Robustness of sequential testing procedures for a family of lifetime distributions
5.1. INTRODUCTION

The pioneering work in the area of sequential tests of statistical hypotheses is due to Wald (1947), who developed sequential probability ratio test (SPRT) for testing a simple hypothesis against a simple alternative. As measures of the performance of SPRT, Wald obtained the expressions for the operating characteristic (OC) and average sample number (ASN) functions. The robustness of the SPRT, when the distribution under consideration has undergone a change, has been studied by various authors while dealing with different probabilistic models useful in reliability/survival analysis. Epstein and Sobel (1955) developed SPRT for testing simple null hypothesis against simple alternative, for the scale parameter of an exponential distribution. Harter and Moore (1976) conducted Monte Carlo study to investigate the robustness of SPRT when the underlying distribution is a Weibull with shape parameter other than one. Montagne and Singpurwalla (1985) generalized the results of Harter and Moore (1976) from Weibull to a class of distributions having an increasing (decreasing) failure rate. They obtained inequalities for OC and ASN functions in order to demonstrate the robustness. Phatarfod (1971) developed SPRT for testing composite hypothesis for the shape parameter of the gamma distribution. For some fixed sample size results concerning the robustness of testing and estimation procedures related to exponential distribution, one may refer to Barlow and Proschan (1967) and Hager, Bain and Antle (1971).
In the present chapter, we consider the family of distributions discussed in Chapter 2. Let the random variable (rv) \( X \) follow the distribution having the pdf
\[
f(x;a,\gamma,\rho,\theta) = \frac{\gamma^\rho e^{-\gamma g(x;\theta)}}{\Gamma(\rho)}; \quad x > a; \quad \gamma > 0, \rho > 0.
\] (5.1.1)

Here, \( g(x;\theta) \) is a function of \( x \) and may also depend on a vector-valued parameter \( \theta \). Moreover, \( g(x;\theta) \) is monotonically increasing in \( x \) with \( g(a;\theta) = 0 \), \( g(\infty;\theta) = \infty \) and \( g'(x;\theta) \) denotes the derivative of \( g(x;\theta) \) with respect to \( x \).

In Sections 5.2 and 5.3, respectively, we develop SPRTS for testing simple hypotheses (versus simple alternatives) for the parameter \( \gamma \) (when \( \rho \) is known) and for the parameter \( \rho \) (when \( \gamma \) is known). The robustness of the proposed SPRTS is studied when the distribution under consideration has undergone a change. In Section 5.4, sequential test for the composite hypothesis regarding \( \rho \) (when \( \gamma \) is unknown) is developed and the OC and ASN functions are obtained. In order to calculate the roots of the equation needed to evaluate OC and ASN functions, we give a method of choosing the initial values in order to apply Newton-Raphson method. Finally, in Section 5.5, a discussion on the numerical findings is provided.

5.2. ROBUSTNESS OF THE SPRT FOR TESTING THE HYPOTHESIS REGARDING \( \gamma \) WHEN \( \rho \) IS KNOWN

Given a sequence of observations \( X_1, X_2, \ldots \) from (5.1.1), suppose one wishes to test the simple null hypothesis \( H_0: \gamma = \gamma_0 \) against the simple alternative \( H_1: \gamma = \gamma_1 (> \gamma_0) \), when ‘\( \rho \)’ is known. The SPRT for testing \( H_0 \) is defined as follows:
Let
\[ Z_i = \ln \left( \frac{f(X_i; \gamma_1, a, \rho, \theta)}{f(X_i; \gamma_0, a, \rho, \theta)} \right) = \ln \left( \frac{\gamma_0^\rho \exp\left( - (\gamma_1 - \gamma_0)g(X_i; \theta) \right)}{\gamma_0^\rho} \right) = \rho \ln \left( \frac{\gamma_1}{\gamma_0} \right) - (\gamma_1 - \gamma_0)g(X_i; \theta). \] (5.2.1)

We choose two numbers A and B such that 0 < B < 1 < A. At the \( n \)th stage, accept \( H_0 \), if \( \sum_{i=1}^{n} Z_i \ln B \leq \sum_{i=1}^{n} \), reject \( H_0 \), if \( \sum_{i=1}^{n} Z_i \ln A \geq \sum_{i=1}^{n} \), otherwise, continue sampling by taking the \( (n+1) \)th observation. If \( \alpha \in (0,1) \) and \( \beta \in (0,1) \) are type I and type II errors, respectively, then according to Wald (1947), A and B are approximately given by
\[ A \approx \frac{1 - \beta}{\alpha} \quad \text{and} \quad B \approx \frac{\beta}{1 - \alpha}. \] (5.2.2)

