CHAPTER 2

Variants of Weyl-Type Theorems

The spectrum of a bounded linear operator $T$ on a Banach space $X$ can be split into subsets in many different ways, depending on the purpose of study. In this chapter, we look more closely to some parts of the spectrum of a bounded operator on a Banach space from the view point of Fredholm theory. In recent years, Fredholm theory has witnessed an explosive development and there is remarkable growth of literature in it. Various variants of Weyl’s theorem have been introduced and studied by many mathematicians. By the efforts of these mathematicians, this theory has been brought to a higher degree of completeness.

In [56], Rakočević introduced property (w), a variant of Weyl’s theorem. Following [56], an operator $T \in B(X)$ possesses property (w) if

$$\sigma_a(T) \setminus \sigma_{usf^-(T)} = E_0(T).$$

Motivated by his work, M. Amouch and M. Berkani [8] defined that an operator $T \in B(X)$ satisfies property (gw) if
\[ \sigma_a(T) \setminus \sigma_{usbf}^-(T) = E(T). \]

Property (gw) extends property (w) in the context of \( B \)-Fredholm theory. It is proved that an operator possessing property (gw) possesses property (w), but the converse is not true in general.

Later in the year 2009, Berkani and Zariouh [18] defined that an operator \( T \in B(X) \) possesses property (b) if

\[ \sigma_a(T) \setminus \sigma_{usf}^{-}(T) = \pi_0(T), \]

and possesses property (gb) if

\[ \sigma_a(T) \setminus \sigma_{usbf}^{-}(T) = \pi(T). \]

Properties (b) and (gb) play roles analogous to Browder’s theorem and generalized Browder’s theorem, respectively. It is shown that an operator possessing property (gb) possesses property (b), but the converse is not true in general.

Continuing their study further, M. Berkani and H. Zariouh [19] defined that an operator \( T \in B(X) \) possesses property (ab) if

\[ \sigma(T) \setminus \sigma_W(T) = \pi_{0}^{a}(T), \]

and possesses property (gab) if

\[ \sigma(T) \setminus \sigma_{BW}(T) = \pi^{a}(T), \]
while \( T \in B(X) \) possesses property (aw) if
\[
\sigma(T) \setminus \sigma_W(T) = E_0^a(T),
\]
and possesses property (gaw) if
\[
\sigma(T) \setminus \sigma_{BW}(T) = E^a(T).
\]
Motivated by the work done in [8], [18] and [19], we define and study new properties, namely property (Bw), property (Baw), property (Bb) and property (Bab). The study has been organized in four sections of the chapter. In Section 1, we study property (Bw). Section 2 deals with property (Baw). Section 3 is devoted to property (Bb) whereas in Section 4, we discuss property (Bab). In the concluding part, we give a diagram summarizing the relations between Weyl-type theorems and various properties.

2.1 Property (Bw)

To begin our study, we introduce the notion of property (Bw). We characterize property (Bw) and obtain for a bounded linear operator defined on a Banach space some necessary and sufficient conditions for the property (Bw).

**Definition 2.1.1** [35]. A bounded linear operator \( T \in B(X) \) is said to satisfy property (Bw) if
\[
\sigma(T) \setminus \sigma_{BW}(T) = E_0(T).
\]
We give an example of an operator satisfying property (Bw):

**Example 2.1.2** [35]. Let $Q \in B(l^2(N))$ be a quasinilpotent operator defined as

$$Q(x_1, x_2, x_3, \ldots) = \left(\frac{1}{2}x_2, \frac{1}{3}x_3, \ldots\right)$$

for all $x = (x_1, x_2, x_3, \ldots) \in \ell^2(N)$ and $N \in B(l^2(N))$ be a nilpotent operator.

Consider the operator $T$ defined on $\ell^2(N) \oplus \ell^2(N)$ by

$$T = Q \oplus N.$$  

Then

$$\sigma(T) = \sigma_W(T) = \sigma_{BW}(T) = \{0\}, \quad E(T) = \{0\}$$

and

$$E_0(T) = \phi.$$  

Thus, $T$ satisfies property (Bw).

Next is an example of an operator which fails to satisfy property (Bw):

**Example 2.1.3** [35]. Let $T \in B(l^2(N))$ be defined as

$$T(x_1, x_2, x_3, \ldots) = \left(\frac{1}{2}x_2, \frac{1}{3}x_3, \ldots\right)$$

for all $x = (x_1, x_2, x_3, \ldots) \in l^2(N)$. 
Then
\[ \sigma(T) = \sigma_W(T) = \sigma_{BW}(T) = E_0(T) = \{0\}. \]

Thus, \( T \) fails to satisfy the property (Bw).

We recall the following proposition by Duggal [23, Proposition 3.9].

**Proposition 2.1.4** [23, Proposition 3.9].

(a) The following statements are equivalent.

(i) \( T \) satisfies generalized Browder’s theorem.

(ii) \( T \) has SVEP at points \( \lambda \not\in \sigma_{BW}(T) \).

(b) \( T \) satisfies generalized Browder’s theorem if and only if \( T \) satisfies Browder’s theorem.

We apply this proposition to establish the following.

**Theorem 2.1.5** [35]. Let \( T \in B(X) \) satisfy property (Bw). Then generalized Browder’s theorem holds for \( T \) and
\[ \sigma(T) = \sigma_{BW}(T) \cup \sigma_{iso}(T). \]

**Proof.** By Proposition 2.1.4, it is sufficient to prove that \( T \) has SVEP at every \( \lambda \not\in \sigma_{BW}(T) \). Assume that \( \lambda \not\in \sigma_{BW}(T) \). The following two cases arise.

**Case 1.** If \( \lambda \not\in \sigma(T) \), then \( T \) has SVEP at \( \lambda \).
Case 2. If \( \lambda \in \sigma(T) \) and \( T \) satisfies property (Bw), then

\[
\lambda \in \sigma(T) \setminus \sigma_{BW}(T) = E_0(T).
\]

This shows that \( \lambda \in \sigma_{iso}(T) \) which implies that \( T \) has SVEP at \( \lambda \). Therefore, generalized Browder’s theorem holds for \( T \).

To prove \( \sigma(T) = \sigma_{BW}(T) \cup \sigma_{iso}(T) \), we consider

\[
\lambda \in \sigma(T) \setminus \sigma_{BW}(T) = E_0(T).
\]

We see that \( \lambda \in \sigma_{iso}(T) \). Thus \( \sigma(T) \subseteq \sigma_{BW}(T) \cup \sigma_{iso}(T) \).

