Chapter-3

A family of lifetime distributions and related estimation and testing procedures for the reliability function under type I and type II censorings
CHAPTER 3
A FAMILY OF LIFETIME DISTRIBUTIONS AND RELATED ESTIMATION AND TESTING PROCEDURES FOR THE RELIABILITY FUNCTION UNDER TYPE I AND TYPE II CENSORINGS

3.1. INTRODUCTION

In the present chapter, we propose a family of distributions, which covers many lifetime distributions as specific cases. The uniformly minimum variance unbiased estimators (UMVUES) and maximum likelihood estimators (MLES) of \( R(t) = P(X > t) \) and \( P = P(X > Y) \) are derived under type I and II censorings. Tests and confidence intervals are also developed.

In Section 3.2, we propose the family of lifetime distributions. In Sections 3.3 and 3.4, respectively, we obtain UMVUES and MLES. In Section 3.5, tests and confidence intervals are developed. Finally, in Section 3.6, the simulation study is performed.

3.2. THE FAMILY OF LIFETIME DISTRIBUTIONS

Let the rv \( X \) follow the distribution having the probability density function (pdf)

\[
f(x; a, \alpha, \beta, \theta) = \frac{\beta}{\alpha} g^{\beta-1}(x; \theta) g'(x; \theta) \exp \left( -\frac{g^\beta(x; \theta)}{\alpha} \right); \quad x > a \geq 0; \quad \alpha > 0, \beta > 0.
\]  

(3.2.1)

Here, \( g(x; \theta) \) is a function of \( x \) and may also depend on a vector-valued parameter \( \theta \). Moreover, \( g(x; \theta) \) is monotonically increasing in \( x \) with \( g(a; \theta) = 0 \), \( g(\infty; \theta) = \infty \) and \( g'(x; \theta) \) denotes the derivative of \( g(x; \theta) \) with respect to \( x \).

We note that (3.2.1) represents a family of lifetime distributions as it covers the following lifetime distributions as the specific cases:
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i. For \( g(x; \theta) = x, a=0 \) and \( \beta = 1 \), we get the one-parameter exponential distribution [Johnson and Kotz (1970, p.166)].

ii. For \( g(x; \theta) = x^p \) (\( p>0 \)), \( \beta = 1 \) and \( a=0 \), it turns out to be Weibull distribution [Johnson and Kotz (1970, p.250)].

iii. For \( g(x; \theta) = x^2 \), \( \beta = 1 \) and \( a=0 \), it gives Rayleigh distribution [Sinha (1986, p.200)].

iv. For \( g(x; \theta) = \log(1+x^b) \) (\( b>0 \)), \( \beta = 1 \) and \( a=0 \), it leads us to Burr distribution [Burr (1942) and Cislak and Burr (1968)].

v. For \( g(x; \theta) = \log\left(\frac{x}{a}\right) \) and \( \beta = 1 \), it turns out to be Pareto distribution [Johnson and Kotz (1970, p.233)].

vi. For \( g(x; \theta) = \log(1+\frac{x}{\nu}) \), \( \nu > 0 \), \( \beta = 1 \) and \( a=0 \), it is called Lomax (1954) distribution.

vii. For \( g(x; \theta) = \log(1+\frac{x^b}{\nu}) \), \( b>0 \), \( \beta = 1 \) and \( a=0 \), it becomes Burr distribution with scale parameter \( \nu \) (>0) [see Tadikamalla (1980)].

viii. For \( g(x; \theta) = \log(1+\frac{x^b}{\delta}) \), \( b>0 \), \( \delta > 0 \), \( \beta = 1 \) and \( a=0 \), it is called log-logistic distribution [see Kleiber (2004)].

ix. For \( g(x; \theta) = x^{\gamma}\exp(\nu x) \), \( \gamma>0,\nu>0 \), \( \beta = 1 \) and \( a=0 \), it gives the modified Weibull distribution of Lai et al. (2003).

x. For \( \beta = 1 \) and \( a=0 \), it leads us to the family of distributions considered by Gurvich et al. (1997).
**xi.** For \( g(x;\theta) = \gamma \exp\left(\frac{X^\beta}{Y^\gamma} - 1\right), \gamma > 0, \nu > 0, \beta = 1 \) and \( a = 0 \), it turns out to be a modified form of Weibull distribution considered by Xie et al. (2002). If we also take \( \gamma = 1 \), this reduces to the lifetime distribution considered by Chen (2000).

**xii.** For \( g(x;\theta) = \mu x + \nu x^2/2, \alpha = \beta = 1 \) and \( a = 0 \), it is called the linear exponential distribution.

**xiii.** For \( g(x;\theta) = (x-a) + \frac{\nu}{\alpha} \log\left(\frac{x^\alpha}{a^\alpha} + \nu\right) \) and \( \beta = 1 \), we get the generalized Pareto distribution of Ljubo(1965).

For the model (3.2.1), the reliability function \( R(t) \) at a specified mission time ‘\( t \)’ is

\[
R(t) = P(X > t) = \exp\left(-\frac{g(\theta)(t;\theta)}{\alpha}\right).
\]

(3.2.2)

### 3.3. UMVUES OF R(t) AND ‘P’ UNDER TYPE I AND TYPE II CENSORINGS

Let \( \alpha \) is unknown but \( \alpha’, \beta \) and \( \theta \) are known. First we consider the estimation based on type II censored data. Suppose \( n \) items are put on a test and the test is terminated after the first \( r \) ordered observations are recorded. Let \( a \leq X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}, \ 0 < r < n, \) be the lifetimes of first \( r \) ordered observations. Obviously, \( (n-r) \) items survived until \( X_{(r)} \).

**Lemma 3.1:** Let \( S_r = \sum_{i=1}^{r} X_{(i)} \), then \( S_r \) is complete and sufficient for the family of distributions given at (3.2.1). Moreover, the pdf of \( S_r \) is

\[
g(s;\alpha,\beta,\theta) = \frac{s^{\alpha r - 1}}{\alpha^{\alpha T(r)}} \exp(-s/\alpha).
\]

(3.3.1)
Proof: From (3.2.1), the joint pdf of \( X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)} \) is
\[
f^* \left( x_{(1)}, x_{(2)}, \ldots, x_{(n)}; a, \alpha, \beta, \theta \right) = \frac{n! \beta^n}{\alpha^n} \prod_{i=1}^{n} g^\beta(x_{(i)}; \theta) g'(x_{(i)}; \theta) \exp \left\{ \sum_{i=1}^{n} \frac{g^\beta(x_{(i)}; \theta)}{\alpha} \right\}.
\] (3.3.2)

