondents in the population who consider the question sensitive. Let the mean and variance for sensitive variable \( X \) be denoted by \( \mu_x \) and \( \sigma_x^2 \) respectively. Further, the value of \( \mu_x \) is also to be estimated in all the models and in light of two parameters (viz., \( \mu_x \) and \( W \)), these models use the split-sample approach. Let the total sample size \( n \) be split into two sub-samples of sizes \( n_1 \) and \( n_2 \) (\( n_1 + n_2 = n \)). Let \( S_i \) be the scrambling variable used to scramble the responses in the
The variables $X_i$, $S_1$ and $S_2$ are assumed to be mutually independent. Let the mean and the variance for $S_i$ be $\theta_i$ and $\sigma_{S_i}^2$, respectively.

*Gupta et al. (2006) model:*

In Gupta et al. (2006) model, the respondent is instructed to provide a truthful response $Z = X$ if the question is deemed non-sensitive, and to provide an additively scrambled response $Z = X + S$ if the question is deemed sensitive.

The reported responses in the two sub-samples under this model are given by

$$Z_i = \begin{cases} X & \text{with probability } y (1 - W) \\ X + S_i & \text{with probability } y W \end{cases}, (i = 1, 2)$$  \hspace{1cm} (5.21)

The mean and variance respectively for $Z_i$ are given by

$$E(Z_i) = \mu_X + W \theta_i, \text{ where } \theta_i = E(S_i), (i = 1, 2),$$  \hspace{1cm} (5.22)

and $\sigma_{Z_i}^2 = \sigma_X^2 + \sigma_{S_i}^2 W + \theta_i^2 W (1 - W)$ \hspace{1cm} (5.23)

From (5.21), $Z_i - X = \begin{cases} 0 & \text{with probability } y (1 - W) \\ S_i & \text{with probability } y W \end{cases}, (i = 1, 2)$  \hspace{1cm} (5.24)

Let $\Delta_{Op}$ denote the modified measure of privacy protection (for models with two sub-samples) of the Gupta et al. (2006) model which is an optional RRT model and $\Delta_{Op(i)}$ denote the $\Delta$ value of $i^{th} (i = 1, 2)$ sub-sample.

Then $\Delta_{Op(i)} = E[(Z_i - X)^2]$

$$= E[(S_i)^2]W$$

$$= W[\sigma_{S_i}^2 + \theta_i^2]$$
Putting the value of $\Delta_{op(1)}$ and $\Delta_{op(2)}$ in $\Delta_{op}$ which is given by

$$
\Delta_{op} = \frac{1}{n} \left[ \Delta_{op(1)} n_1 + \Delta_{op(2)} n_2 \right],
$$

we get

$$
\Delta_{op} = \frac{W}{n} \left[ n_1 (\sigma_{s_1}^2 + \theta_1^2) + n_2 (\sigma_{s_2}^2 + \theta_2^2) \right].
$$

(5.25)

**Gupta et al. (2010) model:**

In Gupta et al. (2010) model, a respondent is asked to answer the sensitive question with probability $(T)$ and with probability $(1 - T)$, the respondent is instructed to provide a truthful response $Z = X$ if the question is deemed non-sensitive, and to provide an additively scrambled response $Z = X + S$ if the question is deemed sensitive.

The reported responses in the two sub-samples under this model are given by

$$
Z_i = \begin{cases} 
X & \text{with probability } T + (1 - T)(1 - W) \\
(X + S_i) & \text{with probability } (1 - T)W 
\end{cases}, (i = 1, 2)
$$

(5.26)

The mean and variance respectively for $Z_i$ are given by

$$
E(Z_i) = \mu_X + \theta_i W (1 - T), \text{ where } \theta_i = E(S_i), \ (i = 1, 2),
$$

(5.27)

and

$$
\sigma_{Z_i}^2 = \sigma_X^2 + \sigma_S^2 [(1 - T)W] + \theta_i^2 [(1 - T)W] [1 - [(1 - T)W]]
$$

(5.28)

From (5.26), $Z_i - X = \begin{cases} 
0 & \text{with probability } T + (1 - T)(1 - W) \\
S_i & \text{with probability } (1 - T)W 
\end{cases}, (i = 1, 2)
$$

(5.29)
Let $\Delta_T$ denote the modified measure of privacy protection (for models with two sub-samples) of the Gupta et al. (2010) model and $\Delta_T(i)$ denote the $\Delta$ value of $i^{th}$ ($i = 1, 2$) sub-sample.