But \( \sigma_{BW}(T) \cup \sigma_{iso}(T) \subseteq \sigma(T) \) for every \( T \in B(X) \) and hence \( \sigma(T) = \sigma_{BW}(T) \cup \sigma_{iso}(T) \).

In the next theorem, we characterize operators possessing property (Bw).

**Theorem 2.1.6** [35]. Let \( T \in B(X) \). Then the following statements are equivalent:

(i) \( T \) satisfies property (Bw),

(ii) generalized Browder’s theorem holds for \( T \) and \( \pi(T) = E_0(T) \).

**Proof.** First, we show that (i) implies (ii). Assume that \( T \) satisfies property (Bw). By Theorem 2.1.5, it is sufficient to prove the equality \( \pi(T) = E_0(T) \).
If \( \lambda \in E_0(T) \), then as \( T \) satisfies property (Bw), we have
\[
\lambda \in \sigma(T) \setminus \sigma_{BW}(T) = \pi(T),
\]
because generalized Browder’s theorem holds for \( T \). Therefore, \( E_0(T) \subseteq \pi(T) \).

If \( \lambda \in \pi(T) = \sigma(T) \setminus \sigma_{BW}(T) = E_0(T) \), then it implies that \( \lambda \in E_0(T) \). Therefore \( \pi(T) \subseteq E_0(T) \) and hence we have the equality \( \pi(T) = E_0(T) \).

To show that (ii) implies (i), we consider \( \lambda \in \sigma(T) \setminus \sigma_{BW}(T) \). Then generalized Browder’s theorem implies that \( \lambda \in \pi(T) = E_0(T) \). Therefore \( \sigma(T) \setminus \sigma_{BW}(T) \subseteq E_0(T) \).

Conversely, if \( \lambda \in E_0(T) \) then \( \lambda \in \pi(T) = \sigma(T) \setminus \sigma_{BW}(T) \). It implies that \( E_0(T) \subseteq \sigma(T) \setminus \sigma_{BW}(T) \). Thus
\[
\sigma(T) \setminus \sigma_{BW}(T) = E_0(T).
\]

Hence \( T \) satisfies property (Bw). \( \square \)

We give a precise description of operators satisfying property (Bw) in terms of SVEP at certain sets.

**Theorem 2.1.7** [35]. Let \( T \in B(X) \). If \( T \) or \( T^\ast \) has SVEP at points in \( \sigma(T) \setminus \sigma_{BW}(T) \), then \( T \) satisfies property (Bw) if and only if \( E_0(T) = \pi(T) \).
The hypothesis $T$ or $T^*$ has SVEP at points in $\sigma(T) \setminus \sigma_{BW}(T) = \sigma(T^*) \setminus \sigma_{BW}(T^*)$ implies that $T$ satisfies generalized Browder’s theorem [23, Proposition 3.9]. Now, if $\pi(T) = E_0(T)$, then $\sigma(T) \setminus \sigma_{BW}(T) = \pi(T) = E_0(T)$. Thus, property (Bw) holds for $T$. Converse is trivially true. \hfill $\Box$

We now recall that operators $S, T \in B(X)$ are said to be injectively interwined, denoted by $S \prec_i T$, if there exists an injection $U \in B(X)$ such that $TU = US$.

For injectively interwined operators, we have the following:

**Theorem 2.1.8** [35]. If $S \prec_i T$ and if $T$ has SVEP at a point $\lambda$, then $S$ has SVEP at $\lambda$.

**Proof**. Let $T$ have SVEP at $\lambda$, let $\mathcal{U}$ be an open neighbourhood of $\lambda$ and let $f : \mathcal{U} \to X$ be an analytic function such that $(S - \mu)f(\mu) = 0$ for every $\mu \in \mathcal{U}$. Then $U(S - \mu)f(\mu) = (T - \mu)uf(\mu) = 0 \Rightarrow uf(\mu) = 0$. Since $U$ is injective, $f(\mu) = 0$, that is, $S$ has SVEP at $\lambda$. \hfill $\Box$

The following is a consequence of Theorem 2.1.7 and Theorem 2.1.8.
Theorem 2.1.9 [35]. Let $S, T \in B(X)$. If $T$ has SVEP and $S \prec_i T$, then $S$ satisfies property (Bw) if and only if $E_0(S) = \pi(S)$.

**Proof.** Suppose that $T$ has SVEP. Since $S \prec_i T$, therefore by Theorem 2.1.8, $S$ has SVEP. Thus, by Theorem 2.1.7, $S$ satisfies property (Bw) if and only if $E_0(S) = \pi(S)$. The result follows. □

It is known that $\sigma_{BW}(T) \subseteq \sigma_W(T)$ for every $T \in B(X)$.

Hence, if $T$ satisfies property (Bw), then

$$\sigma(T) \setminus \sigma_W(T) \subseteq \sigma(T) \setminus \sigma_{BW}(T) = E_0(T).$$

Thus, if $\sigma_{iso}(T) = \phi$, then $\sigma(T) = \sigma_W(T) = \sigma_{BW}(T)$ (and $T$ satisfies Weyl’s theorem and generalized Weyl’s theorem).

For a non-quasinilpotent operator, $T \in B(X)$, a condition guaranteeing $\sigma_{iso}(T) = \phi$ is that $K(T) = \{0\}$.

With these, we establish the following.

Theorem 2.1.10 [35]. Let $T \in B(X)$ be not quasinilpotent and $K(T) = \{0\}$. Then $\sigma(T) = \sigma_W(T) = \sigma_{BW}(T)$ and $T$ satisfies both property (Bw) and generalized Weyl’s theorem.

**Proof.** Let $T \in B(X)$ be not quasinilpotent and $K(T) = \{0\}$. Then $T$ has SVEP, $\sigma(T) = \sigma_W(T)$ is a connected set containing 0 and $\sigma_{iso}(T) = \phi$ [1, Theorem 3.121]. Also if, $K(T) = \{0\}$, then
\[ \sigma(T) = \sigma_{BW}(T) \quad [7, \text{Proposition 2.2}]. \] Therefore \( \sigma(T) = \sigma_W(T) = \sigma_{BW}(T) \). \( T \) has SVEP implies that \( T \) satisfies generalized Browder’s theorem \( [7, \text{Theorem 2.3}] \). Therefore,

\[
\sigma(T) \setminus \sigma_{BW}(T) = \pi(T) = \phi = E_0(T) = E(T).
\]

Hence \( T \) satisfies property (Bw) and generalized Weyl’s theorem (so also Weyl’s theorem).