Integrating out \( x_{(r+1)}, x_{(r+2)}, \ldots, x_{(n)} \) from (3.3.2) over the region \( x_{(r)} \leq x_{(r+1)} \leq \ldots \leq x_{(n)} \), the joint pdf of \( X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(r)} \) comes out to be
\[
h^* \left( x_{(1)}, x_{(2)}, \ldots, x_{(r)}; a, \alpha, \beta, \theta \right) = \frac{\beta^r n(n-1) \cdots (n-r+1)}{\alpha^r} \prod_{i=1}^{r} g^\beta(x_{(i)}; \theta) g'(x_{(i)}; \theta) \exp \left\{ \sum_{i=1}^{r} \frac{S_i}{\alpha} \right\}.
\] (3.3.3)

It follows easily from (3.2.1) that the rv \( U = \frac{g^\beta(x; \theta)}{\alpha} \) has exponential distribution with mean life \( \alpha \). Moreover, if we consider the transformation
\[Z_i = (n-i+1)(U_{(i)} - U_{(i+1)}), \quad i=1,2,\ldots,r; \quad U_0 = 0,
\]
then \( Z_i \)'s are independent and identically distributed (iid) rv's, each having exponential distribution with mean life \( \alpha \). It is easy to see that \( \sum_{i=1}^{r} Z_i = S_r \). Result (3.3.1) now follows from the additive property of gamma distribution [see Johnson and Kotz (1970, p.170)]. It follows from (3.3.3) that \( S_r \) is sufficient for the family of distributions given at (3.2.1). Since the distribution of \( S_r \) belongs to exponential family of distributions, it is also complete [see Rohatgi(1976, p.347)].

The following lemma provides the UMVUE of the power of \( \alpha \).

Lemma 3.2: For \( q \in (-\infty, \infty) \), the UMVUE of \( \alpha^q \) is
\[
\hat{\alpha}^q = \begin{cases} 
\frac{\Gamma(r)}{\Gamma(r-q)} \frac{S_r^q}{\text{q}<r} & \text{q}<r \\
0, & \text{otherwise}.
\end{cases}
\]
**Proof:** From (3.3.1),

\[
E(S_t^n) = \frac{1}{\alpha T(r)} \int_0^\infty S_t^{r-1} \exp(-s_t/\alpha) ds_t
\]

\[
= \frac{\Gamma(r-q)}{\alpha^q T(r)}
\]

and the lemma follows from Lehmann-Scheffé theorem [see Rohatgi (1976, p.357)].

In the following lemma, we provide the UMVUE of the sampled pdf (3.2.1) at a specified point ‘x’.

**Lemma 3.3:** The UMVUE of \( f(x;a,\alpha,\beta,\theta) \) at a specified point ‘x’ is

\[
\hat{f}_n(x;a,\alpha,\beta,\theta) = \begin{cases} 
\frac{(r-1)\beta g^{\beta-1}(x;\theta)g'(x;\theta)}{S_t} \sum_{i=0}^\infty \frac{(-1)^i g^\beta(x;\theta)}{i!} \left[ 1 - \frac{g^\beta(x;\theta)}{S_t} \right]^{i-2} g^\beta(x;\theta) < S_t, \\
0, \text{ otherwise.}
\end{cases}
\]

**Proof:** We can write (3.2.1) as

\[
f(x;a,\alpha,\beta,\theta) = \beta g^{\beta-1}(x;\theta)g'(x;\theta) \sum_{i=0}^\infty \frac{(-1)^i g^\beta(x;\theta)}{i!}.
\]

Using Lemma 1 of Chaturvedi and Tomer (2002) and Lemma 3.2, from (3.3.4), the UMVUE of \( f(x;a,\alpha,\beta,\theta) \) at a specified point ‘x’ is

\[
\hat{f}_n(x;a,\alpha,\beta,\theta) = \beta g^{\beta-1}(x;\theta)g'(x;\theta) \sum_{i=0}^\infty (-1)^i \frac{g^\beta(x;\theta)}{i!} \hat{\alpha}_{n(i+1)}
\]

\[
= \frac{(r-1)\beta g^{\beta-1}(x;\theta)g'(x;\theta)}{S_t} \sum_{i=0}^\infty (-1)^i \left[ 1 - \frac{g^\beta(x;\theta)}{S_t} \right]^{i-2} \left[ \frac{g^\beta(x;\theta)}{S_t} \right]^i
\]

and the lemma holds.
In the following theorem, we obtain the UMVUE of $R(t)$.

**Theorem 3.1:** The UMVUE of $R(t)$ is given by

$$
\hat{R}_n(t) = \left\{ \begin{array}{ll}
\left[ \frac{1}{S_t} \frac{g^\beta(x;\theta)}{S_t} \right]^{-1} & ; \quad g^\beta(x;\theta) < S_t, \\
0 & , \text{ otherwise.}
\end{array} \right.
$$

**Proof:** Let us consider the expected value of the integral $\int_t^\infty \hat{f}_n(x;a,\alpha,\beta,\theta) \, dx$ with respect to $S_t$, i.e.

$$
\int_t^\infty \left\{ \int_0^{r} g(s;a,\alpha,\beta,\theta) \, ds \right\} \hat{f}_n(x;a,\alpha,\beta,\theta) \, dx = \int_0^{r} \left[ \int_t^\infty \hat{f}_n(x;a,\alpha,\beta,\theta) \, dx \right] \cdot g(s;a,\alpha,\beta,\theta) \, ds
$$

$$
= R(t).
$$

We conclude from (3.3.5) that the UMVUE of $R(t)$ can be obtained simply integrating $\hat{f}_n(x;a,\alpha,\beta,\theta)$ from $t$ to $\infty$. Thus, from Lemma 3.3,

$$
\hat{R}_n(t) = \frac{(r-1)\beta}{S_t} \int_t^\infty \frac{g^\beta(x;\theta)g'(x;\theta)}{S_t} \left[ \frac{1}{S_t} \frac{g^\beta(x;\theta)}{S_t} \right]^{-1} \, dx
$$

$$
= (r-1) \left[ \frac{1}{\hat{g}^{\beta}(\theta)} \right]_{g'(0,\theta)}^{\hat{g}^{\beta}(\theta)} [1-y]^{\beta-1} dy
$$

$$
= \left[ -(1-y)^{r-1} \right]_{g'(0,\theta)}^{\hat{g}^{\beta}(\theta)} \frac{1}{S_t}
$$

and the theorem follows.
Let $X$ and $Y$ be two independent rv’s following the classes of distributions $f_1(x; a_1, \alpha_1, \beta_1, \theta_1)$ and $f_2(y; a_2, \alpha_2, \beta_2, \theta_2)$, respectively, where

$$f_1(x; a_1, \alpha_1, \beta_1, \theta_1) = \frac{\beta_1}{\alpha_1} g^{\beta_1}(x; \theta_1) g'(x; \theta_1) \exp \left(-\frac{g^\theta(x; \theta_1)}{\alpha_1}\right); \quad x > a_1 \geq 0, \ \alpha_1 > 0, \ \beta_1 > 0$$

and

$$f_2(y; a_2, \alpha_2, \beta_2, \theta_2) = \frac{\beta_2}{\alpha_2} h^{\beta_2}(y; \theta_2) h'(y; \theta_2) \exp \left(-\frac{h^\theta(y; \theta_2)}{\alpha_2}\right); \quad y > a_2 \geq 0, \ \alpha_2 > 0, \ \beta_2 > 0.$$