Then $\Delta_T(i) = E[(Z_i - X)^2]$

$$= E[(S_i)^2]W (1 - T)$$

$$= W (1 - T)[\sigma_{\delta_i}^2 + \theta_i^2]$$

Putting the value of $\Delta_T(1)$ and $\Delta_T(2)$ in $\Delta_T$ which is given by,

$$\Delta_T = \frac{1}{n}[\Delta_T(1)n_1 + \Delta_T(2)n_2],$$

we get

$$\Delta_T = \frac{W (1 - T)}{n}[n_1(\sigma_{\delta_1}^2 + \theta_1^2) + n_2(\sigma_{\delta_2}^2 + \theta_2^2)].$$

(5.30)

**Two-Stage F-model:**

In Two-Stage F-model, again a split sample approach is used. In each sample, a respondent is asked to scramble their actual response to the sensitive question with probability $(F)$ and with probability $(1 - F)$, the respondent responds using the additive optional strategy, and is instructed to provide a truthful response $Z = X$ if the question is deemed non-sensitive, and to provide an additively scrambled response $Z = X + S$ if the question is deemed sensitive.

The reported response $Z_i$ in the $i^{th}$ ($i = 1, 2$) sub-sample under this model is given by

$$Z_i = \begin{cases} X & \text{with probability } (1 - F)(1 - W) \\ X + S_i & \text{with probability } F + (1 - F)W \end{cases}, (i = 1, 2)$$

(5.31)
The mean and variance respectively for $Z_i$ are given by

$$E(Z_i) = \mu_X + \theta_i[F + (1 - F)W], \text{ where } \theta_i = E(S_i), \quad (i = 1, 2), \quad (5.32)$$

$$\sigma^2_{Z_i} = \sigma_X^2 + \sigma^2_S[F + (1 - F)W] + \theta_i^2[F + (1 - F)W](1 - [F + (1 - F)W]) \quad (5.33)$$

From (5.31), $Z_i - X = \begin{cases} 0 \quad \text{with probability } (1 - F)(1 - W) \\ S_i \quad \text{with probability } F + (1 - F)W \end{cases}, \quad (i = 1, 2) \quad (5.34)$

Let $\Delta_F$ denote the modified measure of privacy protection (for models with two sub-samples) of the Two-Stage $F$-model and $\Delta_{F(i)}$ denote the $\Delta$ value of $i^{th}$ sub-sample.

Then $\Delta_{F(i)} = E[(Z_i - X)^2]$

$$= E[(S_i)^2][F + (1 - F)W]$$

$$= [F + (1 - F)W][\sigma^2_S + \theta_i^2]$$

Putting the value of $\Delta_{F(1)}$ and $\Delta_{F(2)}$ in $\Delta_F$ which is given by

$$\Delta_F = \frac{1}{n}[\Delta_{F(1)}n_1 + \Delta_{F(2)}n_2],$$

we get

$$\Delta_F = \left[\frac{F + (1 - F)W}{n}\right][n_1(\sigma^2_S + \theta_1^2) + n_2(\sigma^2_S + \theta_2^2)]. \quad (5.35)$$

**Three-Stage model:**

In Three-Stage model, a respondent is asked to report the true response with probability $T$ and scramble the actual response with probability $F$ and with
probability \((1 - T - F)\), the respondent responds using the optional strategy, and is, thus, instructed to provide a truthful response \(Z = X\) if the question is deemed non-sensitive, and to provide an additively scrambled response \(Z = X + S\) if the question is deemed sensitive. This is done in both the sub-samples.