**Remark 2.1.11** \([35]\). Let \( T \in B(X) \) be quasinilpotent, then \( \sigma(T) = \sigma_{BW}(T) = \{0\} \). Hence \( T \) satisfies property (Bw) is equivalent to \( T \) satisfies Weyl’s theorem.

**Theorem 2.1.12** \([35]\). Let \( T \in B(X) \) be polaroid and satisfy property (Bw). Then generalized Weyl’s theorem holds for \( T \).

**Proof.** \( T \) is polaroid and satisfies property (Bw) if and only if

\[
\sigma(T) \setminus \sigma_{BW}(T) = E_0(T) \subseteq E(T) = \pi(T) = \sigma(T) \setminus \sigma_{BW}(T),
\]

as \( T \) satisfies generalized Browder’s theorem by Theorem 2.1.5. Thus, \( \sigma(T) \setminus \sigma_{BW}(T) = E(T) \) and hence generalized Weyl’s theorem holds for \( T \). \( \square \)
2.1 Property (Bw)

Theorem 2.1.13 [35]. Let \( T \in B(X) \) be finitely isoloid and satisfy generalized Weyl’s theorem. Then \( T \) satisfies property (Bw).

**Proof.** If \( T \) satisfies generalized Weyl’s theorem, then

\[
\sigma(T) \setminus \sigma_{BW}(T) = E(T).
\]

To show that \( T \) satisfies property (Bw), we need to prove that \( E(T) = E_0(T) \).

If \( \lambda \in E(T) \), then \( \lambda \in \sigma_{\text{iso}}(T) \subseteq E_0(T) \), as \( T \) is finitely isoloid.

We obtain that \( E(T) \subseteq E_0(T) \). Other inclusion is always true and hence \( T \) satisfies property (Bw). \( \square \)

Theorem 2.1.14 [35]. Let \( T \in B(X) \) be finitely polaroid. If \( T \) or \( T^* \) has SVEP, then property (Bw) holds for \( T \).

**Proof.** If \( T \) or \( T^* \) has SVEP, then \( T \) satisfies generalized Browder’s theorem [7, Theorem 2.3]. Suppose \( \lambda \in E_0(T) \). It implies that

\[
\lambda \in \sigma_{\text{iso}}(T) \subseteq \pi_0(T) \subseteq \pi(T),
\]

as \( T \) is finitely polaroid. This shows that \( E_0(T) \subseteq \pi(T) \).

For the reverse inclusion suppose that \( \lambda \in \pi(T) \), then

\[
\lambda \in \sigma_{\text{iso}}(T) \subseteq \pi_0(T) \subseteq E_0(T).
\]
This gives that $\pi(T) \subseteq E_0(T)$ and hence $\pi(T) = E_0(T)$. Thus by Theorem 2.1.6, $T$ satisfies property (Bw). This completes the proof. \hfill \square

A bounded linear operator $T \in B(X)$ is normaloid if

$$\|T\| = r(T) = v(T),$$

where, $\|T\|$ is the usual operator norm of $T$, $r(T)$ is its spectral radius and $v(T)$ is its numerical radius.

A part of an operator is its restriction to a closed invariant subspace. An operator $T \in B(X)$ is totally hereditarily normaloid, $T \in THN$, if every part of $T$ and the inverse of every part of $T$ (whenever it exists), is normaloid.

Totally hereditarily normaloid operators are simply polaroid (that is, isolated points of the spectrum are simple poles of the resolvent) [24, Example 2.2] and have SVEP [24, Theorem 2.8]. An operator $T$ is polynomially $THN$ if there exists a non-constant polynomial $p(\cdot)$ such that $p(T) \in THN$.

With these, we prove the following.

**Theorem 2.1.15** [35]. Let $T \in B(X)$ be a polynomially $THN$ operator. Then $T$ and $T^*$ satisfy property (Bw) if and only if $E(T) = E_0(T)$. 

**Proof.** If \( p(T) \in THN \) for some non-constant polynomial \( p(\cdot) \), then \( p(T) \) has SVEP and \( p(T) \) is simply polaroid. But then \( T \) has SVEP [1, Theorem 2.40] and \( T \) is polaroid [24, Example 2.5]. Hence \( \sigma(T) \setminus \sigma_{BW}(T) = E(T) \). This implies that \( T \) satisfies property (Bw) if and only if \( E(T) = E_0(T) \). \( T \) has SVEP implies that \( T^* \) satisfies generalized Browder’s theorem, that is, \( \sigma(T^*) \setminus \sigma_{BW}(T^*) = \pi(T^*) \). Since \( T \) is polaroid implies \( T^* \) is polaroid, we also have that

\[
E(T) = \sigma(T) \setminus \sigma_{BW}(T) \\
= \sigma(T^*) \setminus \sigma_{BW}(T^*) \\
= \pi(T^*) = E(T^*)
\]

If \( \alpha(T - \lambda I) < \infty \) and \( \lambda \in \sigma(T) \setminus \sigma_{BW}(T) \), then

\[
\alpha(T^* - \lambda I^*) = \beta(T - \lambda I) < \infty.
\]

Hence \( T^* \) satisfies property (Bw) if and only if \( E(T) = E_0(T) \). \( \square \)

### 2.2 Property (Baw)

In this section, we introduce and study another property called property (Baw). The introduction of this class is an outcome of a study of the property (gw) that generalizes the concept of property (w). We present few results giving conditions under which an operator satisfies property (Baw).
Definition 2.2.1 [37]. A bounded linear operator $T \in B(X)$ is said to satisfy property (Baw) if
\[
\sigma_a(T) \setminus \sigma_{usbf^1}(T) = E_0^a(T).
\]

Following is an example of an operator satisfying property (Baw):

Example 2.2.2 [37]. Let $R \in B(l^2(\mathbb{N}))$ be the unilateral right shift and $P \in B(l^2(\mathbb{N}))$ be the operator defined by
\[
P(x_1, x_2, x_3, \ldots) = (0, x_2, x_3, x_4, \ldots)
\]
for all $x = (x_1, x_2, x_3 \ldots) \in l^2(\mathbb{N})$.