We assume that $\alpha_1$ and $\alpha_2$ are unknown, but $a_1, a_2, \beta_1, \beta_2, \theta_1,$ and $\theta_2$ are known. Let $n$ items on $X$ and $m$ items on $Y$ are put on a life test and the truncation numbers for $X$ and $Y$ are $r_1$ and $r_2$, respectively. Let us denote by

$$S_i = \sum_{i=1}^{n} g^h(X_i; \theta_1) + (n-r_1)g^h(X_{(r_1)}; \theta_1)$$

and

$$T_j = \sum_{j=1}^{m} h^h(Y_j; \theta_2) + (m-r_2)h^h(Y_{(r_2)}; \theta_2).$$

In what follows, we obtain the UMVUE of ‘$P$’.

**Theorem 3.2:** The UMVUE of ‘$P$’ is given by

$$\hat{P}_{n} = \left\{ \begin{array}{ll} \frac{(r_2-1)}{s_2} \int_{0}^{\infty} \left[ g^h \left( h^{-1} \left( \frac{(1-z)T_b}{s_b} \right)^{1/\beta_2} \right) \right]^{(r-1)} dz, & g^h(1) < g^h(T_b^{1/\beta_2}) \\ \frac{(r_2-1)}{s_2} \int_{0}^{\infty} \left[ g^h \left( h^{-1} \left( \frac{(1-z)T_b}{s_b} \right)^{1/\beta_2} \right) \right]^{(r-1)} dz, & h^h(T_b^{1/\beta_2}) < g^h(1) \end{array} \right.$$
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**Proof:** It follows from Lemma 3.3 that the UMVUES of \( f_1(x;a_1,\alpha_1,\beta_1,\theta_1) \) and \( f_2(y;a_2,\alpha_2,\beta_2,\theta_2) \) at specified points ‘\( x \)’ and ‘\( y \)’, respectively, are

\[
\hat{f}_{1\text{II}}(x;a_1,\alpha_1,\beta_1,\theta_1) = \begin{cases} 
\frac{(r_1-1)\beta_1 \ g^{\beta_1-1}(x;\theta)g'(x;\theta)}{S_1} \left[ 1 - \frac{g_1^{\beta_1}(x;\theta)}{S_1} \right]^{\eta_1-2} \ g_1^{\beta_1}(x;\theta) < S_1 \\
0, \text{ otherwise} \end{cases} \tag{3.3.6}
\]

and

\[
\hat{f}_{2\text{II}}(y;a_2,\alpha_2,\beta_2,\theta_2) = \begin{cases} 
\frac{(r_2-1)\beta_2 \ h^{\beta_2-1}(y;\theta)h'(y;\theta)}{T_2} \left[ 1 - \frac{h_2^{\beta_2}(y;\theta)}{T_2} \right]^{\eta_2-2} \ h_2^{\beta_2}(y;\theta) < T_2 \\
0, \text{ otherwise} \end{cases} \tag{3.3.7}
\]

From the arguments similar to those adopted in proving Theorem 3.1, it can be shown that the UMVUE of ‘\( P \)’ is given by

\[
\hat{P} = \int_{y=a_2}^{\infty} \int_{x=y}^{\infty} \hat{f}_{1\text{II}}(x;a_1,\alpha_1,\beta_1,\theta_1) \hat{f}_{2\text{II}}(y;a_2,\alpha_2,\beta_2,\theta_2) \ dx \ dy,
\]

which on using (3.3.6) and (3.3.7) gives that

\[
\hat{P} = \frac{(r_1-1)(r_2-1)\beta_1 \beta_2}{S_1 T_1} \int_{y=a_2}^{\infty} \int_{x=y}^{\infty} \left[ 1 - \frac{g_1^{\beta_1}(x;\theta)}{S_1} \right]^{\eta_1-2} \left[ 1 - \frac{h_2^{\beta_2}(y;\theta)}{T_2} \right]^{\eta_2-2} \ g_1^{\beta_1}(x;\theta)h_2^{\beta_2}(y;\theta) \ dx \ dy
\]

\[
= \frac{(r_2-1)\beta_2}{T_2} \int_{y=a_2}^{\infty} \left[ 1 - \frac{g_1^{\beta_1}(y;\theta)}{S_1} \right]^{\eta_1-2} \left[ 1 - \frac{h_2^{\beta_2}(y;\theta)}{T_2} \right]^{\eta_2-2} \ g_1^{\beta_1}(y;\theta)h_2^{\beta_2}(y;\theta) \ dy
\]

\[
= \frac{(r_1-1)\beta_1}{S_1} \int_{y=a_2}^{\infty} \left[ 1 - \frac{g_1^{\beta_1}(y;\theta)}{S_1} \right]^{\eta_1-2} \left[ 1 - \frac{h_2^{\beta_2}(y;\theta)}{T_2} \right]^{\eta_2-2} \ g_1^{\beta_1}(y;\theta)h_2^{\beta_2}(y;\theta) \ dy
\]

The theorem now follows from (3.3.8).
Corollary 3.1: In the case when \(a_1 = a_2 = a\), say, \(\theta_1 = \theta_2 = \theta\), say, \(g(x; \theta) = h(x; \theta)\), but \(\alpha_1 \neq \alpha_2\),

\[
\hat{P}_{ii} = \begin{cases} 
\frac{\beta_2}{\beta_1} (r_2-1) \sum_{i=0}^{r_2-2} (-1)^i \binom{r_2-2}{i} \left( \frac{S^\theta_{r_2}}{T_{r_2}} \right)^{i+1} B \left( (i+1) \frac{\beta_2}{\beta_1}, r_1 \right), & S^{\theta_1} < T^{1/\theta_2}_{r_1}; \\
(r_2-1) \sum_{i=0}^{r_2-1} (-1)^i \binom{r_2-1}{i} \left( \frac{T_{r_2}}{S_{r_1}} \right)^{i} B \left( \frac{\beta_2}{\beta_2}, i+1, r_2 - 1 \right), & T^{1/\theta_2}_{r_1} < S^{\theta_2}. 
\end{cases}
\]

