II
Theoretical Underpinnings and Emerging Issues
An optimal portfolio is more than a long list of good stocks and bonds. It is a balanced whole providing an investor with protections and opportunities with respect to a wide range of contingencies (Markowitz, 1959). The investor would thus progress towards building an integrated portfolio suiting his needs. Portfolio analysis is based on criteria which serve as a guide to the selection of securities. The two most important objectives/criteria common to all investors are:

1. Investors prefer more return over less return.
2. Investors want the return to be stable, consistent, dependable and certain. They prefer certainty over uncertainty. Thus, investors aim at reducing the risk (variability of returns)⁶.

After eliminating all such portfolios which are clearly inferior to other available portfolios (in terms of return or risk) the investor is left with efficient portfolios. Efficient portfolios are the one yielding highest return for a given degree of risk, or providing least risk for a given level of return. The proper choice among efficient portfolios rests on the willingness and ability of an investor to assume risk. Investors often attempt to maximise the expected value of their utility function. They select that portfolio which provides the most suitable combination of risk and return.

The primary objective of this chapter is to discuss the conceptual and theoretical framework for mean-variance efficient portfolio selection and application of quadratic programming to portfolio selection decisions.

**II.1 Conceptual Framework of Mean-Variance Efficient Portfolio Selection**

The Mean- Variance Criterion is by far the most widely known efficiency criterion for investment analysis. A striking advantage of the mean-variance model is that the analyst confines himself to the first two distribution moments – the expected mean returns and the variance of each option being considered. The mean-variance efficiency criterion may be defined as follows:

An security ‘a’ dominates (is preferred to) another security ‘b’, by the mean-variance criterion, if and only if
\[ \mu_a \geq \mu_b \]  
\[ \sigma_a^2 \leq \sigma_b^2 \]

on the condition that at least one of the important inequalities are held. This means that security ‘a’ dominates security ‘b’ if its expected return is greater than (or equal to) that of b, while its variance is smaller than (or equal to) the variance of b.

The mean-variance criterion can also be defined in two other equivalent forms

\[ \mu_a \geq \mu_b; \sigma_a^2 < \sigma_b^2 \]  
\[ \mu_a > \mu_b; \sigma_a^2 \leq \sigma_b^2 \]

These alternative forms have the first necessary condition and the following clear cut requirements. The mean-variance approach provides an optimal criterion which ensures a minimal efficient group when all of the alternative cumulative return distributions intersect.

Far back in the eighteenth century Bernoulli and Cramer reached the conclusion that decisions under conditions of uncertainty could not be made only on the basis of expected (mean) returns. Subsequently, various economists tried to evaluate investments using two indicators based on the distribution of returns. One index reflects the profitability of an investment while the other is based on the dispersion of the distribution of returns and reflects the portfolio risk. The most prevalent profitability measure is the expected (mean) returns of the probability distribution of returns and dispersion is measured through variance of this distribution.

Markowitz (1952) developed a mean-variance model for the selection of portfolios. His analysis blends elegance and simplicity. It offers an intuitive explanation for diversification and a relatively simple computational procedure. Investors desire high returns but are averse to a high variance. After the introduction of Markowitz’s model, numerous portfolio selection models have been developed as an extension and improvisation of the standard mean-variance model. Some models minimised semi variance while some researchers added skewness as an additional variable to portfolio selection models.

Markowitz (1952) argued that expected returns as a criterion alone would lead to absurd results with almost no diversification. Mean and Variance were suggested as
measures of return and risk of the portfolio as a whole. It was argued that minimisation of variance for given expected returns would lead to the right kind of diversification.

Tobin (1958) based his theory of investment choice under conditions of uncertainty on the mean and variance of the distribution of returns. He showed that the mean-variance model is consistent with the von Neumann-Morgenstern postulates of rational behaviour if the utility of wealth is quadratic. He provided two alternate bases for assuming mean and variance as investor’s criteria (1) “the investor evaluates the future of consols only in terms of some two-parameter family of probability distributions” such as a normal distribution or (2) “the assumption that the utility function is quadratic” and returns do not exceed the point at which the quadratic reaches its maximum.

The advantages of using the mean-variance analysis as pointed out by Markowitz are:

a. Computation: It is more economical to trace out a mean-variance efficient set than to maximise the expected utility $E[U(R)]$.

b. Ease to determine investor’s utility function: The modern investing institution with quantitative skills can easily solicit the investor’s preferences among various gambles and summarise them into utility functions.

The Markowitz-Tobin analysis remains the cornerstone of the work in the field of investment analysis.

An approximation of a utility function by a quadratic in a certain neighbourhood is central to the Markowitz (1959) rationale for mean and variance. The justification of mean and variance is an application to portfolio selection of the results and methods of Von Neumann and Morgenstern (1944), L.J. Savage (1954) and R Bellman (1957).

The actions of a rational investor who does not know the future with certainty but has computational and intellectual capacity to act consistently with certain axioms of behaviour are considered. While the rational investor is assumed to be able to compute at essentially zero cost and unlimited speed, it was found his actions follow quite simple rules. The simplicity of these rules makes it easier to approximate portfolio selection actions for those mythical rational investors.