Consider the operator $T$ defined on $l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$ by
\[
T = R \oplus P.
\]

Then
\[
\sigma(T) = D(0, 1),
\]
where $D(0, 1)$ is the closed unit disc in $\mathbb{C}$.

\[
\sigma_a(T) = C(0, 1) \cup \{0\},
\]
where $C(0, 1)$ is the unit circle in $\mathbb{C}$.

\[
\sigma_{usbf^1}(T) = C(0, 1)
\]
and

\[ E_0^a(T) = \{0\}. \]

Thus,

\[ \sigma_a(T) \setminus \sigma_{usbf^-}(T) = E_0^a(T), \]

and hence \( T \) satisfies property (Baw).

Next is an example of an operator which fails to satisfy property (Baw):

**Example 2.2.3** \([37]\). Let \( R \in B(l^2(\mathbb{N})) \) be the right shift and let \( L \) be the weighted unilateral left shift defined by

\[ L(x_1, x_2, x_3, \ldots) = \left( \frac{x_2}{2}, \frac{x_3}{3}, \ldots \right) \]

for all \( x = (x_1, x_2, x_3, \ldots) \in l^2(\mathbb{N}) \).

Consider the operator \( T \) defined on \( l^2(\mathbb{N}) \oplus l^2(\mathbb{N}) \) by

\[ T = R \oplus L, \]

then \( \sigma(T) = D(0, 1) \) is the closed unit disc in \( \mathbb{C} \).

On the other hand \( \sigma_a(T) = \sigma_{usbf^-}(T) = C(0, 1) \cup \{0\} \). However, \( E_0^a(T) = \{0\} \). Thus \( T \) fails to satisfy property (Baw).

Property (Baw) implies generalized a-Browder’s theorem. Precisely, we have the following.
Theorem 2.2.4 [37]. Let $T \in B(X)$. Then the following statements are equivalent:

(i) $T$ satisfies property (Baw);

(ii) generalized a-Browder’s theorem holds for $T$ and

$$\pi^a(T) = E_0^a(T).$$

**Proof.** First, we show that (i) implies (ii). By Proposition 3.10 of [23], it is sufficient to prove that $T$ has SVEP at every $\lambda \not\in \sigma_{usb f^-}(T)$. Let us assume that $\lambda \not\in \sigma_{usb f^-}(T)$. Two cases arise.

**Case 1.** If $\lambda \not\in \sigma_a(T)$, then $T$ has SVEP at $\lambda$.

**Case 2.** If $\lambda \in \sigma_a(T)$ and $T$ satisfies property (Baw), then

$$\lambda \in \sigma_a(T) \setminus \sigma_{usb f^-}(T) = E_0^a(T).$$

Thus $\lambda \in \sigma_{iso}(T)$ which implies that $T$ has SVEP at $\lambda$. So, generalized a-Browder’s theorem holds for $T$.

To prove the equality $\pi^a(T) = E_0^a(T)$, let us consider $\lambda \in E_0^a(T)$. Since $T$ satisfies property (Baw), we have $\lambda \in \sigma_a(T) \setminus \sigma_{usb f^-}(T) = \pi^a(T)$, because generalized a-Browder’s theorem holds for $T$. Thus $E_0^a(T) \subseteq \pi^a(T)$.

For the reverse inclusion, assume that

$$\lambda \in \pi^a(T) = \sigma_a(T) \setminus \sigma_{usb f^-}(T) = E_0^a(T).$$
It implies that $\lambda \in E_0^a(T)$. Therefore $\pi^a(T) \subseteq E_0^a(T)$ and hence $\pi^a(T) = E_0^a(T)$.

To show that (ii) implies (i), we consider $\lambda \in \sigma_a(T) \setminus \sigma_{usbf}^-(T)$. Then generalized a-Browder’s theorem implies that $\lambda \in \pi^a(T) = E_0^a(T)$. Therefore

$$\sigma_a(T) \setminus \sigma_{usbf}^-(T) \subseteq E_0^a(T).$$

Conversely, if $\lambda \in E_0^a(T)$, then $\lambda \in \pi^a(T) = \sigma_a(T) \setminus \sigma_{usbf}^-(T)$.

It implies that $E_0^a(T) \subseteq \sigma_a(T) \setminus \sigma_{usbf}^-(T)$. Hence

$$\sigma_a(T) \setminus \sigma_{usbf}^-(T) = E_0^a(T).$$

We now prove a result for property (Baw) analogous to Theorem 2.1.7.

**Theorem 2.2.5** [37]. *Let $T$ be a bounded linear operator on $X$. If $T$ has SVEP at points in $\sigma_a(T) \setminus \sigma_{usbf}^-(T)$, then $T$ satisfies property (Baw) if and only if $E_0^a(T) = \pi^a(T)$.***

**Proof.** If $T$ has SVEP at points in $\sigma_a(T) \setminus \sigma_{usbf}^-(T)$, then $T$ satisfies generalized a-Browder’s theorem [23, Proposition 3.10]. Thus, by Theorem 2.2.4, $T$ satisfies property (Baw) if and only if $E_0^a(T) = \pi^a(T)$.  

A straightforward application of Theorem 2.2.5 leads to the following.
Corollary 2.2.6 [37]. Let $S, T \in B(X)$. If $T$ has SVEP and $S \prec_i T$, then $S$ satisfies property (Baw) if and only if $E_0^a(S) = \pi^a(S)$.

Theorem 2.2.7 [37]. Let $T \in B(X)$ be left polaroid and satisfy property (Baw). Then generalized a-Weyl’s theorem holds for $T$.

Proof. $T$ is left polaroid and satisfies property (Baw) if and only if

$$
\sigma_a(T) \setminus \sigma_{usbf-}(T) = E_0^a(T)
$$

$$
\subseteq E^a(T)
$$

$$
= \pi^a(T)
$$

$$
= \sigma_a(T) \setminus \sigma_{usbf-}(T),
$$

since $T$ satisfies generalized a-Browder’s theorem by Theorem 2.2.4. Thus $\sigma_a(T) \setminus \sigma_{usbf-}(T) = E^a(T)$ and hence generalized a-Weyl’s theorem holds for $T$.

Theorem 2.2.8 [37]. Let $T \in B(X)$ be finitely $a$-isoloid and satisfy generalized a-Weyl’s theorem. Then $T$ satisfies property (Baw).