Proof: From Theorem 3.2, for \(S^{\theta_1} < T^{1/\theta_2}_{r_1}\),

\[
\hat{P}_{ii} = (r_2-1) \int_0^{S^{\theta_1}_2} \left[ \frac{1}{S^{\theta_2}_1} \left( \frac{(1-z)T_{r_2}}{S_{r_1}} \right)^{\frac{\beta_2}{\beta_1}} \right]^{r_2-1} dz \\
= (r_2-1) \int_0^{S^{\theta_2}_2} \left[ \frac{1}{S^{\theta_1}_1} \left( \frac{uT_{r_2}}{S_{r_1}} \right)^{\frac{\beta_2}{\beta_1}} \right]^{r_2-1} du \\
= \frac{\beta_2}{\beta_1} (r_2-1) \sum_{i=0}^{r_2-2} (-1)^i \binom{r_2-2}{i} \left( \frac{S^{\theta_2}_1}{T_{r_2}} \right)^{i+1} \int_0^{1} v^{i+1} (1-v)^{r_2-1-i} dv
\]

and the first assertion follows. Furthermore, for \(T^{1/\theta_2}_{r_1} < S^{\theta_1}_r\),

\[
\hat{P}_{ii} = (r_2-1) \int_0^{S^{\theta_1}_2} \left[ \frac{1}{S^{\theta_1}_1} \left( \frac{(1-z)T_{r_2}}{S_{r_1}} \right)^{\frac{\beta_2}{\beta_1}} \right]^{r_2-1} dz \\
= (r_2-1) \sum_{i=0}^{r_1-1} (-1)^i \binom{r_1-1}{i} \left( \frac{T_{r_2}}{S_{r_1}} \right)^{i} \int_0^{1} u^{r_2-2} (1-u)^{\frac{\beta_1}{\beta_2}} du
\]

and the second assertion follows.
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Now we consider estimation based on type I censored data. Let \( X_1 \leq X_2 \leq \ldots \leq X_n \) be the failure times of \( n \) items under test from (3.2.1). The test begins at time \( X_0 = a \) and the system operates till \( X_1 = x_1 \) when the first failure occurs. The failed item is replaced by a new one and the system operates till the second failure occurs at time \( X_2 = x_2 \), and so on. The experiment is terminated at time \( t_o \).

**Lemma 3.4:** If \( N(t_o) \) be the number of failures during the interval \([0, t_o]\), then

\[
P[N(t_o) = r|t_o] = \frac{\left\{n g^\beta(t_o;\theta)\right\}^r}{r!} \exp\left\{-\frac{n g^\beta(t_o;\theta)}{\alpha}\right\}.
\]

**Proof:** Let us make the transformations \( W_1 = g^\beta(X_1;\theta), W_2 = g^\beta(X_2;\theta) - g^\beta(X_1;\theta), \ldots, W_n = g^\beta(X_n;\theta) - g^\beta(X_{n-1};\theta) \). The pdf of \( W_1 \) is

\[
h(w_1) = \frac{n}{\alpha} \exp\left(-\frac{nw_1}{\alpha}\right).
\]

Moreover, \( W_2, \ldots, W_n \) are independent and identically distributed as \( W_1 \). Using the monotonicity property of \( g(x;\theta) \),

\[
P[N(t) = r|t_o] = P[X_1 \leq t_o] - P[X_{r+1} \leq t_o] = P[g^\beta(X_1;\theta) \leq g^\beta(t_o;\theta)] - P[g^\beta(X_{r+1};\theta) \leq g^\beta(t_o;\theta)]
\]

\[
= P[W_1 + W_2 + \ldots + W_r \leq g^\beta(t_o;\theta)] - P[W_1 + W_2 + \ldots + W_{r+1} \leq g^\beta(t_o;\theta)]. \tag{3.3.9}
\]

From the additive property of exponentially distributed rv's [see Johnson and Kotz (1970), p.170], \( U = \sum_{i=1}^{n} \frac{W_i}{\alpha} \) follows gamma distribution with pdf
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\[ h(u) = \frac{1}{\Gamma(r)} u^{r-1} e^{-u}; \quad u > 0. \]  \hspace{1cm} (3.3.10)

Using (3.3.10) and a result of Patel, Kapadia and Owen (1976, p.244), we obtain from (3.3.9) that

\[ P[N(t_o) = r|t_o] = \frac{1}{\Gamma(r+1)} \int_{\text{ng}(t;\theta)} e^{-u} \, du - \frac{1}{\Gamma(r)} \int_{\text{ng}(t;\theta)} e^{-u} u^{r-1} \, du \]

\[ = \exp \left\{- \frac{\text{ng}(t_o;\theta)}{\alpha} \right\} \sum_{j=0}^{r} \left\{ \frac{\text{ng}(t_o;\theta)}{\alpha} \right\}^{j} j! - \sum_{j=0}^{r-1} \left\{ \frac{\text{ng}(t_o;\theta)}{\alpha} \right\}^{j} j! \]

and the lemma follows.

In the following lemma, we derive the UMVUE of \( \alpha^{-q} \), where \( q \) is a positive integer.

**Lemma 3.5:** For \( q \) to be a positive integer, the UMVUE of \( \alpha^{-q} \) is given by

\[ \hat{\alpha}_i^{-q} = \frac{r!}{(r-q)!} \left( \frac{\text{ng}(t_o;\theta)}{\alpha} \right)^{-q} \]

**Proof:** It follows from Lemma 3.4 and Fisher-Neyman factorization theorem [see Rohatgi (1976, p.341)] that \( r \) is sufficient for estimating \( \alpha \). Moreover, since the distribution of \( r \) belongs to exponential family, it is also complete [see Rohatgi (1976, p.347)]. The lemma now follows from the result that the \( q \)th factorial moment of distribution of \( r \) is given by

\[ E\{r(r-1) \ldots (r-q+1)\} = \left\{ \frac{\text{ng}(t_o;\theta)}{\alpha} \right\}^{q} \]
In the following lemma, we obtain the UMVUE of the sampled pdf (3.2.1) at a specified point ‘x’.

**Lemma 3.6:** The UMVUE of \( f(x;a,\alpha,\beta,\theta) \) at a specified point ‘x’ is

\[
\hat{f}_i(x;a,\alpha,\beta,\theta) = \left\{ \begin{array}{ll}
\frac{\beta g^{\beta-1}(x;\theta)g'(x;\theta)}{ng^\beta(t_o;\theta)} \left( 1 - \frac{g^\beta(x;\theta)}{ng^\beta(t_o;\theta)} \right)^{r-i} ; & \text{if } g^\beta(x;\theta) < ng^\beta(t_o;\theta) \\
0, & \text{otherwise}
\end{array} \right.
\]

**Proof:** Using Lemma 1 of Chaturvedi and Tomer (2002) and Lemma 3.5, from (3.3.4), the UMVUE of \( f(x;a,\alpha,\beta,\theta) \) at a specified point ‘x’ is

\[
\hat{f}_i(x;a,\alpha,\beta,\theta) = \beta g^{\beta-1}(x;\theta)g'(x;\theta) \sum_{i=0}^{\infty} \left( -\frac{1}{i!} \right) \frac{g^\beta(x;\theta)}{ng^\beta(t_o;\theta)} \frac{1}{i!} \left\{ \frac{r!}{(r-i-1)!} \right\} \left\{ \frac{g^\beta(t;\theta)}{ng^\beta(t_o;\theta)} \right\}^i
\]

and the lemma follows.