The reported response \(Z_i\) in the \(i^{th}\) \((i = 1, 2)\) sub-sample under this model is given by

\[
Z_i = \begin{cases} 
X & \text{with probability } y T \cdot (1 - T - F)(1 - W) \\
X + S_i & \text{with probability } y F \cdot (1 - T - F)W 
\end{cases},\quad (i = 1, 2) \tag{5.36}
\]

The mean and variance respectively for \(Z_i\) are given by

\[
E(Z_i) = \mu_X + \theta_i \cdot \{ F + (1 - T - F)W \}, \quad \text{where } \theta_i = E(S_i), \quad (i = 1, 2), \quad \text{and}
\]

\[
\sigma_{Z_i}^2 = \sigma_X^2 + \left( \sigma_{S_i}^2 + \theta_i^2 \right) \cdot \{ F + (1 - T - F)W \} - \theta_i^2 \cdot \{ F + (1 - T - F)W \}^2 \tag{5.37}
\]

From (5.36), \(Z_i - X = \begin{cases} 
0 & \text{with probability } y T \cdot (1 - T - F)(1 - W) \\
S_i & \text{with probability } y F \cdot (1 - T - F)W 
\end{cases},\quad (i = 1, 2) \tag{5.39}
\]

Let \(\Delta_{TF}\) denote the modified measure of privacy protection (for models with two sub-samples) of the Three-Stage model and \(\Delta_{TF(i)}\) denote the \(\Delta\) value of \(i^{th}\) \((i = 1, 2)\) sub-sample.

Then \(\Delta_{TF(i)} = E[(Z_i - X)^2] \)

\[
= E[(S_i)^2] \cdot \{ F + (1 - T - F)W \}
\]

\[
= \{ F + (1 - T - F)W \} \cdot \{ \sigma_{S_i}^2 + \theta_i^2 \}
\]
Putting the value of $\Delta_{TF\,(1)}$ and $\Delta_{TF\,(2)}$ in $\Delta_{TF}$ which is given by

$$\Delta_{TF} = \frac{1}{n} \left[ \Delta_{TF\,(1)}n_1 + \Delta_{TF\,(2)}n_2 \right],$$

we get

$$\Delta_{TF} = \left[ \frac{F + (1-T-F)W}{n} \right] \left[ n_1 (\sigma_{S_1}^2 + \theta_1^2) + n_2 (\sigma_{S_2}^2 + \theta_2^2) \right].$$

(5.40)

**Comparing the models using the modified Yan et al.’s measure (2009):**

We compare the privacy protection levels of the Gupta et al. (2006) model, Gupta et al. (2010) model, Two-Stage $F$-model and Three-Stage model using the modified Yan et al. (2009)’s measure irrespective of the relation between the variance of the mean estimator of the various pairs of models. We assume that the scrambling variables $S_i$ ($i = 1, 2$) used in each pair of models being compared have the same value of mean and variance, and further that the value of $F$, $T$ and/or $W$ is same in the models being compared. The comparison is made keeping the sub-sample sizes same in the models under consideration and also assuming that $W = 0$ and $W = 1$ are unlikely cases.
Table 5.3:

<table>
<thead>
<tr>
<th>Model 1</th>
<th>Model 2</th>
<th>Comparison</th>
<th>Remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gupta et al. (2006) model</td>
<td>Gupta et al. (2010) model</td>
<td>$\Delta_{op} &gt; \Delta_T$ $\iff T &gt; 0$ which is true</td>
<td>The parameter $T$ can be assumed to be positive as otherwise the Gupta et al. (2010) model reduces to the Gupta et al. (2006) model and in that case the two models would offer the same level of privacy to the respondents.</td>
</tr>
<tr>
<td>Gupta et al. (2006) model</td>
<td>Two-Stage $F$-model</td>
<td>$\Delta_T &gt; \Delta_{op}$ $\iff F(1-W) &gt; 0$ which is true</td>
<td>The parameter $F$ can be assumed to be positive as otherwise the Two-Stage $F$-model reduces to Gupta et al. (2006) model and in that case these models would offer the same level of protection to the respondents.</td>
</tr>
<tr>
<td>Gupta et al. (2006) model</td>
<td>Three-Stage model</td>
<td>$\Delta_{op} &gt; \Delta_T$ $\iff F(1-W) - WT &gt; 0$ which may or may not be true</td>
<td>The parameters $T, F$ can be assumed to be greater than zero as otherwise these models reduce to Gupta et al. (2006) model and the value of $W = 0$ or 1 has already been ruled out.</td>
</tr>
<tr>
<td>Gupta et al. (2010) model</td>
<td>Two-Stage $F$-model</td>
<td>$\Delta_T &gt; \Delta_{op}$ $\iff F(1-W) + WT &gt; 0$ which is true</td>
<td>The parameter $F$ can be assumed to be greater than zero as otherwise the Three-Stage model reduces to Gupta et al. (2010) model and the value of $W = 1$ has been ruled out.</td>
</tr>
<tr>
<td>Gupta et al. (2010) model</td>
<td>Three-Stage model</td>
<td>$\Delta_{op} &gt; \Delta_T$ $\iff F(1-W) &gt; 0$ which is true</td>
<td>The parameter $F$ can be assumed to be greater than zero as otherwise the Three-Stage model reduces to the Two-Stage $F$-model.</td>
</tr>
<tr>
<td>Two-Stage $F$-model</td>
<td>Three-Stage model</td>
<td>$\Delta_T &gt; \Delta_{op}$ $\iff T &gt; 0$ which is true</td>
<td>The parameter $T$ can be assumed to be greater than zero as otherwise the Three-Stage model reduces to the Two-Stage $F$-model.</td>
</tr>
</tbody>
</table>
It may be observed from the above table that $\Delta_f$ is greater than $\Delta_T$, $\Delta_{TF}$, and $\Delta_{op}$.

This seems obvious as in Two-Stage $F$-model more respondents are reporting a scrambled response and it protects their true response, thus, increasing the privacy of the respondents. However, one also notes from the table that $\Delta_f$ is smaller than $\Delta_T$, $\Delta_{op}$, and $\Delta_f$. In Gupta et al. (2010) model, the introduction of a truth element, though introduced while maintaining complete anonymity, does not help protect the privacy of the respondents since a proportion of the respondents here are asked to report their true response.

### 5.4 A new measure of respondent privacy protection

When a respondent is asked a sensitive question, say, how many times have you smoked marijuana, and if the true response of the respondent to this question is a non-zero quantity, then it indicates that the respondent possesses the sensitive characteristic, that is he/she has smoked marijuana at least once. The true response of the respondent to this question indicates the number of times marijuana was smoked. In a survey if the respondent reports his/her response and if the interviewer can determine that the respondent’ actual response is non-zero (not necessarily the actual response), then it indicates the prevalence of the sensitive characteristic in the respondent which too may cause embarrassment to the respondent. Thus, if a respondent knows that using his/her reported response, the interviewer can determine with a high probability that the respondent possesses the sensitive characteristic, it can lead to refusal to respond and non-cooperation from the respondent due to the stigma that the sensitive questions carry. So, in a survey one should try to minimize the probability that the respondent possesses the sensitive characteristic given his/her reported response.
We follow Lanke (1976) approach which was given for the binary case. The measure of protection introduced therein was given by

\[ \Delta = \max \{ P(A|Yes), P(A|No) \} . \]

The underlying idea in the Lanke (1976) measure and the one we introduce below is that after a respondent responds using the randomization device, it would be stigmatizing for the respondent if he/she is categorized with high probability as one who possesses the sensitive characteristic \( A \). Thus, in Lanke (1976) the model with the smaller value of \( \Delta \) was regarded as the one which offered more privacy protection to the respondents.

In this section, keeping this idea as the fundamental basis, we introduce a measure of privacy protection of respondents in the quantitative setup.

**Definition 5.1:** For a quantitative response model with sensitive characteristic \( A \), and \( Z_{=0} \) and \( Z_{\neq 0} \) respectively denoting that the reported response is zero and non-zero, respondent’s privacy protection is defined as \( \gamma = \max \{ P(A|Z_{=0}), P(A|Z_{\neq 0}) \} . \)

**Definition 5.2:** Given two randomized response models \( R_1 \) and \( R_2 \), Model \( R_1 \) offers more privacy protection to the respondents than Model \( R_2 \) if \( \gamma_{R_1} < \gamma_{R_2} \).