The OC function is approximately given by
\[ L(\gamma) \approx \frac{A^{1-B} - 1}{A^{1-B} - B^B}, \] (5.2.3)
where \( t_0 \) is the non-zero solution of the equation
\[ E\left( e^{t_0 Z_i} \right) = 1. \] (5.2.4)

Using the fact that \( g(X_i; \theta) \) follows a gamma distribution with scale parameter \( \frac{1}{\gamma} \) and shape parameter \( \rho \), we obtain from (5.2.1) and (5.2.4) that
\[ \int_0^\infty \exp \left( \left\{ \rho \ln \left( \frac{\gamma_1}{\gamma_0} \right) - (\gamma_1 - \gamma_0)g(x; \theta) \right\} t_0 \right) \frac{\gamma^\rho \exp\left( \frac{x; \theta}{\gamma}g'(x; \theta) \right)}{\Gamma(\rho)} \exp(-\gamma g(x; \theta)) dx = 1, \]
or,
\[
\gamma = \frac{(\gamma_1 - \gamma_o)T_o}{\{\gamma_1 / \gamma_o\}^{\alpha} - 1}.
\] (5.2.5)

The ASN function is approximately given by
\[
E(N|\gamma) \approx \frac{L(\gamma) \ln B + [1-L(\gamma)] \ln A}{E(Z|\gamma)},
\]
which on using (5.2.1) gives that
\[
E(N|\gamma) \approx \frac{L(\gamma) \ln B + [1-L(\gamma)] \ln A}{\rho \ln \left\{ \frac{\gamma_1}{\gamma_o} \right\} - \rho \left( \frac{\gamma_1 - \gamma_o}{\gamma_o} \right)}.
\] (5.2.6)

From (5.2.6), the ASN function under \( H_0 \) and \( H_1 \) is given [ see Siegmund (1985, P.13, Remarks 2.20)], respectively, by
\[
E_o(N) \approx \frac{(1-\alpha) \ln B + \alpha \ln A}{\rho \ln \left\{ \frac{\gamma_1}{\gamma_o} \right\} - \rho \left( \frac{\gamma_1 - \gamma_o}{\gamma_o} \right)},
\] (5.2.7) and
\[
E_i(N) \approx \frac{\beta \ln B + (1-\beta) \ln A}{\rho \ln \left\{ \frac{\gamma_1}{\gamma_o} \right\} - \rho \left( \frac{\gamma_1 - \gamma_o}{\gamma_1} \right)}.
\] (5.2.8)

The maximum value of the ASN occurs for \( \gamma = \tilde{\gamma} \), where \( \tilde{\gamma} \) is the solution of
\[
E(Z|\tilde{\gamma}) = 0 \quad \text{and this value is given by [ see Rohatgi (1976, p. 635)]}
\]
\[
E_i(N) \approx \frac{(-\ln A \ln B)}{E(Z_i|\tilde{\gamma})}.
\] (5.2.9)

It is easy to see that
\[
\tilde{\gamma} = \frac{\gamma_o - \gamma_1}{\ln \left\{ \frac{\gamma_o}{\gamma_1} \right\}}.
\] (5.2.10)
Chapter 5: Robustness of Sequential Testing Procedures for a family of Lifetime Distributions

and

\[ E(Z_i^2 | \gamma) = \rho \left[ \ln \left( \frac{\gamma_1}{\gamma_o} \right) \right]^2. \]  \hspace{1cm} (5.2.11)

Using (5.2.11) in (5.2.9), we obtain that

\[ E_i(N) \approx \frac{(-\ln A \ln B)}{\rho \left[ \ln \left( \frac{\gamma_1}{\gamma_o} \right) \right]^2}. \]  \hspace{1cm} (5.2.12)

Now we study the robustness of the SPRT. Let us suppose that the parameter \( \rho \) has undergone a change and the pdf (5.1.1) becomes \( f(x; a, \gamma, \kappa, \theta) \), which is obtained on replacing \( \rho \) by \( k \). The OC function of the SPRT is

\[ L(\gamma) \approx \frac{A^b - 1}{A^b - B^b}, \]  \hspace{1cm} (5.2.13)

where, in order to study the robustness of the SPRT, we consider \('h' as the solution of the equation

\[ \frac{\gamma^k}{\Gamma(k)} \left( \frac{\gamma_1}{\gamma_o} \right)^{\gamma h} \int_a^\infty \exp \left\{ -\left( \gamma-h \gamma_o + h \gamma_1 \right) g(x_i; \theta) \right\} g^{k-1}(x_i; \theta) g'(x_i; \theta) dx_i = 1. \]  \hspace{1cm} (5.2.14)