In the following theorem, we derive the UMVUE of \( R(t) \).

**Theorem 3.3:** The UMVUE of \( R(t) \) is given by

\[
\hat{R}_i(t) = \left\{ \begin{array}{ll}
\frac{\beta g^{\beta-1}(t;\theta)g'(t;\theta)}{ng^\beta(t_o;\theta)} \left( 1 - \frac{g^\beta(t;\theta)}{ng^\beta(t_o;\theta)} \right)^{r-i} ; & \text{if } g^\beta(t;\theta) < ng^\beta(t_o;\theta) \\
0, & \text{otherwise}
\end{array} \right.
\]

**Proof:** From the arguments similar to those adopted in the proof of Theorem 3.1, using Lemma 3.6,

\[
\hat{R}_i(t) = \int_{t}^{\infty} \hat{f}_i(x;a,\alpha,\beta,\theta)dx
\]
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\[ = \frac{r \beta}{\eta_\beta(t_o; \theta)} \int_t^1 g^{\beta-1}(x; \theta)g'(x; \theta) \left\{ 1 - \frac{g^\beta(x; \theta)}{\eta_\beta(t_o; \theta)} \right\}^r \, dx \]

\[ = r \int_{\frac{\eta(t_o; \theta)}{\eta_\beta(t_o; \theta)}}^1 (1-y)^{-1} \, dy \]

and the theorem follows.

In what follows, we obtain UMVUE of ‘P’. Suppose n items on X and m items on Y are put through a life test and \( t_o \) and \( t_{oo} \) are their truncation times, respectively. Let \( r_1 \) items on X and \( r_2 \) items on Y fail before times \( t_o \) and \( t_{oo} \), respectively.

**Theorem 3.4:** The UMVUE of ‘P’ is given by

\[ \hat{P}_1 = \begin{cases} \frac{r_1}{\eta_\beta(t_o; \theta)} \int_0^{1} \left\{ 1 - \frac{g^\beta(h^{-1}(m h^{\beta_1}(t_{oo}, \theta_2) z)^{1/\beta_1})}{\eta^\beta(t_o; \theta)} \right\}^r (1-z)^{-1} \, dz, \\
\frac{r_2}{\eta_\beta(t_o; \theta)} \int_0^{1} \left\{ 1 - \frac{g^\beta(h^{-1}(m h^{\beta_1}(t_o, \theta_1) z)^{1/\beta_1})}{\eta^\beta(t_o; \theta)} \right\}^r (1-z)^{-1} \, dz, \\
g^{-1}(\eta^\beta(t_o, \theta_1))^{1/\beta_1} < h^{-1}(m h^{\beta_1}(t_{oo}, \theta_2))^{1/\beta_1}. \end{cases} \]

**Proof:** Using the arguments similar to those applied in the proofs of Theorem 3.1 and Lemma 3.6,

\[ \hat{P}_1 = \int_{y=a_1}^{y=a_2} \int_{x=x_1}^{x=x_2} \hat{f}_{i1}(x; a_1, \alpha_1, \beta_1, \theta_1) \hat{f}_{i2}(y; a_2, \alpha_2, \beta_2, \theta_2) \, dx \, dy \]

\[ = \frac{r_1 r_2 \beta_2}{n \eta^\beta(t_o, \theta) h^{\beta_2}(t_{oo}, \theta_2)} \int_{y=a_1}^{y=a_2} \int_{x=x_1}^{x=x_2} g^{\beta_1-1}(x; \theta_1) h^{\beta_2-1}(y; \theta_2) g'(x; \theta_1) h'(y; \theta_2) \\
\left[ 1 - \frac{g^\beta(x; \theta_1)}{\eta^\beta(t_o; \theta_1)} \right]^{\beta_1-1} \left[ 1 - \frac{h^\beta(y; \theta_2)}{m h^{\beta_1}(t_{oo}, \theta_2)} \right]^{\beta_2-1} \, dx \, dy \]
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\[ \frac{r_2 \beta_2}{mh^{\beta_2}(t_{oo};\theta)} \int_{y=a_2}^\infty \frac{1}{ng^\beta(t_o;\theta)} (1-u)^{\alpha-1}y^{\beta+1}(y;\theta)h(y;\theta) \left[ 1 - \frac{h^{\beta_2}(y;\theta)}{mh^{\beta_2}(t_{oo};\theta)}\right]^{-\alpha-1} du dy \]

\[ = \frac{r_2 \beta_2}{mh^{\beta_2}(t_{oo};\theta)} \min\left\{ g^\beta(t_o;\theta), h^\beta(t_o;\theta) \right\} h(y;\theta) h'(y;\theta) \left[ 1 - \frac{h^{\beta_2}(y;\theta)}{mh^{\beta_2}(t_{oo};\theta)}\right]^{-\alpha-1} dy. \]  

(3.3.11)

The theorem now follows from (3.3.11).

**Corollary 3.2:** In the case when \( a_1 = a_2 = a \), say, \( \theta_1 - \theta_2 = \theta \), say, \( \beta_1 = \beta_2 = \beta \), say, 
\( g(x;\theta) = h(x;\theta) \), \( t_o = t_{oo} \), but \( \alpha_1 \neq \alpha_2 \), 

\[ \hat{P}_1 = \begin{cases} 
\sum_{i=0}^{m} (-1)^i \binom{m}{i} B(i+1, r_2), & m < n \\
\sum_{i=0}^{n} (-1)^{i+1} \binom{n}{i} B(i+1, r_2+1), & n < m.
\end{cases} \]

**Proof:** From Theorem 3.4, for \( m < n \),

\[ \hat{P}_1 = r_2 \int_0^1 \left( 1 - \frac{m}{n}z \right)^{r_2} (1-z)^{r_2-1} dz \]

\[ = r_2 \sum_{i=0}^{n} (-1)^i \binom{n}{i} \int_0^1 z^i (1-z)^{r_2-1} dz \]

and the first assertion follows. For \( n < m \),

\[ \hat{P}_1 = r_2 \int_0^{n/m} \left( 1 - \frac{m}{n}z \right)^{r_2} (1-z)^{r_2-1} dz \]

\[ = r_2 \frac{n}{m} \int_0^1 (1-u)^{r_2} \left( 1 - \frac{n}{m}u \right)^{r_2-1} du \]
\[
\begin{align*}
= r_2 \sum_{i=0}^{r-1} (-1)^i \binom{r_2-1}{i} \left( \frac{n}{m} \right)^{i+1} \int_0^1 (1-u)^t du
\end{align*}
\]

and the second assertion follows.