In all the models, let the mean and variance of \( X \) be denoted by \( \mu_X \) and \( \sigma_X^2 \) respectively. Let the total sample size \( n \) be split into two sub-samples of sizes \( n_1 \) and \( n_2 \) \( (n_1 + n_2 = n) \). Let \( S_i \) be the scrambling variable used to scramble the responses in the \( i^{th} \) sub-sample \( (i = 1, 2) \). Assume that the scrambling variable \( S_i \) in the \( i^{th} \) sub-sample \( (i = 1, 2) \) takes only positive values. The variables \( X, S_1 \) and \( S_2 \) are assumed to be mutually independent. Let the mean and the variance for \( S_i \) be \( \theta_i \) and \( \sigma_{S_i}^2 \) respectively.
Let $\pi$ denote the true proportion of the respondents with sensitive characteristic $A$. Let $Z_0$ and $Z_{\neq 0}$ denote the event that the reported response of the respondent is non-zero and the reported response is zero respectively.

**Gupta et al. (2006) model:**

Now, we introduce $\pi$ in the probability categories as appearing in equation (5.21), involved in the Gupta et al. (2006) model. When the probabilities involve $\pi$, then the reported response $Z_i$ in the $i^{th}$ ($i = 1, 2$) sample is as given below:

$$Z_i = \begin{cases} 
X & \text{with probability } (1 - W)\pi \\
0 & \text{with probability } (1 - W)(1 - \pi) \\
X + S_i & \text{with probability } W\pi \\
S_i & \text{with probability } (1 - \pi)W
\end{cases}$$

(5.41)

Note that $P(Z_{\neq 0} | A) = 1$, $P(Z_0 | A) = 0$, $P(Z_{\neq 0} | \overline{A}) = 1 - W$ and $P(Z_0 | \overline{A}) = W$.

Using Bayes Theorem, we get

$$P(A | Z_{\neq 0}) = \frac{P(Z_{\neq 0} | A)P(A)}{P(Z_{\neq 0} | A)P(A) + P(Z_0 | \overline{A})P(\overline{A})} = \frac{\pi}{\pi + (1 - \pi)W}. $$

$$P(A | Z_0) = \frac{P(Z_0 | A)P(A)}{P(Z_0 | A)P(A) + P(Z_0 | \overline{A})P(\overline{A})} = 0$$

We denote $\gamma$ of the Gupta et al. (2006) model by $\gamma_{op}$.

Note that $\gamma_{op} = \frac{\pi}{\pi + (1 - \pi)W}$. 
Gupta et al. (2010) model:

Keeping Gupta et al. (2010) as the underlying model, we introduce \( \pi \) in the probability categories where \( \pi \) denotes the true proportion of the respondents with sensitive characteristic. It may be observed that the true response of a respondent will be non-zero if and only if he/she possesses the sensitive characteristic. All probability categories as appearing in equation (5.26), viz., \( T, (1 - T)W \) and \( (1 - T)(1 - W) \) can be further classified into two categories each involving \( \pi \). For example: \( T \) can be classified into \( T \pi \) and \( T(1 - \pi) \). When the probabilities involve \( \pi \), then the reported response \( Z_i \) in the \( i^{th} \) \((i = 1, 2)\) sample is as given below:

For \( i = 1, 2 \),

\[
Z_i = \begin{cases} 
X & \text{with probability } \pi T + (1 - T)(1 - W)\pi \\
0 & \text{with probability } T(1 - \pi) + (1 - T)(1 - W)(1 - \pi) \\
X + S_i & \text{with probability } W(1 - T)\pi \\
S_i & \text{with probability } (1 - T)(1 - \pi)W 
\end{cases}
\]

(5.42)

We note that the posterior probability \( P(Z_{x0} | A) = 1 \). This is so because if a respondent possesses the sensitive characteristic, then he/she is supposed to either report the truth (which is non-zero) or the scrambled response (which is sum of the true response and a positive scrambling variable, thus, non-zero). So in either situation, the respondent reports a non-zero response.