Using (5.1.1) and denoting by \( \phi_i = \frac{\rho}{k} \), we obtain from (6.2.15) that

\[ \frac{\gamma^k}{\Gamma(k)} \left( \frac{\gamma_1}{\gamma_o} \right)^{\gamma h} \int_a^\infty \exp \left\{ -\left( \gamma-h \gamma_o + h \gamma_1 \right) g(x_i; \theta) \right\} g^{k-1}(x_i; \theta) g'(x_i; \theta) dx_i = 1, \]

or,

\[ \left[ \begin{array}{c}
\left( \frac{\gamma_1}{\gamma_o} \right)^{\gamma h} \\
\gamma - h \gamma_o + h \gamma_1 \\
\gamma
\end{array} \right] = 1, \]
or,

\[
\gamma = \left\{ \frac{h(\gamma_o - \gamma_1)}{1 - \left( \frac{\gamma_1}{\gamma_o} \right)^{\gamma h}} \right\}.
\]  (5.2.15)

While dealing with the expression (5.2.15), we faced the practical problem of choosing the initial values in order to get the roots ‘h’ through Newton Raphson method. We give below a method to ‘assess’ the roots. To this end, we rewrite (5.2.15) as

\[
h \phi \ln \left( \frac{\gamma_1}{\gamma_o} \right) = \ln \left\{ 1 - \frac{h(\gamma_o - \gamma_1)}{\gamma} \right\}.
\]

Using the expansion for \( \ln(1-x) \) and retaining the terms up to third degree in ‘h’ and simplifying, we get

\[
\frac{h^2}{3\gamma^3} (\gamma_o - \gamma_1)^3 + \frac{h}{2\gamma^2} (\gamma_o - \gamma_1)^2 + \left( \frac{\gamma_o - \gamma_1}{\gamma} \right) + \phi \ln \left( \frac{\gamma_1}{\gamma_o} \right) = 0,
\]

or,

\[
h = \frac{-3\gamma}{4(\gamma_o - \gamma_1)} \pm \frac{3\gamma}{2(\gamma_o - \gamma_1)} \left\{ -\frac{13}{12} - \frac{4\phi \ln \left( \frac{\gamma_1}{\gamma_o} \right)}{3(\gamma_o - \gamma_1)} \right\}^{1/2}.
\]  (5.2.16)

We use the values of ‘h’ given by (5.2.16) as initial values for solving (5.2.15) through Newton–Raphson method.

5.3. ROBUSTNESS OF SPRT FOR TESTING THE HYPOTHESIS REGARDING \( \rho \) WHEN \( \gamma \) IS KNOWN

Let \( X_1, X_2, ... \) be a sequence of independent and identically distributed (i.i.d.) observations from (5.1.1). Our goal is to test the simple hypothesis \( H_o: \rho = \rho_o \) against
the simple alternative \( H_1: \rho = \rho_1(> \rho_o) \), when \( \gamma \) and \( \theta \) are known. We propose the following SPRT:

Define

\[
Z_i = \ln \frac{f(X;\rho, a, \gamma, \theta)}{f(X;\rho_o, a, \gamma, \theta)} \\
= \ln \Gamma(\rho) - \ln \Gamma(\rho_o) + (\rho_1 - \rho_o) \ln \gamma + (\rho_1 - \rho_o) \ln \left[ g(X;\theta) \right].
\]  

(5.3.1)

Let us denote by 

\[
g(n, \rho_o, \rho_1) = n \left[ \ln \Gamma(\rho) - \ln \Gamma(\rho_o) \right] + n(\rho_1 - \rho_o) \ln \left[ g(x;\theta) \right].
\]

At the \( n^{th} \) stage, accept \( H_o \) if

\[
\sum_{i=1}^{n} \ln \left[ g(X;\theta) \right] \leq (\rho_1 - \rho_o)^{-1} \left[ \ln B - g(n, \rho_o, \rho_1) \right],
\]

reject \( H_o \) if

\[
\sum_{i=1}^{n} \ln \left[ g(X;\theta) \right] \geq (\rho_1 - \rho_o)^{-1} \left[ \ln A - g(n, \rho_o, \rho_1) \right],
\]

and continue sampling by taking the \((n+1)^{th}\) observation if

\[
(\rho_1 - \rho_o)^{-1} \left[ \ln B - g(n, \rho_o, \rho_1) \right] < \sum_{i=1}^{n} \ln \left[ g(X;\theta) \right] < (\rho_1 - \rho_o)^{-1} \left[ \ln A - g(n, \rho_o, \rho_1) \right].
\]