### 3.4. MLES OF R(t) AND ‘P’ UNDER TYPE I AND TYPE II CENSORINGS

In order to compare the performance of UMVUES and MLES, we consider the case when ‘a’, \( \beta \) and \( \theta \) are known, but \( \alpha \) is unknown. We first consider estimation based on type II censoring.

**Lemma 3.7:** The MLE of \( \alpha \) is

\[
\tilde{\alpha}_{II} = \frac{S_r}{r}.
\]

**Proof:** From (3.3.3), the log-likelihood is

\[
\log L(\alpha | x) = r \log \beta - r \log \alpha + \log \{n(n-1)...(n-r+1)\} + (\beta-1) \sum_{i=1}^r \log \{g(x_i; \theta)\} + \sum_{i=1}^r \log \{g'(x_i; \theta)\} \cdot \frac{S_r}{\alpha}.
\]

The result now follows on differentiating (3.4.1) with respect to \( \alpha \), equating the differential coefficient to zero and solving the equation for \( \alpha \).

**Lemma 3.8:** The MLE of \( f(x; a, \alpha, \beta, \theta) \) at a specified point ‘x’ is

\[
\tilde{f}_{II} (x; a, \alpha, \beta, \theta) = \left( \frac{rf}{S_r} \right) g^{(1)}(x; \theta) g'(x; \theta) \exp \left( -r \frac{g^\beta(x; \theta)}{S_r} \right).
\]

**Proof:** The proof follows from (3.2.1), Lemma 3.7 and invariance property of the MLE.
Theorem 3.5: The MLE of \( R(t) \) is given by

\[
\tilde{R}_n(t) = \exp \left\{ -\frac{\sum g^\beta(x;\theta)}{S_t} \right\}.
\]

Proof: From invariance property of MLE,

\[
\tilde{R}_n(t) = \int \tilde{f}_n(x; a, \alpha, \beta, \theta) \; dx,
\]

which, on using Lemma 3.8, gives that

\[
\tilde{R}_n(t) = \frac{\sum g^\beta(x;\theta)}{S_t} \int g^{\beta - 1}(x;\theta) g'(x;\theta) \exp \left( -\frac{\sum g^\beta(x;\theta)}{S_t} \right) \; dx
\]

\[
= \int \frac{\sum g^\beta(x;\theta)}{S_t} e^{-\gamma} \; dy
\]

and the theorem follows.

Theorem 3.6: The MLE of \('P'\) is given by

\[
\tilde{P}_n = \int_0^\infty \exp \left\{ -\frac{\sum r_i g^\beta(h^{-1}(\frac{T_{r_1}}{r_2})^{1/2})}{S_i} \right\} e^{\gamma} \; dv.
\]

Proof: From invariance property of MLE,

\[
\tilde{P}_n = \int_y^\infty \int_y^\infty \tilde{f}_{1n}(x; a_1, \alpha_1, \beta_1, \theta_1) \tilde{f}_{2n}(y; a_2, \alpha_2, \beta_2, \theta_2) \; dx \; dy,
\]

which, on using Lemma 3.8, gives that
\[ \tilde{P}_{ii} = \left( \frac{r_1 r_2 \beta_2}{S_{y_{a_2} - y}} \right) \int_{y = a_2}^{\infty} \int_{x = y}^{\infty} \left\{ g^{\beta_1}(x; \theta) g'(x; \theta) \exp \left( -\frac{r_1 g^\beta(x; \theta)}{S_{r_1}} \right) \right\} dx dy \]

\[ \cdot \left\{ h^{\beta_1}(y; \theta) h'(y; \theta) \exp \left( -\frac{r_1 h^\beta(y; \theta)}{S_{r_2}} \right) \right\} dy \]

\[ = \left( \frac{r_1 \beta_2}{T_{r_2}} \right) \int_{y = a_2}^{\infty} \int_{x = y}^{\infty} e^{-u} h^{\beta_1}(y; \theta) h'(y; \theta) \exp \left( -\frac{r_1 h^\beta(y; \theta)}{S_{r_2}} \right) du dy \]

\[ = \left( \frac{r_1 \beta_2}{T_{r_2}} \right) \int_{y = a_2}^{\infty} \exp \left( -\frac{r_1 g^\beta(y; \theta)}{S_{r_1}} \right) h^{\beta_1}(y; \theta) h'(y; \theta) \exp \left( -\frac{r_1 h^\beta(y; \theta)}{S_{r_2}} \right) dy \]

and the theorem follows.

**Corollary 3.3:** In the case when \( a_1 = a_2 = \alpha \), say, \( \theta_1 = \theta_2 = \theta \), say, \( \beta_1 = \beta_2 = \beta \), say, \( g(x; \theta) = h(y; \theta) \), but \( \alpha_1 \neq \alpha_2 \),

\[ \tilde{P}_{ii} = \left( \frac{r_1 S_{r_1}}{r_2 S_{r_2} + r_1 T_{r_1}} \right) \]

Now we consider estimation base on type I censoring.

**Lemma 3.9:** The MLE of \( \alpha \) is

\[ \tilde{\alpha}_i = \frac{ng^\beta(t_0; \theta)}{r} \]

**Proof:** From Lemma 3.4, the log-likelihood is

\[ \log L(\alpha | r) = r \log n - r \log \alpha + r \log \left[ g^\beta(t_0; \theta) \right] - \log r! - n \frac{g^\beta(t_0; \theta)}{\alpha} \]

and the result follows.
Lemma 3.10: The MLE of $f(x;a,\alpha,\beta,\theta)$ at a specified point ‘x’ is

$$
\tilde{f}_i(x;a,\alpha,\beta,\theta) = \left\{ \frac{r \beta g^{-1}(x;\theta)g'(x;\theta)}{ng'(t_o;\theta)} \right\} \exp \left\{ \frac{-rg^\beta(x;\theta)}{ng'(t_o;\theta)} \right\}.
$$

Proof: The lemma follows from (3.2.1) and Lemma 3.9.

Theorem 3.7: The MLE of $R(t)$ is given by

$$
\tilde{R}_1(t) = \exp \left\{ \frac{-rg^\beta(t;\theta)}{ng'(t_o;\theta)} \right\}.
$$

Proof: From Lemma 3.10,

$$
\tilde{R}_1(t) = \int \tilde{f}_i(x; a,\alpha,\beta,\theta) \, dx
$$

$$
= \left\{ \frac{r \beta}{ng'(t_o;\theta)} \right\} \int g^{-1}(x;\theta)g'(x;\theta) \exp \left\{ \frac{-rg^\beta(x;\theta)}{ng'(t_o;\theta)} \right\} \, dx
$$

$$
= \int e^y \, dy
$$

and the theorem follows.