An investor holds, as possible, only a finite number of hypotheses about the nature of the world. At the beginning of each period $t$ the investor splits wealth $W_t$
between consumption $C_t$ and investment $I_t$. The investor allocates $I_t$ to various securities $X_1, X_2, ..., X_n$. The returns of various securities $r_1, r_2, ..., r_n$ determine the next period’s wealth $W_{t+1} = \sum_{i=1}^{n} r_i X_i$. This wealth is again split $W_{t+1} = C_{t+1} + I_{t+1}$. This process is repeated. A rational investor’s objective is to obtain a good sequence of consumption during his lifetime.

Four axioms and a theorem consistent with them were developed by Markowitz (1959) altering Von Neumann and Morgenstern (1944) strategy. Bayesian Interface implied single period utility maximisation and quadratic approximations were used to prove the superiority of mean and variance criteria. Whenever the individual’s utility function can be reasonably approximated by a (concave) quadratic, one of the portfolios which minimises variance for some value of expected return provides almost the maximum obtainable expected utility. Every investor has his own utility function. Each examines the curve showing efficient combinations of mean and variance. Each selects the portfolio most appropriate to his own needs. Each obtains a portfolio as good as the very best available for him.

Young and Trent (1969) conclude that “empirical evidence indicates that even though a number of the monthly and annual distributions deviate significantly from normality, the approximation involving only the mean and variance produces quite accurate estimates of the geometric means of these distributions.”

Samuelson (1970) and Ohlson (1975) presented conditions under which mean and variance are asymptotically sufficient for optimum decisions as the length of holding periods/ intervals between portfolio revisions approaches zero. Levy and Markowitz (1979) compared actual expected utility and mean-variance approximations for various utility functions, historical distributions and mean-variance approximations. The mean-variance criterion $f_e (E, V)$ was found to be as good as any of the other approximations. For most of the utility functions, the mean-variance approximation did quite well, especially for annual returns on diversified portfolios and for monthly returns even on undiversified portfolio.

Dexter, Yu and Ziemba (1980) concluded that decision based on mean-variance approximation yield expected utility values which were virtually identical to those obtained from the true optimal solution. Pulley (1983) found that investors maximising
expected logarithmic utility functions would hold virtually the same portfolio as an investor maximising certain mean-variance function. Also, the goodness of mean-variance approximations were found to be robust for different holding periods, different ratios of non-invested to invested wealth and different subjective return distributions. Investors could achieve virtually identical results by maximising functions of mean and variance alone.

Kroll, Levy and Markowitz (1984) found that for various utility functions and the historical returns for three different set of securities, when a portfolio is chosen from any of the infinite number of portfolios from the standard constraint set, the best mean-variance efficient portfolio has almost the maximum obtainable expected utility.

Simaan (1986) evaluated the goodness of the mean-variance approximation by the amount which an investor will be willing to pay for obtaining an optimum portfolio. The results showed a larger domain for which mean-variance strategies led to a fairly good approximate choice of expected utility maximisation. Ederington (1986) also considered whether the goodness of mean-variance approximation was due to near quadraticness of the utility function (U) or near normality of probability distributions. It was found that \( f_e (E,V) \) is mean-variance approximation if (U) is nearly quadratic, whereas another approximation \( f_n (E,V) \) is approximate if the distributions are normal.

The universal conclusion of these studies is that mean-variance approximations provide almost maximum expected utility. The mean-variance analysis can be justified by the assumptions concerning the joint distribution of security returns. Chamberlain (1988) shows that if the joint distribution of returns is “spherically symmetrical” then the probability distribution of a portfolio is known once its mean and variance are known. Joint normal is a special case of spherically symmetrical distribution. Ross (1978) showed that even a broader class of joint distributions implied that the mean-variance capital asset pricing models are precisely right if \( \sum_{i=1}^n X_i = 1 \) is the only constraint. If the form of joint distributions of return is such as to justify mean-variance analysis, then argument based on quadratic approximation is not needed. Alternatively, if the joint distributions are not of the required form, then the quadratic approximation argument applies.
Merton (1973, 1992) developed an analog of the continuous time mean-variance analysis. He developed portfolio theory assuming that the investor could trade continuously in time. His model provided the analyses with means and variances of returns on the derivative securities, and co-variances between each of these and each of the other securities of the analysis.

II.1.1 Assumptions of Mean-Variance Criterion

The use of mean-variance criterion to characterize investor’s behaviour implies a set of assumptions:

1. Investors in the securities market are risk-averse. This implies that the investors have concave (to y-axis) utility functions.
2. The distribution of returns approximates a normal distribution. Investors in a competitive capital market tend to include a large number of individual securities in their portfolio. This makes the portfolio returns to normalise.