Proof. If $T$ satisfies generalized a-Weyl’s theorem, then $\sigma_a(T) \setminus \sigma_{usbf-}(T) = E^a(T)$. 

2.3 Property (Bb)

We need to prove that \( E^a(T) = E^a_0(T) \). Suppose that \( \lambda \in E^a(T) \). It implies that \( \lambda \in \sigma^{iso}_a(T) \subseteq E^a_0(T) \), as \( T \) is finitely a-isoloid. Thus \( E^a(T) \subseteq E^a_0(T) \). Other inclusion is always true and hence the result follows.

\[ \square \]

2.3 Property (Bb)

In this section, we introduce and study property (Bb) in connection with Weyl-type theorems. We prove that if \( T \) is a bounded linear operator acting on a Banach space \( X \), then \( T \) satisfies property (Bb) if and only if generalized Browder’s theorem holds for \( T \) and \( \pi(T) = \pi_0(T) \). We also prove that property (Bw) implies property (Bb), but the converse is not true in general.

**Definition 2.3.1** [38]. A bounded linear operator \( T \in B(X) \) is said to satisfy property (Bb) if

\[
\sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T).
\]

**Example 2.3.2** [38]. The operator defined in Example 2.1.2 satisfies property (Bb) as \( \sigma(T) = \sigma_W(T) = \sigma_{BW}(T) = \{0\} \) and \( E_0(T) = \pi_0(T) = \phi \).
Example 2.3.3 [38]. Let $I_1$ be the identity on $\mathbb{C}$. Let $T_1$ be defined on $l^2(\mathbb{N})$ by

$$T_1(x_1, x_2, \ldots) = \left(0, \frac{1}{3}x_1, \frac{1}{3}x_2, \ldots\right)$$

for all $x = (x_1, x_2, \ldots) \in l^2(\mathbb{N})$.

Let $T = I_1 \oplus T_1$. Then $\sigma(T) = \left\{\lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{3}\right\} \cup \{1\}$.

It is shown in [65] that

$$\sigma_{BW}(T) = \left\{\lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{3}\right\}.$$

Thus we have that $\sigma(T) \setminus \sigma_{BW}(T) = \{1\}$.

But $\{1\}$ is not a pole of finite rank. Therefore, $T$ fails to satisfy property (Bb).

Next we give a condition for the equivalence of the property (Bw) and property (Bb).

Theorem 2.3.4 [38]. Let $T \in B(X)$. Then $T$ satisfies property (Bw) if and only if $T$ satisfies property (Bb) and $\pi_0(T) = E_0(T)$.

Proof. Suppose $T$ satisfies property (Bw), then

$$\sigma(T) \setminus \sigma_{BW}(T) = E_0(T).$$

If $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$, then $\lambda \in \sigma_{iso}(T)$ and $T - \lambda I$ is B-Fredholm operator of index zero. From [14, Theorem 2.3] we have $\lambda \in \pi(T)$. 
Also, $\lambda \in E_0(T)$ implies that $\dim N(T - \lambda I) < \infty$, therefore $\lambda \in \pi_0(T)$. We obtain that

$$
\sigma(T) \setminus \sigma_{BW}(T) \subseteq \pi_0(T).
$$

Now if $\lambda \in \pi_0(T)$, then $\lambda \in E_0(T)$ because $\pi_0(T) \subseteq E_0(T)$ for every $T \in B(X)$. Since $T$ satisfies property (Bw), we have $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Thus $\pi_0(T) \subseteq \sigma(T) \setminus \sigma_{BW}(T)$. Hence,

$$
\sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T).
$$

That is, $T$ satisfies property (Bb) and $\pi_0(T) = E_0(T)$.

Conversely, assume that $T$ satisfies property (Bb) and $\pi_0(T) = E_0(T)$.

Then $\sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T)$ and $\pi_0(T) = E_0(T)$.

So, $\sigma(T) \setminus \sigma_{BW}(T) = E_0(T)$ and hence $T$ satisfies property (Bw).

The following example shows that in general property (Bb) does not imply property (Bw).

**Example 2.3.5** [38]. Let $T \in B(l^2(\mathbb{N}))$ be defined by

$$
T(x_1, x_2, x_3 \ldots) = \left( \frac{x_2}{3}, \frac{x_3}{4}, \frac{x_4}{5} \ldots \right)
$$

for all $x = (x_1, x_2, x_3, \ldots) \in l^2(\mathbb{N})$. 

Then
\[ \sigma(T) = \sigma_{BW}(T) = \{0\}, \]
\[ E_0(T) = \{0\}, \]
and
\[ \pi_0(T) = \phi. \]
This implies that \( \sigma(T) \setminus \sigma_{BW}(T) \neq E_0(T) \) and \( \sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T) \). Therefore, \( T \) satisfies property (Bb) but \( T \) does not satisfy property (Bw).

An analogous result to that of Theorem 2.1.4 can be proved in a similar way for property (Bb).

**Theorem 2.3.6** [38]. Let \( T \in B(X) \) satisfy property (Bb). Then generalized Browder’s theorem holds for \( T \) and
\[ \sigma(T) = \sigma_{BW}(T) \cup \sigma_{iso}(T). \]

We now establish a characterization of operators satisfying property (Bb).

**Theorem 2.3.7** [38]. Let \( T \in B(X) \). Then the following statements are equivalent:

(i) \( T \) satisfies property (Bb),
(ii) generalized Browder’s theorem holds for \( T \) and
\[ \pi(T) = \pi_0(T). \]
**Proof.** First, we show that (i) implies (ii). By Theorem 2.3.6, generalized Browder’s theorem holds for $T$, therefore it is sufficient to prove the equality $\pi(T) = \pi_0(T)$. We know $\pi_0(T) \subseteq \pi(T)$ for all $T \in B(X)$.

For the reverse inclusion, let $\lambda \in \pi(T) = \sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T)$. This shows that $\pi(T) \subseteq \pi_0(T)$. Therefore, we have the equality $\pi(T) = \pi_0(T)$.

We now prove that (ii) implies (i). By the hypothesis that generalized Browder’s theorem holds for $T$, we have

$$\sigma(T) \setminus \sigma_{BW}(T) = \pi(T)$$

$$= \pi_0(T).$$

Hence $T$ satisfies property (Bb). \hfill \Box

**Theorem 2.3.8** [38]. Let $T \in B(X)$. If $T$ or $T^*$ has SVEP at points in $\sigma(T) \setminus \sigma_{BW}(T)$, then $T$ satisfies property (Bb) if and only if $\pi_0(T) = \pi(T)$.