Theorem 3.8: The MLE of ‘P’ is given by

$$
\tilde{P}_1 = \int_0^\infty \exp \left\{ \frac{r g^h(h^{-1} m h \beta^2(t_o,\theta) v)^{1/2}}{ng'(t_o,\theta)} \right\} e^v \, dv.
$$

Proof: From Lemma 3.10,

$$
\tilde{P}_1 = \int_\gamma^\infty \int_y^\infty \tilde{f}_1(x; a_1,\alpha_1,\beta_1,\theta_1) \tilde{f}_2(y; a_2,\alpha_2,\beta_2,\theta_2) \, dx \, dy
$$
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\[
\begin{align*}
&= \left\{ \frac{r_x r_y \beta_1 \beta_2}{n m g^{\beta_1}(t_0; \theta) h^{\beta_2}(t_\infty; \theta)} \right\} \int_0^\infty \int_0^\infty g^{\beta_1-1}(x; \theta) g'(x; \theta) \\
&\times \exp \left\{ -\frac{r_x g^{\beta_1}(x; \theta)}{n g^{\beta_1}(t_0; \theta)} \right\} h^{\beta_2-1}(y; \theta) h'(y; \theta) \exp \left\{ -\frac{r_y h^{\beta_2}(y; \theta)}{m h^{\beta_2}(t_\infty; \theta)} \right\} \\
&\times \int_0^\infty \int_0^\infty g^{\beta_1}(y; \theta) g'(y; \theta) \\
&\times \exp \left\{ -\frac{r_x g^{\beta_1}(y; \theta)}{n g^{\beta_1}(t_0; \theta)} \right\} h^{\beta_2-1}(y; \theta) h'(y; \theta) \exp \left\{ -\frac{r_y h^{\beta_2}(y; \theta)}{m h^{\beta_2}(t_\infty; \theta)} \right\}
\end{align*}
\]

and the theorem follows.

**Corollary 3.4:** In the case when \( a_1 = a_2 = a \), say, \( \theta_1 = \theta_2 = \theta \), say, \( \beta_1 = \beta_2 = \beta \), say,

\[
g(x; \theta) = h(x; \theta), \quad t_0 = t_\infty, \quad \text{but } \alpha_1 \neq \alpha_2,
\]

\[
\hat{P}_i = \frac{r_x n}{r_x n + r_y m}.
\]

**REMARKS 3.1**

(i) In the literature, researchers have dealt with the estimation of \( R(t) \) and \( 'P' \), separately. If we look at the proofs of Theorems 3.1-3.8, we observe that the UMVUE(S) / MLE(S) of power(s) of parameter(s) is (are) used to obtain UMVUE(S) / MLE(S) of the sampled pdf(s), which is (are) subsequently used to estimate \( R(t) \) and \( 'P' \). Thus, for both the estimation problems, the basic role is played by the estimator(s) of power(s) of parameter(s). In this way, we have justified estimation of power(s) of parameter(s).

(ii) In the literature, the researchers have derived the UMVUES / MLES of \( 'P' \) for the case when \( X \) and \( Y \) follow the same distribution (may be with different parameters). We have obtained these estimators for all the three situations, when
X and Y follow the same distribution having all the parameters same other than α's, when X and Y have the same distribution with different parameters and when X and Y follow different distributions.

(iii) In the present approaches of obtaining UMVUES and MLES, one does not need the expressions of R(t) and ‘P’.

(iv) The problems of obtaining MLES when more parameters are unknown can be dealt on similar lines. One just needs as many differentials (with respect to unknown parameters) of likelihood function as the number of unknown parameters and their simultaneous solutions. The MLE of any parametric function can be obtained by plugging the MLES in place of unknown parameters.

(v) It follows from Lemma 3.2 that \( \text{Var}(\hat{\alpha}_{II}) = \frac{ra^2}{(r-1)^2} \rightarrow 0 \) as \( r \rightarrow \infty \). Moreover, from Lemma 3.7, \( E(\bar{a}_{II}) = \frac{ra}{r-1} \rightarrow \alpha \) as \( r \rightarrow \infty \) and \( \text{Var}(\bar{a}_{II}) = \frac{\alpha^2}{r} \rightarrow 0 \) as \( r \rightarrow \infty \). Thus, \( \hat{\alpha}_{II} \) and \( \bar{a}_{II} \) are consistent estimators of \( \alpha \). Since, \( \hat{f}_{II}(x; a, \alpha, \beta, \theta) \), \( \bar{f}_{II}(x; a, \alpha, \beta, \theta) \), \( \hat{R}_{II}(t) \), \( \bar{R}_{II}(t) \), \( \hat{P}_{II} \) and \( \bar{P}_{II} \) are continuous functions of consistent estimators, they are also consistent estimators.

### 3.5. TESTS AND CONFIDENCE INTERVALS

Let us consider the problem of testing of the hypothesis \( H_0 : \alpha \geq \alpha_o \) against \( H_1 : \alpha < \alpha_o \) at the level of significance \( \alpha' \in (0, 1) \) under type II censoring. The results for the complete samples can be obtained by putting \( r=n \). From (3.3.3),

\[
\frac{f^*(x_{(1)}, x_{(2)}, ..., x_{(r)}; a, \alpha, \beta, \theta)}{f^*(x_{(1)}, x_{(2)}, ..., x_{(r)}; a, \alpha, \beta, \theta)} = \left( \frac{\alpha_1}{\alpha_o} \right)^n \exp \left( \frac{1}{\alpha_1} - \frac{1}{\alpha_o} \right) S_r \]

(3.5.1)
For \( \alpha > a_t \), the right hand side of (3.5.1) is a non-decreasing function of \( s_t \). Thus the family of distributions \( f(x;\alpha,a,\beta,\theta) \) given at (3.2.1) has a monotone likelihood ratio in \( s_t \).

From a result of Rohatgi (1976, p. 420), the uniformly most powerful (UMP) critical region is given by

\[
S_t \leq k(\alpha_o, a_t),
\]

where \( k(\alpha_o, a_t) \) is determined by the size condition, namely

\[
P\{S_t \leq k(\alpha_o, a_t)|\alpha_o\} = \alpha^*.
\]

Denoting by \( Y = \frac{2S_t}{\alpha} \), It follows from Lemma 3.1 that \( Y \) follows chi-square distribution with \( 2r \) degrees of freedom (d.f.). Denoting by \( \xi_p(q) \), the 100p\% point of chi-square distribution with q d.f., i.e.

\[
P[\chi^2(q) \leq \xi_p(q)] = p.
\]

From (3.5.3) and (3.5.4),

\[
k(\alpha_o, a_t) = \frac{a_o}{2} \xi_o(2r).
\]