II.1.2 Utility Foundations of Mean-Variance Criterion

Efficiency criteria are based on underlying assumptions regarding investor’s preferences. The restrictions on the shape of investor’s utility functions and the shape of distribution of returns make it a more efficient criterion in reducing the size of the efficient set. It also makes the criterion appropriate for a smaller class of investors. The main assumptions about utility function of the investor in case of mean-variance criterion are:

1. Every investor always prefers more wealth to less wealth i.e. $U'(x) \geq 0$.
2. The marginal utility of money declines i.e. $U''(x) \leq 0$.
3. Quadratic Utility functions: The utility function is assumed to be of the specific form.
   \[ U(x) = a + bx + cx^2 \]  \hspace{1cm} (2.5)
4. Conclave utility functions: This assumption indicated general risk aversion. The efficient set generated under this assumption is relevant for a broad class of risk averse investors. When risk aversion is assumed the random variables must belong to the same family of distributions with two parameters, each of which is an independent function of the mean and the variance. This will be true for a normal distribution of returns.
II.1.3 Graphical Representation of the Mean-Variance Criterion

The mean-variance criterion by its very nature is two-dimensional. The efficiency analysis using this criterion can be illustrated on a graphical plane.

![Efficiency Frontier Diagram](image)

**Figure II.1 Markowitz's Efficient Frontier**

The horizontal axis denotes risk/variance while the vertical axis represents expected returns. Only the options comprising the segment AB can be regarded as efficient. The remaining portion of the envelope curve and all the options in the interior of the circle are inefficient. The locus of efficient points AB is referred as efficiency frontier in figure II.1.

![Indifference Curves Diagram](image)

**Figure II.2 Indifference Curves of the Investor (Diversifier)**

An investor increases his utility when expected return is increased or variance is decreased. Indifference curves (IC) of the investors are introduced. The utility of an investor is increased by an upward movement of the IC, with the option of higher returns or a leftward movement of the IC, with smaller variance. The indifference
curves are drawn convex to the origin on the assumption that additional increments of variance require increasingly larger increments of expected return to compensate the individual. Also, the Indifference curves never intersect each other as in figure II.2.

![Figure II.3 Optimal Portfolio Selection](image)


An investor’s portfolio of securities depends upon his tastes. He will select the option which allows him to reach the highest indifference curve and thus the maximum utility. By superimposing the indifference map on the opportunity set of investments the graphical representation of the mean-variance model is possible. This is shown in figure II.3. Depending upon the slope of the indifference curve any option on the efficient locus AB may be the optimal investment of a particular investor.

II.2 Theoretical Mean-Variance Efficient Portfolio Selection Models

II.2.1 The Standard Mean-Variance Portfolio Selection Model

In the standard portfolio selection model, an investor has to choose a fraction $X_1, X_2, \ldots, X_n$ invested in $n$ securities subject to constraints

$$\sum_{i=1}^{n} X_i = 1 \quad (2.6)$$

$$X_i \geq 0 \quad i=1,\ldots,n \quad (2.7)$$

The returns on individual securities are assumed to be jointly distributed random returns. The expected (mean) return on the portfolio is

$$E = \sum_{i=1}^{n} X_i \mu_i \quad (2.8)$$

where, $\mu_i$ represents the return on a security.
The variance of return $V$ on the portfolio is

\[ V = \sum_{i=1}^{n} \sum_{j=1}^{n} X_i X_j \sigma_{ij} \]  

(2.9)

where,

\[ \sigma_{ij} = E[(r_i - \mu_i)(r_j - \mu_j)] \]  

(2.10)

$\sigma_{ij}$ is the covariance between $r_i$ and $r_j$.

The portfolio $X_1, X_2, ..., X_n$ which meets the two constraints (equation 2.6 and 2.7) is the feasible solution of the standard model. It is regarded as obtainable/legitimate portfolio. An obtainable EV combination is inefficient if another obtainable combination has either higher mean and no higher variance, or less variance and no less mean return. Efficient EV combinations are the ones which are not inefficient.

II.2.2 Standard Analysis with Upper Bounds

The maximum amount to be invested in any one security is sometimes restricted by law or by some policy measure for some institutional players such as mutual funds. Allowing for this restriction for achieving a diversified portfolio then equation 2.11 is added to the constraint set:

\[ 0 \leq X_i \leq U_i \quad i=1,\ldots,n \]  

(2.11)

The upper bound constraint will be redundant and feasible portfolio will remain unaffected if the upper bound restriction is more than the amount available for investment for all the securities. Then the obtainable set is unaffected. It will also be unaffected if some upper bound restrictions are less than amount available for investment, and the corresponding weight of security in the solution are less than the upper bound constraint. As the upper bound limit is decreased the set of obtainable Mean-Variance efficient combinations will eventually shrink. A model becomes infeasible if no portfolio can meet its constraints.

II.2.3 The Tobin-Sharpe-Lintner Model

Portfolios are chosen subject to the following additional constraints:

\[ \sum_{i=1}^{n} X_i = 1 + X_{n+1} \]  

(2.12)

\[ X_{n+1} \geq -1 \]  

(2.13)

where, $U_{n+1}$ is the upper bound on the amount borrowed.
In the analyses of Tobin (1958), Sharpe (1964), and Lintner (1965) the variable referred as $X_{n+1}$ is the amount borrowed (if positive) or amount lent (if negative). The rate of return received (with certainty) by the investor on money borrowed/lent is the risk-free rate and is denoted by $r_0$.