**Proof.** The hypothesis $T$ or $T^*$ has SVEP at points in $\sigma(T) \setminus \sigma_{BW}(T) = \sigma(T^*) \setminus \sigma_{BW}(T^*)$, implies that $T$ satisfies generalized Browder’s theorem [23, Proposition 3.9]. Hence by Theorem 2.3.7, we have that $T$ satisfies property (Bb) if and only if $\pi_0(T) = \pi(T)$. \hfill \Box
Following corollary can be deduced from Theorem 2.3.8.

**Corollary 2.3.9** [38]. Let $S, T \in B(X)$. If $T$ has SVEP and $S \prec_i T$, then $S$ satisfies property (Bb) if and only if 
$$\pi_0(S) = \pi(S).$$

**Theorem 2.3.10** [38]. Let $T \in B(X)$ be not quasinilpotent and $K(T) = \{0\}$. Then $\sigma(T) = \sigma_W(T) = \sigma_{BW}(T)$ and $T$ satisfies both property (Bb) and generalized Weyl’s theorem.

**Proof.** If $T \in B(X)$ is not quasinilpotent and $K(T) = \{0\}$, then $T$ has SVEP, $\sigma(T) = \sigma_W(T)$ is a connected set containing $0$ and $\sigma_{iso}(T) = \phi$ [1, Theorem 3.121]. $T$ has SVEP implies $T$ satisfies generalized Browder’s theorem [7, Theorem 2.3]. Thus,

$$\sigma(T) \setminus \sigma_{BW}(T) = \pi(T) = \phi = \pi_0(T) = E_0(T) = E(T).$$

Hence $T$ satisfies property (Bb) and generalized Weyl’s theorem, (so also Weyl’s theorem). \qed

**Remark 2.3.11** [38]. If $T \in B(X)$ is quasinilpotent, then $\sigma(T) = \sigma_{BW}(T) = \{0\}$. Hence $T$ satisfies property (Bb) is equivalent to $T$ satisfies Browder’s theorem.

Finitely polaroid operators are polaroid but not conversely. In the following, we give a condition under which the converse is also true.
Theorem 2.3.12 [38]. A polaroid operator satisfying property (Bb) is finitely polaroid.

Proof. Let $\lambda \in \sigma_{iso}(T)$. Since $T$ is polaroid, we have $\lambda \in \pi(T)$. Now as $T$ satisfies property (Bb), by Theorem 2.3.7, we have $\pi(T) = \pi_0(T)$. Thus $\lambda \in \pi_0(T)$, which implies that $T$ is finitely polaroid. 

Theorem 2.3.13 [38]. Let $T \in B(X)$ be finitely polaroid. Then $T$ satisfies property (Bb) if

(i) $T$ satisfies generalized Weyl’s theorem,

or

(ii) $T$ or $T^*$ has SVEP.

Proof. In both the cases $T$ satisfies generalized Browder’s theorem ([14, Corollary 2.6] and [7, Theorem 2.3]). By Theorem 2.3.7, we need to prove $\pi(T) = \pi_0(T)$.

Suppose $\lambda \in \pi(T)$, then $\lambda \in \sigma_{iso}(T) \subseteq \pi_0(T)$, as $T$ is finitely polaroid. We obtain that $\pi(T) \subseteq \pi_0(T)$. Other inclusion is always true. Thus $\pi(T) = \pi_0(T)$ and hence $T$ satisfies property (Bb). 

Theorem 2.3.14 [38]. Let $T \in B(X)$ be a polaroid and satisfy property (Bb). Then generalized Weyl’s theorem holds for $T$. 

**Proof.** $T$ is polaroid and satisfies property (Bb) if and only if

\[
\sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T)
\]

\[
\subseteq E(T)
\]

\[
= \pi(T)
\]

\[
= \sigma(T) \setminus \sigma_{BW}(T),
\]

as $T$ satisfies generalized Browder’s theorem by Theorem 2.3.7. Thus $\sigma(T) \setminus \sigma_{BW}(T) = E(T)$ and hence generalized Weyl’s theorem holds for $T$. \hfill \Box

The condition that $T$ is polaroid cannot be dropped from the Theorem 2.3.14 as the following example shows.

**Example 2.3.15** [38]. Let $T \in B(X)$ be a nilpotent operator such that $R(T)$ is not closed. Let $Q \in B(X)$ be a quasi-nilpotent operator which is not nilpotent.

Let $S = T \oplus Q \in B(X \oplus X)$.

Then

\[
\sigma(S) = \sigma_W(S) = \sigma_{BW}(S) = E(S) = \{0\},
\]

\[
E_0(S) = \pi_0(S) = \pi(S) = \phi.
\]

Property (Bb) is satisfied but generalized Weyl’s theorem is not satisfied as the operator is not polaroid.
From Theorem 2.3.13 and Theorem 2.3.14, we conclude the following remark.

**Remark 2.3.16** [38]. If $T$ is a finitely polaroid operator, then $T$ satisfies property (Bb) if and only if $T$ satisfies generalized Weyl’s theorem.

### 2.4 Property (Bab)

Our aim in this section is to introduce and study property (Bab). We show that $T$ satisfies property (Bab) if and only if generalized a-Browder’s theorem holds for $T$ and $\pi^a(T) = \pi^a_0(T)$. We also give a condition for the equivalence of the property (Baw) and property (Bab).

**Definition 2.4.1** [38]. A bounded linear operator $T \in B(X)$ is said to satisfy property (Bab) if

$$\sigma_a(T) \setminus \sigma_{ushf^-(T)} = \pi^a_0(T).$$

**Example 2.4.2** [38]. The operator $T$ defined in Example 2.2.2 satisfies property (Bab) as $\sigma_a(T) = C(0, 1) \cup \{0\}$ and $\sigma_{ushf^-(T)} = C(0, 1)$, where $C(0, 1)$ is the unit circle in $\mathbb{C}$. However, $\pi^a_0(T) = \{0\}$. Thus property (Bab) holds for $T$.