Here, \( A \) and \( B \) are same as that defined at (5.2.2). The OC function is given by

\[
L(\rho) \approx \frac{A^t - 1}{A^t - B^t}.
\]

(5.3.2)

where \( t_o \) is the non-zero solution of

\[
E(e^{t_o X}) = 1.
\]

(5.3.3)

Once again, using the fact that \( g(x;\theta) \) follows a gamma distribution with scale parameter \( 1/\gamma \) and shape parameter \( \rho \), we obtain from (5.3.3) that
Chapter 5: Robustness of Sequential Testing Procedures for a family of Lifetime Distributions

\[
\left[ \frac{\Gamma(\rho_o)}{\Gamma(\rho_1)} \right]^t \left[ \frac{\Gamma(\rho+t_o(\rho_1-\rho_o))}{\Gamma(\rho)} \right] = 1. \tag{5.3.4}
\]

It is interesting to note that \( t_o \), and hence, the OC function is free from \( \gamma \). The ASN function is approximately given by

\[
E(N|\rho) \approx \frac{[L(\rho) \ln B + (1-L(\rho)) \ln A]}{E(Z_i|\rho)} \tag{5.3.5}
\]

Using a result of Gradshteyn and Ryzhik (1965, p.576) that

\[
\int_0^\infty x^{\rho-1}e^{-x}\ln x\,dx = \Gamma'(\mu), \tag{5.3.6}
\]

where \( \Gamma'(x) = \frac{d}{dx}\Gamma(x) \), we obtain from (5.3.1) that

\[
E(Z_i|\rho) = \ln \Gamma(\rho_o) - \ln \Gamma(\rho_1) + (\rho_o - \rho_1) \Gamma'(\rho). \tag{5.3.7}
\]

From (5.3.5) and (5.3.7), the ASN functions under \( H_o \) and \( H_1 \) is given, respectively, by

\[
E_o(N) \approx \frac{(1-\alpha)\ln B + \alpha \ln A}{\ln \Gamma(\rho_o) - \ln \Gamma(\rho_1) + (\rho_o - \rho_1) \Gamma'(\rho)} \tag{5.3.8}
\]

and

\[
E_i(N) \approx \frac{\beta \ln B + (1-\beta)\ln A}{\ln \Gamma(\rho_o) - \ln \Gamma(\rho_1) + (\rho_o - \rho_1) \Gamma'(\rho)}. \tag{5.3.9}
\]

The maximum value of the ASN function is at the point \( \rho = \tilde{\rho} \), where \( \tilde{\rho} \) is the solution of

\[
E(Z_i|\tilde{\rho}) = 0, \text{ i.e.,}
\]

\[
\Gamma'(\tilde{\rho}) = \frac{\left[ \ln \{\Gamma(\rho_i)\} - \ln \{\Gamma(\rho_o)\} \right]}{(\rho_i - \rho_o)} \tag{5.3.10}
\]

and this maximum value is given by
Chapter 5: Robustness of Sequential Testing Procedures for a family of Lifetime Distributions

\[
E_{\rho}(N) \approx \left( -\ln B \ln A \right) \frac{E(Z^2_{i|\rho})}{E(Z^2_{i|\bar{\rho}})}. \tag{5.3.11}
\]

Utilizing (5.3.1), (5.3.6) and a result of Gradshteyn and Ryzhik (1965, p.578), we get

\[
E(Z^2_{i|\bar{\rho}}) = 2 \left\{ \ln \frac{\Gamma(\rho_0)}{\Gamma(\rho_1)} \right\}^2 + \frac{(\rho_1 - \rho_0)^2}{\Gamma(\bar{\rho})} \xi(2, \rho-1),
\]

where, \( \xi(z, q) \) is Riemann’s Zeta function defined by

\[
\xi(z, q) = \sum_{n=0}^{\infty} \frac{1}{(q+n)^z}.
\]