Limited borrowing can be modelled by adding the following constraint

$$X_{n+1} \leq U_{n+1}$$  \hspace{1cm} (2.14)

II.2.4 Black’s Model

Portfolios are subjected to the following constraint in Black’s Model:

$$\sum_{i=1}^{n} X_i = 1$$  \hspace{1cm} (2.15)

The non negativity constraint is omitted. Negative $X_i$ are short positions in the $i^{th}$ security and positive $X_i$ are long positions. Black’s model is a typical set of obtainable EV combinations with the feasible set as two halves of the same parabola.

II.2.5 Recent developments in Mean-Variance Portfolio Selection Models

Alexander and Baptista (2002) related Value at Risk (VaR) to mean-variance analysis and examined the economic implications arising from mean-VaR framework. As an approximation, the use of mean-VaR criteria is found to be consistent with expected utility maximisation framework. Economic implications indicate that the risk exposure of a highly risk-averse agent, as measured by standard deviation, increases when VaR is used as the relevant measure of risk.

$$\min_{\text{well}} \{ \sigma[r_w] - E[r_w] \}$$ \hspace{1cm} (2.16)

$$\text{s.t.} E[r_w] = \bar{E}$$ \hspace{1cm} (2.17)

Zhou and Li (2000) formulated a continuous-time mean-variance portfolio selection model as a bi-criteria optimisation problem. The objective was to maximise the expected terminal return and minimise the variance of the terminal wealth. The original problem is embedded into a stochastic Linear-Quadratic (LQ) control problem.

$$\text{Minimise} \ (J_1(u(.)),J_2(u(.))) \equiv (-Ex(T),Var x(T))$$ \hspace{1cm} (2.18)

$$\text{subject to} \ \{ u(.) \in L_{r}^2 (0,T;R^n), (x(.),u(.)) \}$$ \hspace{1cm} (2.19)

The following tractable stochastic LQ problem was proposed,
\[ \text{Minimise } J(u(\cdot); \mu, \lambda) = E \{ \mu x(T)^2 - \lambda x(T) \} \]  
(2.20)  

subject to \( u(\cdot) \in L^2_u(0,T; R^m), x_\cdot, u(\cdot) \)  
(2.21)

Ogryczak (2000) developed a multiple criteria linear programming model for portfolio selection. The model is based on the preference axioms for choice under risk. It is shown that the classical mean-risk approaches solved using linear programming models correspond to specific solution techniques applied to this multiple criteria model.

\[
\max (z_1, z_2, \ldots, z_m)  
\text{subject to } x \in Q, \quad (2.22)
\]

\[
y_i = r_i x \text{ for } i = 1, 2, \ldots, m, \quad (2.23)
\]

\[
z_i \leq \sum_{k=1}^i y^{\tau(k)}_{\tau(i)}, \text{ for } \tau \in \prod_i, i = 1, 2, \ldots, m \quad (2.24)
\]

\[
z_i \leq \sum_{k=1}^i y^{\tau(k)}_{\tau(i)}, \text{ for } \tau \in \prod_i, i = 1, 2, \ldots, m \quad (2.25)
\]

Steinbach (2001) described the interplay between objective and constraints in a number of single-period variants, including semi variance models. Theoretical analysis of multi period models based on scenario trees is also presented.

\[
\max \mu \rho(x) - \frac{1}{2} R(x)  
\text{subject to } e^T x = 1 \quad (2.27)
\]

Another variant of the portfolio selection problem where the investor’s goal is split between objective and reward condition was also presented.

\[
\min \frac{1}{2} R(x)  
\text{subject to } e^T x = 1, \quad (2.29)
\]

\[
\rho(x) = \rho \quad (2.31)
\]

Zhao and Ziemb (2001) have derived the global mean-variance efficient frontier and optimal portfolio policies for dynamic investments. The objective function is equal to the variance of the portfolio value for a given mean return \( R \). The efficient
frontier is uniquely determined by the mean and the standard deviation of the contingent state price.

\[ V^2 = \min_{\tilde{R}} E[\tilde{R}^2] - R^2 \]  
\[ \text{s.t. } E[\tilde{R}^2] = R \]  
\[ E[\xi \tilde{R}] = 1 \]

Li, Qin and Kar (2010) extended the work of Liu and gave variations of mean-variance model in uncertain environment. The model has been formulated as a single objective programming model as follows,

\[ \max_{\xi} \left[ \sum_{i=1}^{n} x_i \xi_i + x_2 \xi_2 + \ldots + x_n \xi_n \right] - \lambda \left[ \sum_{i=1}^{n} (\tilde{R}_i - R_i)x_i \right] \]  
\[ \text{s.t. } x_1 + x_2 + \ldots + x_n = 1 \]  
\[ x_i \geq 0 \text{ for } i = 1, 2, \ldots, n \]

where, \( \lambda \) represents the degree of risk aversion of the investors.