Further, we also note that if a respondent does not possess the sensitive characteristic, then he/she is supposed to either report the truth (which is zero) or the scrambled response (which is \( 0 + S_i = S_i \), a non-zero value with the probability \( (1 - T)(1 - \pi)W \)). So the probability of a respondent reporting a non-zero response and simultaneously not possessing the sensitive characteristic is \( (1 - T)(1 - \pi)W \). Thus, \( P(Z_{x0} | \overline{A}) = (1 - T)W \).
Using Bayes Theorem, we get

\[
P(A|Z_{z=0}) = \frac{P(Z_{z=0}|A)P(A)}{P(Z_{z=0}|A)P(A) + P(Z_{z=0}|\bar{A})P(\bar{A})} = \frac{\pi}{\pi + (1-\pi)(1-T)W} \quad (5.43)
\]

It may be observed that \( P(Z_{z=0}|A) = 0 \). This is so because if a respondent possesses the sensitive characteristic, then he is supposed to either report the truth (which is non-zero) or the scrambled response (which is also non-zero). So in either situation, he reports a non-zero response.

Further, note that if a respondent does not possess the sensitive characteristic, he is supposed to either report the truth (which is zero with the probability \( (1-\pi)[T + (1-T)(1-W)] \)) or the scrambled response (which is non-zero). So, a respondent who does not possess the sensitive characteristic and reports a zero response does so with the probability \( (1-\pi)[T + (1-T)(1-W)] \). Thus,

\[
P(Z_{z=0}|\bar{A}) = T + (1-T)(1-W)
\]

Thus, again using Bayes Theorem, we get

\[
P(A|Z_{z=0}) = \frac{P(Z_{z=0}|A)P(A)}{P(Z_{z=0}|A)P(A) + P(Z_{z=0}|\bar{A})P(\bar{A})} = 0
\]

From (5.43), it may be observed that as \( T \) increases, the probability \( P(A|Z_{z=0}) \) increases and \( \gamma_T = \frac{\pi}{\pi + (1-\pi)(1-T)W} \). Thus, the Gupta et al. (2010) model which uses smaller value of \( T \) offers more protection to the respondent than the one.
**Two-Stage F-model:**

As before, we introduce \( \pi \) in the probability categories involved in Two-Stage F-model, where \( \pi \) denotes the true proportion of the respondents with sensitive characteristic and all probability categories as appearing in equation (5.31) can be further classified into two categories each involving \( \pi \). For example: \( F \) can be classified into \( F \pi \) and \( F(1-\pi) \). When the probabilities involve \( \pi \), then the reported response \( Z_i \) in the \( i^{th} \) \((i = 1, 2) \) sample is as given below:

\[
Z_i = \begin{cases} 
X & \text{with probability } (1-F)(1-W)\pi \\
0 & \text{with probability } (1-F)(1-W)(1-\pi) \\
X + S_i & \text{with probability } F\pi + W(1-F)\pi \\
S_i & \text{with probability } F(1-\pi) + (1-F)(1-\pi)W
\end{cases}
\]

Using the same arguments as in Gupta et al. (2010) model, we have for the Two-Stage F-model,

\[
P(Z_{x0}|A) = 1 \text{ and } P(Z_{\neq 0}|A) = F + (1-F)W.
\]

Thus,

\[
P(A|Z_{x0}) = \frac{P(Z_{x0}|A)P(A)}{P(Z_{x0}|A)P(A) + P(Z_{\neq 0}|A)P(A)} = \frac{\pi}{\pi + (1-\pi)[F + (1-F)W]}
\]

Again, by a similar argument as made for Gupta et al. (2010) model,

\[
P(A|Z_{\neq 0}) = \frac{P(Z_{\neq 0}|A)P(A)}{P(Z_{\neq 0}|A)P(A) + P(Z_{x0}|A)P(A)} = 0.
\]
We denote $\gamma$ of the Two-Stage $F$-model by $\gamma_F$ and it is given by

$$\gamma_F = \frac{\pi}{\pi + (1 - \pi)[F + (1 - F)W]}.$$  

**Three-Stage model:**

Now, we introduce $\pi$ in the probability categories as appearing in equation (5.36), involved in the Three-Stage model. When the probabilities involve $\pi$, then the reported response $Z_i$ in the $i^{th}$ $(i = 1, 2)$ sample is as given below:

For $i = 1, 2$,

$$Z_i = \begin{cases} 
X & \text{with probability } \gamma T \pi + (1 - T - F)(1 - W)\pi \\
0 & \text{with probability } \gamma (1 - \pi) + (1 - T - F)(1 - W)(1 - \pi) \\
X + S_i & \text{with probability } \gamma F \pi + W(1 - T - F)\pi \\
S_i & \text{with probability } \gamma (1 - \pi) + (1 - T - F)(1 - \pi)W 
\end{cases}  \quad (5.45)$$

Clearly, $P(Z_{x0}|A) = 1$ and $P(Z_{\neq 0}|\overline{A}) = F + (1 - T - F)W$.

Thus,

$$P(A|Z_{x0}) = \frac{P(Z_{x0}|A)P(A)}{P(Z_{x0}|A)P(A) + P(Z_{\neq 0}|\overline{A})P(\overline{A})} = \frac{\pi}{\pi + (1 - \pi)[F + (1 - T - F)W]}$$

Also,

$$P(A|Z_{=0}) = \frac{P(Z_{=0}|A)P(A)}{P(Z_{=0}|A)P(A) + P(Z_{=0}|\overline{A})P(\overline{A})} = 0$$
We denote $\gamma$ of the Three-Stage model by $\gamma_{TF}$ and it is given by

$$\gamma_{TF} = \frac{\pi}{\pi + (1 - \pi)[F + (1 - T - F)W]}.$$  

**Comparing the models using the modified Lanke’s measure (1976):**

Irrespective of the relation between the variance of the mean estimator of the various pairs of models and assuming that the scrambling variables $S_i$ $(i = 1, 2)$ used in the each pair of models being compared have the same value of mean and variance and are positive and the value of sub-sample sizes are same, we compare four models: Gupta et al. (2006) model, Gupta et al. (2010) model, Two-Stage $F$-model, and the Three-Stage model. The models are compared assuming that the value of $F$, $T$ and/or $W$ is same and that $W = 0$ and $W = 1$ are improbable cases.

**Table 5.4:**


<table>
<thead>
<tr>
<th>Model 1</th>
<th>Model 2</th>
<th>Comparison</th>
<th>Remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gupta et al. (2006) model</td>
<td>Gupta et al. (2010) model</td>
<td>$\gamma_T &gt; \gamma_{op} \iff T &gt; 0$ which is true</td>
<td>The parameter $T$ can be assumed to be positive otherwise the Gupta et al. (2010) model reduces to the Gupta et al. (2006) model.</td>
</tr>
<tr>
<td>Gupta et al. (2006) model</td>
<td>Two-Stage $F$-model</td>
<td>$\gamma_{op} &gt; \gamma_F \iff F(1 - W) &gt; 0$ which is true</td>
<td>The parameter $F$ can be assumed to be positive otherwise the Two-Stage $F$-model reduces to Gupta et al. (2006) model and in that case these models would offer the same level of protection to the respondents.</td>
</tr>
<tr>
<td>Model (Year)</td>
<td>Model Type</td>
<td>Inequality</td>
<td>Description</td>
</tr>
<tr>
<td>-------------</td>
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<td>------------</td>
<td>-------------</td>
</tr>
<tr>
<td>Gupta et al. (2006) model</td>
<td>Three-Stage model</td>
<td>$\gamma_{op} &gt; \gamma_{TF}$&lt;br&gt;$F(1-W) - WT &gt; 0$&lt;br&gt;which may or may not be always true</td>
<td></td>
</tr>
<tr>
<td>Gupta et al. (2010) model</td>
<td>Two-Stage F-model</td>
<td>$\gamma_T &gt; \gamma_F$&lt;br&gt;$F(1-W) + WT &gt; 0$&lt;br&gt;which is true</td>
<td>The parameters $T, F$ can be assumed to be greater than zero as otherwise these models reduce to Gupta et al. (2006) model.</td>
</tr>
<tr>
<td>Gupta et al. (2010) model</td>
<td>Three-Stage model</td>
<td>$\gamma_T &gt; \gamma_{TF} \Leftrightarrow F(1-W) &gt; 0$&lt;br&gt;which is true</td>
<td>The parameter $F$ can be assumed to be greater than zero as otherwise the Three-Stage model reduces to Gupta et al. (2010) model.</td>
</tr>
<tr>
<td>Two-Stage F-model</td>
<td>Three-Stage model</td>
<td>$\gamma_{TF} &gt; \gamma_F \Leftrightarrow T &gt; 0$&lt;br&gt;which is true</td>
<td>The parameter $T$ can be assumed to be greater than zero as otherwise the Three-Stage model reduces to the Two-Stage F-model.</td>
</tr>
</tbody>
</table>