Now we study the robustness of the SPRT. Let us now suppose that the distribution (5.1.1) has undergone a change and it becomes \( f(x; a, \nu, \rho, \theta) \), which is obtained on replacing \( \gamma \) by \( \nu \). The OC function of the SPRT is

\[
L(\rho) \approx \frac{A^{h-1}}{B^{h-1}},
\]

where ‘h’ is the solution of the equation

\[
\int_{a}^{b} \left\{ \frac{f(x; a, \gamma, \rho_1, \theta)}{f(x; a, \gamma, \rho_0, \theta)} \right\}^{h} f(x; a, \nu, \rho, \theta) \, dx = 1. \tag{5.3.12}
\]

using (5.3.1), we obtain from (5.3.12) that

\[
\left\{ \frac{\Gamma(\rho_0)}{\Gamma(\rho_1)} \right\}^{h} \frac{\gamma^{b(\rho_1-\nu_0)}}{\Gamma(\rho)} \int_{0}^{\infty} y^{\rho(\rho_1-\nu_0)-1} e^{-\gamma y} \, dy = 1,
\]

or,

\[
\left\{ \frac{\Gamma(\rho_0)}{\Gamma(\rho_1)} \right\}^{h} \left\{ \frac{\Gamma(\rho + h(\rho_1-\nu_0))}{\Gamma(\rho)} \right\} \phi_2^{\rho(\rho_1-\nu_0)} = 1, \tag{5.3.13}
\]

where \( \phi_2 = \left\{ \frac{\gamma}{\nu} \right\} \). The expression for the ASN function is same as that given at (5.3.5),
\( E(Z|\rho) = \ln \left\{ \Gamma(\rho_o)/\Gamma(\rho_i) \right\} + (\rho_i - \rho_o) \ln \phi_2 + (\rho_i - \rho_o)\Gamma'(\rho). \) \hspace{1cm} (5.3.14)

The expressions (5.3.13) and (5.3.4) are not of much use for the purpose of calculating the OC and ASN functions. For practical purpose, we use the approximation [see Phatarfod (1971, p.876 (2.3))]

\[
\ln \Gamma(x) \approx \ln \left( \sqrt{2\pi x} \right) - x + (x-1/2) \ln x,
\] \hspace{1cm} (5.3.15)

i.e.

\[
\Gamma'(x) \approx \ln(x-1/(2x)).
\] \hspace{1cm} (5.3.16)

Expression (5.3.13) now gives that

\[
\left\{ h(\rho_o - 1/2) \ln \rho_o - (\rho_i - 1/2) \ln \rho_i + (\rho_i - \rho_o) \ln \phi_2 \right\} - (\rho - 1/2) \ln \rho + \left[ \rho + h(\rho_i - \rho_o) - 1/2 \right] \ln \left[ \rho + h(\rho_i - \rho_o) \right] = 0.
\] \hspace{1cm} (5.3.17)

Using the expansion for \( \ln(1+x) \), retaining the terms up to third-degree in \( h \) and simplifying, we obtain from (5.3.17) that

\[
\frac{(\rho+1)(\rho_i - \rho_o)^3}{6\rho^3} h^2 - \frac{(2\rho+1)(\rho_i - \rho_o)^2}{6\rho^2} h
\]

\[
- \left\{ (\rho_o - 1/2) \ln \rho_o - (\rho_i - 1/2) \ln \rho_i + (\rho_i - \rho_o) \ln \rho + \left( 1 + \ln \phi_2 - \frac{1}{2\rho} \right) (\rho_i - \rho_o) \right\} = 0,
\]

or,

\[
h = \frac{3\rho}{(\rho+1)(\rho_i - \rho_o)} \left[ (2\rho+1) \pm \frac{\rho}{(\rho_i - \rho_o)} \left( \frac{(2\rho+1)^3(\rho_i - \rho_o)^3}{16\rho^2} \right) \right]
\]

\[
+ \frac{4(\rho+1)(\rho_i - \rho_o)}{6\rho} \left\{ (\rho_o - 1/2) \ln \rho_o - (\rho_i - 1/2) \ln \rho_i + (\rho_i - \rho_o) \left( 1 + \ln \phi_2 - \frac{1}{2\rho} \right) \right\} \right\}^{1/2}.
\]
5.4. SPRT FOR TESTING THE COMPOSITE HYPOTHESIS REGARDING $\rho$ WHEN $\gamma$ IS UNKNOWN

Based on a sequence $X_1, X_2, \ldots$ of iid observations from (5.1.1), our goal is to test the composite hypothesis $H_0: \rho = \rho_0$ versus $H_1: \rho = \rho_1$, where ‘$\gamma$’ is assumed to be unknown. We proceed as follows:

Let us make the transformations

$$Y_r = \frac{g(X_{r+1}; \theta)}{\sum_{i=1}^{r} g(X_i; \theta)}; \quad r=1,2, \ldots, (n-1). \quad (5.4.1)$$