Lai, Wang, Xu, Zhu and Fang (2002) proposed a model for portfolio selection based on the semi absolute deviation measure of risk, which can be converted into a linear interval programming model. The uncertain returns of assets in capital markets are considered as intervals.

\[ W_j(x) = \min \left\{ 0, \sum_{j=1}^{n} (r_{ij} - R_j)x_j \right\} \]  

Lim and Zhou (2002) present a continuous-time, mean-variance portfolio selection problem in a complete market with random interest rate, appreciation rates, and volatility coefficients. Recent developments in stochastic linear-quadratic (LQ) optimal control and backward stochastic differential equations (BSDEs) have been used to tackle the portfolio selection problem.

\[ \min J_{MVP}(u(.)) := E \left[ \frac{1}{2} (x(T) - d)^2 \right], \]  
\[ \text{subject to :} \]  
\[ Ex(T) = d, \]  
\[ (x(.), u(.)) \]

The mean-variance efficient frontier for this problem is a perfect square, indicating that risk-free investments are still possible when interest rates are random.
Goldfarb and Iyengar (2003) provided a robust analog of the Markowitz’s mean-variance optimisation problem. The objective function was to minimise the worst case variance of the portfolio subject to the constraint that the worst case expected return in the portfolio is at least $\alpha$.

$$\min \max_{(\phi \in \Phi, \phi \in \bar{\Phi})} Var[\phi],$$  \hspace{1cm} (2.41)

$$\text{subject to } \min \ E[\phi] \geq \alpha,$$  \hspace{1cm} (2.42)

$$1^T \phi = 1$$  \hspace{1cm} (2.43)

Robust portfolios were found to perform 200% better than the classical portfolios when tested on real life data.

Yin and Zhou (2003) study the discrete-time version of Markowitz’s mean-variance portfolio selection problem where the market parameters depend on the market model that jumps among a finite number of states. A discrete-time Markov modulated portfolio selection model is presented. An aggregated process with smaller state space is introduced and the underlying portfolio selection is formulated as a two-time-scale problem.

$$\text{Minimise } J^\epsilon(x_0, \ell_0, u^\epsilon) = E[x_{T/\epsilon}^\epsilon - z]^2$$  \hspace{1cm} (2.44)

$$\text{subject to: } x_0^\epsilon = x_0, \alpha_0^\epsilon = \ell_0, E x_{T/\epsilon}^\epsilon = z$$  \hspace{1cm} (2.45)

$$\text{and } (x^\epsilon, u^\epsilon) \in A^\epsilon$$  \hspace{1cm} (2.46)

The transition matrix of the discrete-time Markov chain $\alpha^\epsilon_\delta$ is given by

$$P^\epsilon = P + \epsilon Q,$$  \hspace{1cm} (2.47)

$$P = \text{diag}(P^1, \ldots, P^\epsilon)$$  \hspace{1cm} (2.48)

Markowitz’s work on mean-variance efficient portfolio selection has ever since been extended by many researchers all over the world. Mean-Variance efficient portfolios have resulted in maximising the expected utility (quadratic) of the investor. This forms the reason for taking up this research endeavour. The application of bicriteria linear programming, multiple criteria linear programming, stochastic linear quadratic optimal controls, backward stochastic equations, Markov chain process and the like on the mean-variance portfolio has been witnessed.
On the basis of the foregoing discussion it may be derived that in the current capital market situation with abundant data availability and latest computational techniques, there is a need to develop a portfolio selection model analysing efficiency across multiple financial variables. A portfolio selection model that can realistically incorporate the multiple goals and constraints of today’s investor needs to be devised. The vast literature available in the field of portfolio optimisation can pave way in understanding the dynamics of investor’s behaviour. In the present state of emerging stock market activity, ever increasing aspirations of the investors and their numerous limitations, an effort needs to be made to model them into a Linear-Quadratic Programming framework.

II.3 Quadratic Programming Model: Theoretical Framework

The earliest linear programming was developed by Leonid Kantorovich in 1939 for use during World War II to plan expenditures and returns in order to reduce costs to the army and increase losses to the enemy. The method was kept secret until 1947 when George B. Dantzig published the simplex method and John von Neumann developed the theory of duality as a linear optimization solution, and applied it in the field of game theory.

A special case of nonlinear programming problem, quadratic programming includes the case where (1) the objective function is a concave quadratic function and (2) the constraints are all linear. The formulation of Kuhn-Tucker conditions results in a set of linear expressions for the solution. This set of linear expressions makes the solution by simplex method also possible.

Quadratic Programming is the optimisation of a quadratic function subject to linear equality and inequality constraints. It arises in least squares problems under constraints. It also arises in case of multiple objective decisions making where the deviation of actual decisions from ideal values is evaluated using quadratic distance measure.

Quadratic Programming (QP) has been widely used in formulation of mean-variance optimisation of investment decisions in an uncertain environment. The constraints are linear and the objective function is quadratic. An investor reconciles the conflicting desires of maximising expected portfolio return, represented by linear
portfolio return term, and minimising the portfolio risk, represented by the quadratic portfolio variance term, in the objective function.

The QP multi-objective problem (for \( v, u \in \mathbb{R}^n, < v, u > = \sum_{i=1}^{n} v^i u^i \)):

\[
\min \left\{ < x - x^d, Q(x - x^d) > | H^T x \leq h \right\},
\]

\( x^d \) is the desired value of \( x \in \mathbb{R}^n, Q \) is a symmetric positive definite matrix penalizing the deviations of \( x \) from \( x^d \). The matrix \( H \in \mathbb{R}^{n \times \iota} \) and the column vector \( h \in \mathbb{R}^\iota \) represent the linear inequality constraints with

\[
H = [\varphi^1 \varphi^2 \ldots \varphi^\iota],
\]

where, \( \varphi^i \) is the \( n \)-dimensional column vector of coefficients for the \( i^{th} \) constraint.