**Example 2.4.3** [38]. Let $R$ be the unilateral right shift operator defined on $l^2(\mathbb{N})$ and $T$ be the operator defined on $l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$ by $T = 0 \oplus R$. Then $\sigma(T) = D(0, 1)$, $\sigma_a(T) = C(0, 1) \cup \{0\}$,
\(\pi_a(T) = \{0\}\) and \(\{0\} \notin \sigma_{usbf}(T)\). It implies that \(\{0\} \in \sigma_a(T) \setminus \sigma_{usbf}(T)\), but \(\pi^a_0(T) = \phi\).

Therefore, \(T\) does not satisfy property (Bab).

We now give a condition for the equivalence of property (Baw) and property (Bab).

**Theorem 2.4.4** [38]. An operator \(T \in B(X)\) possesses property (Baw) if and only if \(T\) possesses property (Bab) and \(E^a_0(T) = \pi^a_0(T)\).

**Proof.** Suppose that \(T\) possesses property (Baw) and suppose \(\lambda \in \sigma_a(T) \setminus \sigma_{usbf}(T)\), then \(T - \lambda I\) is upper semi-B-Fredholm operator of index less than or equal to zero and \(\lambda \in E^a_0(T)\). As \(\dim N(T - \lambda I) < \infty\), \(T - \lambda I\) is upper semi-Fredholm operator of index less than or equal to zero \([8, \text{Lemma 2.2}]\). Now \(\lambda \in E^a_0(T)\) implies that \(\lambda \in \sigma^{i_{so}}(T)\). Therefore, from \([16, \text{Theorem 2.8}]\), we have \(\lambda \in \pi^a_0(T)\). Thus \(\sigma_a(T) \setminus \sigma_{usbf}^-(T) \subseteq \pi^a_0(T)\).

If \(\lambda \in \pi^a_0(T) \subseteq E^a_0(T)\), then as \(T\) possesses property (Baw), \(\lambda \in \sigma_a(T) \setminus \sigma_{usbf}^-(T)\). Therefore, \(\pi^a_0(T) \subseteq \sigma_a(T) \setminus \sigma_{usbf}^-(T)\).

Thus, \(\sigma_a(T) \setminus \sigma_{usbf}^-(T) = \pi^a_0(T)\). Hence \(T\) possesses property (Bab) and \(E^a_0(T) = \pi^a_0(T)\).

Conversely, if \(T\) possesses property (Bab) and \(E^a_0(T) = \pi^a_0(T)\), then \(\sigma_a(T) \setminus \sigma_{usbf}^-(T) = E^a_0(T)\). Thus \(T\) possesses property (Baw). \(\square\)
Theorem 2.4.5 [38]. Let $T \in B(X)$ satisfy property (Bab). Then generalized a-Browder’s theorem holds for $T$ and $\sigma_a(T) = \sigma_{\text{usbf}^-}(T) \cup \sigma_{\text{iso}}^a(T)$.

Proof. By Proposition 3.10 of [23], it is sufficient to prove that $T$ has SVEP at every $\lambda \notin \sigma_{\text{usbf}^-}(T)$. Let us assume that $\lambda \notin \sigma_{\text{usbf}^-}(T)$. Two cases arise.

Case 1. If $\lambda \notin \sigma_a(T)$, then $T$ has SVEP at $\lambda$.

Case 2. If $\lambda \in \sigma_a(T)$ and $T$ satisfies property (Bab) then $\lambda \in \sigma_a(T) \setminus \sigma_{\text{usbf}^-}(T) = \pi_0^a(T)$. This shows that $\lambda \in \sigma_{\text{iso}}^a(T)$, which implies $T$ has SVEP at $\lambda$.

Therefore, generalized a-Browder’s theorem holds for $T$.

To prove that $\sigma_a(T) = \sigma_{\text{usbf}^-}(T) \cup \sigma_{\text{iso}}^a(T)$, we observe that

$$
\lambda \in \sigma_a(T) \setminus \sigma_{\text{usbf}^-}(T) = \pi_0^a(T).
$$

It implies that $\lambda \in \sigma_{\text{iso}}^a(T)$. Therefore

$$
\sigma_a(T) \subseteq \sigma_{\text{usbf}^-}(T) \cup \sigma_{\text{iso}}^a(T).
$$

But $\sigma_{\text{usbf}^-}(T) \cup \sigma_{\text{iso}}^a(T) \subseteq \sigma_a(T)$ for every $T \in B(X)$ and hence $\sigma_a(T) = \sigma_{\text{usbf}^-}(T) \cup \sigma_{\text{iso}}^a(T)$.

It is easy to derive the following result analogous to that established in Theorem 2.2.4.
**Theorem 2.4.6** [38]. Let $T \in B(X)$. Then the following statements are equivalent:

(i) $T$ satisfies property (Bab);

(ii) generalized $a$-Browder’s theorem holds for $T$ and

$$\pi^a(T) = \pi^a_0(T).$$

**Proof.** First, we prove that (i) implies (ii). By Theorem 2.4.5, it is sufficient to prove the equality $\pi^a(T) = \pi^a_0(T)$.

Let $\lambda \in \pi^a(T)$. Since $T$ satisfies generalized $a$-Browder’s theorem, it implies that

$$\lambda \in \sigma_a(T) \setminus \sigma_{usbf}^{-}(T) = \pi^a_0(T),$$

because $T$ satisfies property (Bab). Thus $\pi^a(T) \subseteq \pi^a_0(T)$.

Other inclusion is always true and hence we have the equality $\pi^a(T) = \pi^a_0(T)$.

To show that (ii) implies (i), we consider

$\lambda \in \sigma_a(T) \setminus \sigma_{usbf}^{-}(T)$. Then generalized $a$-Browder’s theorem implies that $\lambda \in \pi^a(T) = \pi^a_0(T)$.

It means that $\sigma_a(T) \setminus \sigma_{usbf}^{-}(T) \subseteq \pi^a_0(T)$.

Conversely, if $\lambda \in \pi^a_0(T)$ then $\lambda \in \pi^a(T) = \sigma_a(T) \setminus \sigma_{usbf}^{-}(T)$.

Thus $\pi^a_0(T) \subseteq \sigma_a(T) \setminus \sigma_{usbf}^{-}(T)$ and hence

$$\pi^a_0(T) = \sigma_a(T) \setminus \sigma_{usbf}^{-}(T).$$

$\square$
**Remark 2.4.7** [38]. Property (Bab) implies property (Bb), but the converse is not true in general. Indeed, if we consider the operator defined in Example 2.4.3, then $T$ does not satisfy property (Bab). However, $T$ satisfies property (Bb), as

$$\sigma(T) = \sigma_{BW}(T) = D(0,1)$$

and

$$\pi_0(T) = \phi.$$  

**Theorem 2.4.8** [38]. Let $T \in B(X)$. If $T$ has SVEP at points in $\sigma_a(T) \setminus \sigma_{usbf^-}(T)$, then $T$ satisfies property (Bab) if and only if $\pi^a_0(T) = \pi^a(T)$.