Since $g(X_i; \theta)$’s are iid rv’s, each having gamma distribution with scale parameter $1/\gamma$ and shape parameter $\rho$, utilizing the additive property of gamma distribution [see Johnson and Kotz (1970, p.181)], we conclude from (5.4.1) that $Y_r$ follows a beta distribution of the first kind, with pdf given by

$$f(y_r; \rho) = \frac{1}{B(\rho, r\rho)} y_{r-1}^{\rho-1}(1-y_r)^{r\rho-1}; \quad 0 \leq y_r \leq 1. \quad (5.4.2)$$

We first establish the independence of $Y_r$’s. To this end, denoting by

$$U_r = g(X_{r+1}; \theta)/\sum_{i=1}^{r} g(X_i; \theta),$$

We can write

$$Y_r = U_r / (1 + U_r), \quad r = 1,2, \ldots, (n-1). \quad (5.4.3)$$

From (5.4.3), the independence of $Y_r$’s follow, if we can prove that $U_r$ and $U_{r+1}$ are independent. We note that the joint distribution of $V= g(X_r; \theta)$ and $W=\sum_{i=1}^{r} g(X_i; \theta)$ is
Chapter 5: Robustness of Sequential Testing Procedures for a Family of Lifetime Distributions

\[ f(v,w) = \frac{\rho^\rho v^{\rho-1} w^{(r-1)\rho-1} \exp(-(v+w)^\gamma)}{\Gamma(\rho)\Gamma((r-1)\rho)} \]  

Making the transformations \( S = V+W \) and \( Z=V/W \), from (5.4.4), the joint distribution of \( S \) and \( Z \) comes out to be

\[ g(s, z) = \left\{ \frac{s^{\rho-1} \exp(-\gamma s)}{\Gamma(r\rho)} \right\} \left\{ \frac{z^{\rho-1}}{B(\rho, (r-1)\rho)(1+z)^\rho} \right\} \]  

(5.4.5)

From (5.4.5), we conclude that the rv’s \( g(X_{i+1}; \theta), \sum_{i=1}^{r} g(X_i; \theta) \) and \( U_{r+1} \) are mutually independent. This establishes the independence of \( U_r \) and \( U_{r+1} \). It is to be noted here that the rv’s \( Y_i \)'s are not identically distributed. From (5.4.2), the likelihood of observing \( Y_1, Y_2, ..., Y_{n-1} \) is

\[ R_n = \frac{\Gamma(\rho_0)}{\Gamma(\rho_1)} \prod_{r=1}^{n-1} \frac{\Gamma(r\rho_0)\Gamma((r+1)\rho_1)}{\Gamma(r\rho_1)\Gamma((r+1)\rho_0)} y_r^{(\rho_1-\rho_0)} (1-y_r)^{\rho_1-\rho_0} \]

\[ = \frac{\Gamma(\rho_0)}{\Gamma(\rho_1)} \prod_{r=1}^{n} \frac{\Gamma(n \rho_0)}{\Gamma(n \rho_1)} \prod_{r=1}^{n-1} y_r^{(\rho_1-\rho_0)} (1-y_r)^{\rho_1-\rho_0} . \]  

(5.4.6)

The sequential test is carried out as follows:

Let us denote by \( Z_n = \ln R_n \). For \( A \) and \( B \) defined at (5.2.2), at the \( n^{th} \) stage, accept \( H_o \), if \( Z_n \leq \ln B \), reject \( H_o \), if \( Z_n \geq \ln A \), otherwise, continue sampling by taking the \( (n+1)^{th} \) observation. In terms of the original observations, the continuation region is given by

\[ \ln B + n \ln \frac{\Gamma(\rho_1)}{\Gamma(\rho_0)} + f(n) < (\rho_1-\rho_0) \sum_{r=1}^{n} \ln (g(x_i; \theta)/\bar{g}(n)) \]

\[ < \ln A + n \ln \frac{\Gamma(\rho_1)}{\Gamma(\rho_0)} + f(n), \]  

(5.4.7)
Chapter 5: Robustness of Sequential Testing Procedures for a family of Lifetime Distributions

where

\[ f(n) = n(\rho_1 - \rho_0) \ln n + \ln \left( \frac{\Gamma(n \rho)}{\Gamma(n \rho_0)} \right) \quad (5.4.8) \]

and

\[ \overline{g}(n) = n^{-1} \sum_{i=1}^{n} g(x_i; \theta). \]