Investment decision making under uncertainty may be formulated as the simultaneous optimisation of the expected (mean value) of the total return and the associated risk, measured as the variance of the return. A trade-off between maximisation of return and minimisation of risk exists. The more risk-seeking an investor, greater will be the emphasis on expected return. The more risk-averse and cautious an investor, the more important is minimisation of variance of returns. The classical quadratic mean-variance portfolio optimisation approach to portfolio selection (Markowitz, 1959; Sharpe, 1970 and Elton and Gruber, 1992) laid the foundation for this research work.

The return vector \( r \in \mathbb{R}^n \) on \( n \) investments is considered.

\[
r = \epsilon[r] + \rho
\]

where \( \epsilon \) denotes expectation,

\[
\rho \sim N(0, \Lambda_{\rho^2}, \Lambda_{\rho^2} = \epsilon[(\rho)\rho^T]).
\]

Let \( \omega \in \mathbb{R}^n \) denote the weights, in terms of proportion of total amount to be allocated to each investment. The expected return on the portfolio return may be calculated as

\[
\epsilon[< \omega, r >] = \epsilon[< \omega, \epsilon[r] + \rho >] = < \epsilon[r], \omega >.
\]

\[
P = \text{diag}(P^1, \ldots, P^\iota)
\]

Risk is measured as the statistical variance of the portfolio return calculated as
\[ \text{var} < \omega, r > = \varepsilon[< \omega, r > - \varepsilon[< \omega, r >]]^2 \]
\[ = \varepsilon[< \omega, r >]^2 - 2 \varepsilon[< \omega, r > \varepsilon[< \omega, r >]] + [\varepsilon[< \omega, r >]]^2 \]
\[ = < \omega, \varepsilon[\rho \rho^T] \omega > . \quad (2.55) \]

Investor’s attitude to risk determines the importance attached to maximisation of portfolio return versus the minimisation of risk as measured by variance.

Let \( \alpha \in [0,1] \) be a fixed scalar and let \( < u, v > = u^T v \). An optimal mean-variance portfolio is given by the combination of the mean of the portfolio and its variance.

\[ \max \{ \alpha < \varepsilon[r], \omega > -(1-\alpha) < \omega, \rho, \omega > \mid <1, \omega > = 1, \omega \geq 0 \} \quad (2.56) \]

where, \( 1 \) is the vector with all unit elements. The portfolio return is maximised simultaneously with minimisation of risk associated with it. The total investment budget is restricted and the weights allocated to each investment as a proportion of the budget are non-negative.

**II.3.1 The Kuhn-Tucker Conditions**

Based upon the classical calculus method of solution the Kuhn-Tucker conditions provide the basis for recognizing candidates for an optimal solution to a non-linear programming problem. The Kuhn-Tucker conditions assume that the nonlinear problem has been formulated as a Lagrange multiplier expression, and differentiated with respect to \( x_j (j = 1, 2, \ldots, n) \) and \( \lambda_i (i = 1, 2, \ldots, m) \). These conditions provide only a test for optimality and not a procedure for finding out the solution. If the general problem includes two decision variables and two constraints:

\[ y = (x_1, x_2) \quad (2.57) \]

Subject to

\[ g_1(x_1, x_2) \leq b_1 \quad (2.58) \]
\[ g_2(x_1, x_2) \leq b_2 \quad (2.59) \]

and

\[ x_1, x_2 \geq 0 \quad (2.60) \]

the Lagrangian expression is

\[ L = f(x_1, x_2) - \lambda_1 [g_1(x_1, x_2) - b_1] - \lambda_2 [g_2(x_1, x_2) - b_2] \quad (2.61) \]

The classical approach to solution of the Lagrange multiplier expression requires solving the following equations simultaneously,
The Kuhn-Tucker conditions, which must be satisfied to yield candidates for an optimal solution, are:

Given:

\[
\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j}, \quad j = 1, 2, \ldots, n
\]  

(2.63)

then:

\[
x^*_j \left( \frac{\partial L}{\partial x_j} \right) = 0, \quad j = 1, 2, \ldots, n
\]  

(2.64)

\[
\frac{\partial L}{\partial x_j} \leq 0, \quad j = 1, 2, \ldots, n
\]  

(2.65)

\[
\lambda_i [g_i(x^*_1, x^*_2, \ldots, x^*_n) - b_i] = 0, \quad i = 1, 2, \ldots, m
\]  

(2.66)

\[
g_i(x^*_1, x^*_2, \ldots, x^*_n) - b_i \leq 0, \quad i = 1, 2, \ldots, m
\]  

(2.67)

\[
x^*_j \geq 0, \quad \lambda_i \geq 0
\]  

(2.68)

The preceding conditions are interpreted by the following restatements:

If \( x^*_j > 0 \), then \( \frac{\partial L}{\partial x_j} = 0, \quad j = 1, 2, \ldots, n \)  