For a number \( \gamma \in (0,1) \), suppose we wish to test the hypothesis \( H^* : R(t) \leq \gamma \) against \( H^* : R(t) > \gamma \). Using (3.2.2), this is equivalent to \( H_o : \alpha \leq \frac{g^\beta(t;\theta)}{-\log(\gamma)} \) against \( H_1 : \alpha > \frac{g^\beta(t;\theta)}{-\log(\gamma)} \). Thus (3.5.3), gives UMP size \( \alpha^* \) test with \( \alpha_o = \frac{g^\beta(t;\theta)}{-\log(\gamma)} \) and

\[
k(\alpha_o, a_t) = \frac{g^\beta(t;\theta)\xi_o(2r)}{-2\log(\gamma)}.
\]

Now we consider the problems of constructing confidence intervals for \( \alpha \). The confidence intervals are obtained by using the already defined pivotal quantity \( Y = \frac{2S_t}{\alpha} \). Denoting by \( G_{2r}(\cdot) \), the cumulative distribution function of a \( \chi^2_{2r} \) rv,
\[ P(Y \leq c) = G_{2r}(c), \]

or,

\[ P\left( \frac{2S_t}{c} \leq \alpha \right) = G_{2r}(c). \quad (3.5.6) \]

From (3.5.6),

\[
\left( \frac{2S_t}{\xi_{2r}(2r)}, \infty \right)
\]

is the left one-sided confidence interval for \( \alpha \) with confidence coefficient \( 1 - \alpha^* \).

Similarly, \( \left( 0, \frac{2S_t}{\xi_a(2r)} \right) \) is the right one-sided confidence interval for \( \alpha \). A two-sided confidence interval for \( \alpha \) is given by

\[
\left( \frac{2S_t}{\xi_{a/2}(2r)}, \frac{2S_t}{\xi_{a/2}(2r)} \right).
\]

Since the reliability function (3.2.2) is a monotone function of \( \alpha \), a two-sided confidence interval for \( R(t) \) is

\[
\left[ \exp \left( \frac{-g^\theta(t;\theta)\xi_{1-a/2}(2r)}{2S_t} \right), \exp \left( \frac{-g^\theta(t;\theta)\xi_{a/2}(2r)}{2S_t} \right) \right].
\]
3.6. SIMULATION STUDIES

We have shown under REMARKS 3.1(v) that $\tilde{a}_n$, $\tilde{c}_n$, $\hat{f}_n(x; a, \alpha, \beta, \theta)$, $\tilde{f}_n(x; a, \alpha, \beta, \theta)$, $\tilde{R}_n(t)$, $\tilde{R}_n(t)$, $\tilde{P}_n$ and $\tilde{P}_n$ are consistent estimators. In order to verify these results, we have drawn sample of size $n=50$ from (3.2.1) with $g(x; \theta) = x^p$, $a=0$, $p=2$, $\beta=1$ and $\alpha=1$. In Fig. 3.1 and Fig. 3.2, respectively, we have plotted $\hat{f}_n(x; a, \alpha, \beta, \theta)$ and $\tilde{f}_n(x; a, \alpha, \beta, \theta)$ for different values of $r=5(5)30$ and 50 under type II censorship. We conclude from the figures that as $r$ increases, the curves of $\hat{f}_n(x; a, \alpha, \beta, \theta)$ and $\tilde{f}_n(x; a, \alpha, \beta, \theta)$ come close to the curve of $f(x; a, \alpha, \beta, \theta)$. This justifies the consistency property of the estimators.

![Fig. 3.1: The curves of $f(x; a, \alpha, \beta, \theta)$ (bold) and $\hat{f}_n(x; a, \alpha, \beta, \theta)$, (dotted)]
For the case when $\alpha$ is unknown, we have conducted a simulation experiment using the bootstrap resampling technique of the following sample of size 50, generated from (3.2.1) with $g(x; \theta) = x$, $a=0$, $\beta=1$ and $\alpha=2500$. 


Assuming that the data represents life spans of items in hours. For different values of $t$, we have computed $\hat{R}(t)$, $\hat{R}(t)$, bias, mean square error (MSE)/Variance, 95% confidence length and corresponding coverage percentage at $r=10, 20, 25$ and 35 under type II censoring. All the computations are based on 500 bootstrap replications and the results under type II censoring are reported in Table 3.1.
Table 3.1: Simulation results for estimation of $R(t)$ under type II censoring

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<th>$\hat{R}_{II}$</th>
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First row indicates the estimate, the second row indicates the bias, the third row indicates MSE, the fourth row indicates 95% bootstrap confidence length and the fifth row indicates the coverage percentage.
In order to estimate $P$, when $X$ and $Y$ follow the same distribution, for the case when $\alpha_1$ and $\alpha_2$ are unknown but the other parameters are known, we have conducted simulation experiments using bootstrap re-sampling technique for samples of sizes $(n, m) = (40, 30), (40, 40), (50, 40)$ and $(60, 60)$ across different $(r_1, r_2) = (10, 10), (10, 15), (15, 15)$ and $(25, 25)$. The samples are generated from (3.2.1) with $g(x; p_1) = x^{p_1}$, $h(y; p_2) = y^{p_2}$, $a_1 = a_2 = 0$, $p_1 = p_2 = 2$, $\beta_1 = \beta_2 = 1$, $\alpha_1 = 1$ and $\alpha_2 = 2$. The computations are based on 500 bootstrap replications. We have computed $\hat{P}_I$, $\hat{P}_II$, bias, MSE, 95% confidence length and corresponding coverage percentage under type II censoring and the results are presented in Table 3.2.

**Table 3.2: Simulation Results for estimation of $P$ on Type-II censoring**

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First row indicates the estimate, the second row indicates the bias, the third row indicates MSE, the fourth row indicates 95% bootstrap confidence length and the fifth row indicates the coverage percentage.
For the case when $X$ and $Y$ follow the different distributions, when all the parameters are unknown, the following samples (each of size 50) are generated from the distributions of $X$ and $Y$. The samples are generated from (3.2.1) with 
\[ g(x; \theta) = x^{p_1}, \quad p_1 = 3, \quad \beta_1 = 1, \quad \alpha_1 = 2000, \]
\[ h(y; \theta) = \log\left(\frac{y}{a}\right), \quad a = 1000 \text{ and } \alpha_2 = 0.5. \]


Let $t_0 = 1500$ be the truncation time for $X$ and $t_\infty = 1300$ for $Y$, so we have $r_1 = 18$ and $r_2 = 19$.

The MLES of the parameters of $X$ are $\hat{\theta}_1 = 4.29$ and $\hat{\theta}_1 = 1223.828$ and the MLES of the parameters of $Y$ are $\hat{\alpha}_2 = 0.6847$ and $\hat{\alpha} = 1002.18$. We have $P = 0.549$ with absolute error $< 2.0e-07$.

Using Theorem 3.8, $\hat{\theta}_1 = 0.6025$ with absolute error $< 1.1e-05$. 

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