Using the approximation [see Phatarfod (1971, p. 876 (2.3))]

\[ \ln \Gamma(x) \approx \ln \left( \sqrt{2\pi} \right) - x + (x-1/2) \ln x, \quad (5.4.9) \]

we obtain from (5.4.8) that

\[ f(n) \approx n \left[ (\rho_1 - \rho_0) + \rho_0 \ln \rho_0 - \rho_1 \ln \rho_1 \right] + \frac{1}{2} \ln \left( \frac{\rho_1}{\rho_0} \right). \quad (5.4.10) \]

From (5.4.7), we note that the boundaries for \( (\rho_1 - \rho_0) \sum_{r=1}^{n} \ln (g(x_r; \theta)/\overline{g}(n)) \) are not straight lines. However, on making use of (5.4.10), the boundaries can be reduced to straight lines.

Utilizing (5.4.6), for a fixed positive integer \( n \), the moment generating function of \( Z_n \) comes out to be

\[ M_n(t) = E \left\{ \exp(tZ_n) \right\} \]

\[ = \left[ \frac{\Gamma(n \rho_1)}{\Gamma(n \rho_0)} \right]^{n} \left[ \frac{\Gamma(\rho_0)}{\Gamma(\rho_1)} \right]^{n} \prod_{r=1}^{n} \left[ \frac{B(t(\rho_1 - \rho_0) + \rho, rt(\rho_1 - \rho_0) + r \rho_0)}{B(\rho, r \rho_0)} \right] \]

\[ = \left[ \frac{\Gamma(n \rho_1)}{\Gamma(n \rho_0)} \right]^{n} \left[ \frac{\Gamma(\rho_0)}{\Gamma(\rho_1)} \right]^{n} \left[ \frac{\Gamma(n \rho) \Gamma(t(\rho_1 - \rho_0) + \rho)}{\Gamma^a(\rho) \Gamma(nt(\rho_1 - \rho_0) + n \rho)} \right]. \quad (5.4.11) \]
Denoting by $N$, the stopping rule associated with the sequential procedure, we know Wald’s fundamental identity [see Wald (1947, P. 160)]

$$E\left[ \exp(tZ_N + \ln(M_n(t)))^{-1} \right] = 1,$$  \hspace{1cm} (5.4.12)

where $M_n(t)$ is obtained from (5.4.11) replacing $n$ by $N$. Of course, the identity (5.4.12) is exact but is of no practical use. Once again, utilizing the approximation (5.4.9), we after some algebraic manipulations that

$$\ln(M_n(t))^{-1} \approx \ln \left[ \frac{\rho^\rho}{\Gamma(\rho)} \left( \frac{\Gamma(t(\rho_1-\rho_o)+\rho)}{(t(\rho_1-\rho_o)+\rho)^{\rho/(\rho_o-\rho)}} \right)^{t^{-N}} \right]$$

$$+ \ln \left[ \frac{\rho}{(t(\rho_1-\rho_o)+\rho)} \left( \frac{\rho_1}{\rho_o} \right)^{t^{1/2}} \right].$$  \hspace{1cm} (5.4.13)

Making substitution from (5.4.13) in (5.4.12), we get

$$E\left[ e^{i\omega N \{\lambda(t)\}} \right] \approx \tau(t),$$  \hspace{1cm} (5.4.14)

where

$$\lambda(t) = \frac{\rho^\rho}{\Gamma(\rho)} \left( \frac{\Gamma(t(\rho_1-\rho_o)+\rho)}{(t(\rho_1-\rho_o)+\rho)^{\rho/(\rho_o-\rho)}} \right)^t \left( \frac{\rho_1}{\rho_0} \Gamma(\rho_o) \right)^{t^{-1}}$$  \hspace{1cm} (5.4.15)

and

$$\tau(t) = \left( \frac{(t(\rho_1-\rho_o)+\rho)}{\rho} \left( \frac{\rho_o}{\rho_1} \right)^{t^{1/2}} \right).$$  \hspace{1cm} (5.4.16)

Let $t = t(\rho)$ be the non-zero solution of the equation $\lambda(t) = 1$, i.e. from (5.4.15),

$$G(t(\rho_1-\rho_o)+\rho) + t\left[ G(\rho_o)-G(\rho_1) \right] = G(\rho),$$  \hspace{1cm} (5.4.17)

where

$$G(x) = \ln \Gamma(x) - x \ln x.$$
Once again using (5.4.9), we obtain from (5.4.17) that

$$\rho = \frac{(t(\rho_1 - \rho_0) + \rho)}{\left(\frac{\rho_1}{\rho_0} - 1\right)}.$$  \hspace{1cm} (5.4.18)