(2.69)

If \( x^*_j = 0 \), then \( \frac{\partial L}{\partial x_j} \leq 0, \quad j = 1, 2, \ldots, n \)  

(2.70)

If \( \lambda_i > 0 \), then \( g_i(x^*_1, x^*_2, \ldots, x^*_n) - b_i = 0, \quad i = 1, 2, \ldots, m \)  

(2.71)

If \( \lambda_i = 0 \), then \( g_i(x^*_1, x^*_2, \ldots, x^*_n) - b_i \leq 0, \quad i = 1, 2, \ldots, m \)  

(2.72)

and all \( x^*_j \geq 0, \lambda_i \geq 0 \)  

(2.73)

Thus the Kuhn-Tucker conditions are defined for each constraint \( i \), by

(i) Binding constraint (boundary point optimization)

If \( \lambda_i > 0 \), then \( g_i - b_i = 0 \), i.e., \( g_i = b_i \), for \( (x^*_1, x^*_2, \ldots, x^*_n) \).  

(2.74)

(ii) Non-binding constraint (interior point optimization)

If \( \lambda_i = 0 \), then \( g_i - b_i \leq 0 \), i.e., \( g_i \leq b_i \), for \( (x^*_1, x^*_2, \ldots, x^*_n) \).  

(2.75)

The satisfaction of Kuhn-Tucker conditions only yields the acceptable candidates for an optimal solution. These candidates must further be evaluated to determine the global optimal solution.

If \( f(x_1, x_2, \ldots, x_n) \) is a concave function and if \( g_i(x_1, x_2, \ldots, x_n) \) for \( i = 1, 2, \ldots, m \), are convex functions then the results obtained by employing the Lagrangian function,
which satisfy the Kuhn-Tucker conditions, will result in an optimal solution. The application of this extension of Kuhn-Tucker theorem applies to quadratic programming. If the objective function is known to be a quadratic cone function, subject to all linear constraints (yielding a convex feasible solution space), then satisfaction of Kuhn-Tucker conditions will yield an optimal solution. This corollary was presented and proven by Kuhn and Tucker.

A quadratic programming problem may be stated as

\[
\begin{align*}
\text{Maximise} \quad & y = f(x_1, x_2) \\
\text{subject to} \quad & g_1(x_1, x_2) \leq b_1 \\
& x_1 \geq 0, x_2 \geq 0 \Rightarrow -x_1 \leq 0, -x_2 \leq 0
\end{align*}
\]

The problem may be restated as

\[
\begin{align*}
\text{Maximise} \quad & y = f(x_1, x_2) \\
\text{subject to} \quad & g_1(x_1, x_2) - b_1 \leq 0 \\
& h_1(x_1) \leq 0 \quad \text{where} \quad h_1(x_1) = -x_1 \\
& h_2(x_2) \leq 0 \quad \text{where} \quad h_2(x_2) = -x_2.
\end{align*}
\]

The Lagrangian expression is given as:

\[
L = f(x_1, x_2) - \lambda_1 [g_1(x_1, x_2) - b_1] - \mu_1 [h_1(x_1)] - \mu_2 [h_2(x_2)]
\]

where,

\[
\lambda_1 = \text{the Lagrange multiplier for } g_1, \\
\mu_1 = \text{the Lagrange multiplier for } h_1 \text{ (the non negativity requirement for } x_1), \\
\mu_2 = \text{the Lagrange multiplier for } h_2 \text{ (the non negativity requirement for } x_2)
\]

The Kuhn-Tucker conditions may thus be presented as:

\[
\begin{align*}
\frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial g_1}{\partial x_1} + \mu_1 = 0 \\
\frac{\partial L}{\partial x_2} = \frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial g_1}{\partial x_2} + \mu_2 = 0 \\
g_1 - b_1 \leq 0 \\
\lambda_1 (g_1 - b_1) = 0 \\
\mu_1 x_1 = 0 \\
\mu_2 x_2 = 0 \\
x_1, x_2, \lambda_1, \mu_1, \mu_2 \geq 0
\end{align*}
\]
The inequality can further be converted to equality by introducing a slack variable. In general, the Kuhn-Tucker conditions for quadratic programming problem including \( n \) decision variables and \( m \) constraints are given as follows:

\[
\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j} + \mu_j = 0, \quad j = 1, 2, \ldots, n \quad (2.91)
\]

\[
g_i - b_j + s_i = 0, \quad i = 1, 2, \ldots, m \quad (2.92)
\]

\[
\lambda_i s_i = 0, \quad i = 1, 2, \ldots, m \quad (2.93)
\]

\[
\mu_j x_j = 0, \quad j = 1, 2, \ldots, n \quad (2.94)
\]

\[
all \ x_j, \lambda_i, \mu_j, s_i \geq 0 \quad (2.95)
\]

This general problem can be solved using the simplex procedure by including the linear equations in the simplex tableau. The final requirement to form an initial basic solution is to add artificial variables to all the linear expressions. These equations can then be solved in terms of the artificial variables to form the objective function which is to be minimised.