It may be observed from the above table, that $\gamma_f$ is smaller than $\gamma_T$, $\gamma_{TF}$ and $\gamma_{op}$. Thus, giving us that the Two-Stage $F$-model offers more privacy to the respondents than the Gupta et al. (2006) model, Gupta et al. (2010) model and the Three-Stage model. This seems obvious as in Two-stage $F$-model more respondents are reporting a scrambled response and it protects their true response, thus, increasing the privacy of the respondents. However, one also notes from the table that $\gamma_T$ is smaller than $\gamma_{TF}$, $\gamma_{op}$ and $\gamma_f$. In Gupta et al. (2010) model, the introduction of a truth element, though introduced while maintaining complete anonymity, does not help protect the privacy of the respondents as the respondents are then asked to report their true response.

It was shown in Mehta et al. (2012) that the Two-Stage $F$-model does perform better than the Gupta et al. (2010) model for values of $F$ satisfying a condition. But
irrespective of the value of the variance of the mean estimator, the Two-Stage F-model does offer more privacy protection to the respondents in light of definition 5.2.

5.5 Conclusion

In this Chapter, we have considered two important aspects that are associated with RRT. In RRT survey methodology, respondent’s trust in the survey technique plays a crucial role and the success of the RRT depends a lot on it as else the respondents will not follow the instructions of the RRT completely. The presence of a section of respondents who are hesitant to respond truthfully and hence defy instructions, can significantly affect the variance of the mean estimator negatively. This impact can be quite substantial if the proportion of such respondents who defy instructions is large. The respondents may feel that the researcher can achieve descrambling at an individual level which is not so in reality. Thus extra care should be taken by the researcher in educating the respondents that the RRT model being used can not reveal the true response of the respondent.

Also, this chapter discusses the issue of privacy protection of respondents using two measures. The first is a modified version of the measure introduced in Yan et al. (2009) which was for quantitative models with one sample. The second measure which is proposed here is based on Lanke (1976) but is in quantitative set up and works under the assumption that a respondent will feel embarrassed if upon giving a response, he/she is classified as one possessing the sensitive characteristic. We have compared four models, viz., Gupta et al. (2006) model, Gupta et al. (2010) model, Two-Stage F-model and Three-Stage model. The models are compared only in terms of privacy protection levels and the variance has not been taken into account for comparing the models. The findings obtained after comparing these models using the two measures are same. Among these models, the Two-Stage F-model of Mehta et al. (2012) protects respondent privacy the most. This is logically viable, since a known proportion of respondents are instructed to scramble their responses and thus the model, offers more privacy protection to the respondents. It was observed in Chapter
that the Two-Stage $F$-model performs more efficiently than the Gupta et al. (2006)
model for large values of $W$, which is the scenario when one relies on a large value of
$T$. So for questions which are highly sensitive, the Two-Stage $F$-model may be used
to elicit response from respondents. Asking respondents to scramble their responses
to a highly sensitive question will help improve respondent cooperation and also
reduce the variance of the mean estimator. On the other hand, the introduction of the
truth element leads to some loss of privacy in Gupta et al. (2010) model.