For the values of $t(\rho)$ and $\rho$ satisfying (5.4.18), the OC function is given by

$$L(\rho) \approx \frac{(A^{\mu(\rho)} - \tau(t(\rho)))}{(A^{\mu(\rho)} - B^{\mu(\rho)})}.$$  

Now we obtain the expression for the ASN function. To this end, differentiating (5.4.14) with respect to $t$, we get

$$E\left(Z_{\mu}e^{tZ_{\mu}} \left\{\lambda(t)\right\}^{N \lambda(t)} e^{e^{tZ_{\mu}}\tau'(t)}\right) \approx \tau'(t),$$

which on substituting $t=0$ gives that

$$E(N) \approx \frac{E(Z_{\mu}) - \tau'(0)}{\lambda'(0)}.$$  

From (5.4.16),

$$\tau'(0) = (1/2) \left[\ln \left(\frac{\rho_1}{\rho_0}\right) - \frac{(\rho_1 - \rho_0)}{\rho} \right]$$

and ignoring the excess over boundaries

$$E(Z_{\mu}) \approx L(\rho) \ln B + \left[1 - L(\rho)\right] \ln A.$$  

Furthermore, using approximation (5.4.9) in (5.4.15) and taking limit as $t \to 0$,

we get

$$\lambda'(0) \approx (1/2) \left[\ln \left(\frac{\rho_1}{\rho_0}\right) - \frac{(\rho_1 - \rho_0)}{\rho} \right].$$
4.5. NUMERICAL FINDINGS

We first consider the problem of testing simple versus simple hypotheses regarding $\gamma$ when $\rho$ is known. Let us consider $H_0: \gamma = 5$ versus $H_1: \gamma = 10$ taking $\alpha = \beta = 0.05$. For different values of $\gamma$, the real roots ‘t’ are obtained from (5.2.16) and are taken as initial values to solve the original equation (5.2.15) through Newton-Raphson method. For $\phi_1 = 1, 1.0811$ and $0.9091$, the OC and ASN functions are computed. In order to study the robustness of the SPRT, we have plotted OC and ASN curves in Fig. 5.1 and Fig. 5.2, respectively. It is evident from the figures that for $\phi_1 > 1$ (<1), the OC curve shifts to the left (right) of the curve corresponding to $\phi_1 = 1$, whereas the ASN function curve shifts below (above) to the curve corresponding to $\phi_1 = 1$. Both the curves are highly sensitive for changes in $\rho$.

Now we consider the problem of testing the hypothesis $H_0: \rho = 2$ versus $H_1: \rho = 5$, when $\gamma$ is known, taking $\alpha = \beta = 0.05$. For different values of $\rho$ and $\phi_2 = 1, 1.05$ and $0.97$, the real root ‘t’ are obtained from (5.3.13) through iterative process and are used to compute OC and ASN functions. In order to study the robustness of the SPRT, we have plotted OC and ASN curves in Fig. 5.3 and Fig. 5.4, respectively. It is evident from the figures that for $\phi_2 > 1$ (<1), the OC curve shifts to the right (left) of the curve corresponding to $\phi_2 = 1$, whereas the ASN function curve shifts above (below) to the curve corresponding to $\phi_2 = 1$. Although, both the curves are sensitive to changes in $\gamma$ but the sensitivity is comparatively less for the case when $\phi_2 < 1$.

Finally, we consider the problem of testing the hypotheses for $\rho$, when $\gamma$ is unknown. We consider $H_0: \rho = 10$ versus $H_1: \rho = 15$ taking $\alpha = \beta = 0.05$. For different values of $\rho$, the real roots of $t = t(\rho)$ are obtained by solving $\lambda(t) = 1$ [from (5.4.15)] through iterative
procedures and these values are used to compute OC and ASN functions. The OC and ASN curves are presented in Fig. 5.5 and Fig. 5.6, respectively.

**Fig. 5.1: OC function curves for testing the hypothesis regarding $\gamma$ when $\rho$ is known**

**Fig. 5.2: ASN function curves for testing the hypothesis regarding $\gamma$ when $\rho$ is known**
Fig. 5.3: OC function curves for testing the hypothesis regarding $\rho$ when $\gamma$ is known

Fig. 5.4: ASN function curves for testing the hypothesis regarding $\rho$ when $\gamma$ is known
Fig. 5.5: OC function curves for testing the hypothesis regarding $\rho$ when $\gamma$ is unknown

Fig. 5.6: ASN function curves for testing the hypothesis regarding $\rho$ when $\gamma$ is unknown