**Proof.** Since $T$ has SVEP at points in $\sigma_a(T) \setminus \sigma_{usbf^-}(T)$, therefore $T$ satisfies generalized a-Browder’s theorem [23, Proposition 3.10]. Hence, if $\pi^a_0(T) = \pi^a(T)$, then

$$\sigma_a(T) \setminus \sigma_{usbf^-}(T) = \pi^a(T) = \pi^a_0(T).$$

Thus, property (Bab) holds for $T$. Converse is trivially true and hence the result follows. \hfill $\Box$

Following corollary can be deduced from Theorem 2.4.8.

**Corollary 2.4.9** [38]. Let $S, T \in B(X)$. If $T$ has SVEP and $S \prec_i T$, then $S$ satisfies property (Bab) if and only if $\pi^a_0(S) = \pi^a(S)$. 
Finitely left-polaroid are left-polaroid but the converse is not true in general. In the next result, we give condition under which the converse is also true.

**Theorem 2.4.10** [38]. *A left-polaroid satisfying property (Bab) is finitely left-polaroid.*

**Proof.** Let $\lambda \in \sigma_a^{iso}(T)$. As $T$ is left-polaroid, $\lambda \in \pi^a(T)$. $T$ satisfies property (Bab), therefore by Theorem 2.4.6, we have $\pi^a(T) = \pi_0^a(T)$. Thus, $\lambda \in \pi_0^a(T)$ and hence $T$ is finitely left-polaroid.

**Theorem 2.4.11** [38]. *Let $T \in B(X)$ be finitely left-polaroid and satisfy generalized a-Weyl’s theorem. Then $T$ satisfies property (Bab).*

**Proof.** From [16, Corollary 3.3], we have that $T$ satisfies generalized a-Browder’s theorem. Suppose $\lambda \in \pi^a(T)$, then $\lambda$ is isolated in $\sigma_a(T)$, that is, $\lambda \in \sigma_a^{iso}(T) \subseteq \pi_0^a(T)$, as $T$ is finitely left-polaroid. Thus $\pi^a(T) \subseteq \pi_0^a(T)$. Other inclusion is always true. Therefore $\pi^a(T) = \pi_0^a(T)$ and hence $T$ satisfies property (Bab).

**Theorem 2.4.12** [38]. *Let $T \in B(X)$ be left-polaroid and satisfy property (Bab). Then generalized a-Weyl’s theorem holds for $T$.***
\textbf{Proof.} \( T \) is left-polaroid and satisfies property (Bab) if and only if
\[
\sigma_a(T) \setminus \sigma_{ushf^-}(T) = \pi_0^a(T)
\]
\[
\subseteq E^a(T)
\]
\[
= \pi^a(T)
\]
\[
= \sigma_a(T) \setminus \sigma_{ushf^-}(T),
\]
since \( T \) satisfies generalized a-Browder’s theorem by Theorem 2.4.5. Thus \( \sigma_a(T) \setminus \sigma_{ushf^-}(T) = E^a(T) \) and hence generalized a-Weyl’s theorem holds for \( T \).

From Theorem 2.4.11 and Theorem 2.4.12, we conclude the following remark.

\textbf{Remark 2.4.13} [38]. If \( T \) is a finitely left-polaroid, then \( T \) satisfies property (Bab) if and only if \( T \) satisfies generalized a-Weyl’s theorem.

\textbf{Conclusion}

In this last part, we give a summary of the known properties introduced in [8], [18], [19], [56] and in this chapter. We use the abbreviations (gw), (w), (gaw), (aw), (Baw) and (Bw), to signify that an operator \( T \in B(X) \) satisfies property (gw), property (w), property (gaw), property (aw), property (Baw) and property
Similarly, the abbreviations (gb), (b), (gab), (ab), (Bab) and (Bb) have analogous meaning with respect to Browder’s theorem or the properties introduced in [18] and [19] or the new properties introduced in this chapter.

The following table summarizes the meaning of various properties.

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Property</th>
<th>Description</th>
<th>Abbreviation</th>
<th>Property</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>gw</td>
<td>(\sigma_a(T) \setminus \sigma_{usbf}(T) = E(T))</td>
<td></td>
<td>gb</td>
<td>(\sigma_a(T) \setminus \sigma_{usbf}(T) = \pi(T))</td>
<td></td>
</tr>
<tr>
<td>w</td>
<td>(\sigma_a(T) \setminus \sigma_{usf}(T) = E_0(T))</td>
<td></td>
<td>b</td>
<td>(\sigma_a(T) \setminus \sigma_{usf}(T) = \pi_0(T))</td>
<td></td>
</tr>
<tr>
<td>gaw</td>
<td>(\sigma(T) \setminus \sigma_{BW}(T) = E^a(T))</td>
<td></td>
<td>gab</td>
<td>(\sigma(T) \setminus \sigma_{BW}(T) = \pi^a(T))</td>
<td></td>
</tr>
<tr>
<td>aw</td>
<td>(\sigma(T) \setminus \sigma_{W}(T) = E^a_0(T))</td>
<td></td>
<td>ab</td>
<td>(\sigma(T) \setminus \sigma_{W}(T) = \pi^a_0(T))</td>
<td></td>
</tr>
<tr>
<td>Baw</td>
<td>(\sigma_a(T) \setminus \sigma_{usbf}(T) = E^a_0(T))</td>
<td></td>
<td>Bab</td>
<td>(\sigma_a(T) \setminus \sigma_{usbf}(T) = \pi^a_0(T))</td>
<td></td>
</tr>
<tr>
<td>Bw</td>
<td>(\sigma(T) \setminus \sigma_{BW}(T) = E_0(T))</td>
<td></td>
<td>Bb</td>
<td>(\sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T))</td>
<td></td>
</tr>
<tr>
<td>Bb</td>
<td>(\sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T))</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
In the following diagram, which extends the similar diagram presented in [19], arrows signify implications between various Weyl-type theorems, Browder-type theorems and properties.