**II.4 Emerging Issues and Challenges in Indian Equity Markets**

Over the last few years, there has been a rapid change in the Indian securities market, more so in the secondary market. Advanced technology and online-based transactions have modernized the stock exchanges. In terms of the number of companies listed and total market capitalization, the Indian equity market is considered large relative to the country’s stage of economic development.

The crises of recent years in the midst of the worst financial catastrophe of recent decades marked by massive credit failures, banks and brokerage meltdowns and government bailouts have impacted the Capital markets immensely. Since the beginning of 2011, the two adverse developments were witnessed in the advanced economies: (1) a much slower rate of recovery than expected, and (2) an increase in fiscal and financial risks. The data reflects a sluggish growth in the advanced economies. The world economy was hit by a number of shocks. Japan was shaken by the devastating earthquake and tsunami. Political turmoil was witnessed in the oil producing nations. The growth in private demand has got stalled. Downside risks have increased with fiscal uncertainty, housing market weakness, renewed financial stress and subdued business and consumer sentiment.
These global developments undermined the prospects of self-sustaining recovery in India. The sovereign debt crisis and prolonged slowdown in European Union nations and United States has impacted India’s growth prospects. Gross Domestic Product (GDP) growth, declining index of industrial production (IIP), rising inflation, fiscal management, management of external sector transactions, depreciation of Indian rupee are some important issues raised in the backdrop of the global slowdown.

The effects are more pronounced on India’s financial markets particularly securities market than on real economy. The resources mobilised through public and rights issue in the primary market, market capitalisation to GDP ratio, average daily market turnover, Net Foreign Institutional Investor’s investment in debt and equity, all have fallen. The financial markets experienced higher than normal levels of volatility and uncertainty during 2011.

Securities and Exchange Board of India (SEBI) has concerns over the changes and improvements required in the market structure in the view of the fast pace of technological developments. It has formed Technical Advisory Committees (TAC’s) to frame appropriate policies arising out of technological advancements in the areas of wireless trading, co-location, algorithmic trading, smart order routing and application programming interface.

In the light of developments in the secondary market, measures for improving market safety, efficiency, transparency and integrity assume importance. The reduction of transaction costs, simplification and transparency in the systems and procedures of legal framework, quick and efficient handling of investor grievances and a strong regulatory framework especially for intermediaries and mutual funds are a few important areas requiring attention. Some of the emerging issues in the Indian capital market may be enumerated as below.

II.4.1 Risk Management

In the present day scenario the risk management framework for the cash and derivatives segment needs to be reviewed. Changes in the risk management/margin system must be incorporated. The regulatory framework for risk management for the cash and derivative segment requires an in-depth analysis. Attempts must be made for reduction of the transmission of risk from other segments. Investor protection measures related to risk management should also be reviewed.
II.4.2 Disclosures and Accounting Standards

The disclosure requirements in offer documents, application forms, advertisements or any other mode of mass communication are standardised keeping in view the protection of interests of the investor and improving the overall efficiency of the market. The continuous disclosure requirements pertaining to listing of equity or debt of an issuer are framed by SEBI. Also, the disclosure requirements of intermediaries registered with SEBI have been framed.

The continuous disclosure requirements of listed companies and the valuation methods and standard norms of intermediaries operating in the capital market need to be continuously reviewed in the light of changing market scenario. The operational and systemic risks in the primary securities market need to be addressed in these disclosures.

An effort is needed to ensure smooth implementation of accounting standards and statements of Institute of Chartered Accountants of India (ICAI) pertaining to disclosures in the capital market. Coordination between SEBI and ICAI by constituting study teams for providing inputs to the Accounting Standard Board (ASB) for evolving new accounting standards and reviewing the existing ones would be a positive step in improving the disclosure and accounting standards.

II.4.3 Investor Protection and Education

Investor protection and education activities directly undertaken by SEBI or through any agency to utilize the SEBI investor protection and education fund must be carefully monitored. Different target groups of investors at varied locations across the length and breadth of the country should be covered. Also, some method to evaluate the effectiveness of the investor education programme needs to be devised.

II.4.4 Wireless Trading and Co-location

Millions of Wi-Fi internet access points with broadband connectivity have allowed investors to sign on to their brokerage accounts on laptops and mobiles at different locations. This has resulted in lower costs of transactions. With real time data streaming in it has made the trading of stocks more flexible and convenient from the point of view of an investor. However, increased access requires improved security systems. For robustness of the system, creation of security tokens and passwords, use of smart cards, encryptions, biometric devices, improvements in cell networks and
enhancements in the end user devices is important. Systems where users are allowed to shut down user accounts when the devices are stolen must be created.

Extending the internet trading platform in May 2009, SEBI released a proposed framework for using wireless trading subject to security and encryption safeguards. This has provided a boost to equity trading in India. It has widened the scope of trading by giving an opportunity to the tech savvy young urban population of mobile subscribers in the equity markets. Measures for user identification, authentication and access control are important and require frequent upgradation in the changing computing environment.

In September 2010, Bombay Stock Exchange (BSE) was the first to launch mobile trading in India soon followed by National Stock Exchange (NSE). All the brokers providing Internet trading, who complied with the security norms issued by SEBI were allowed to provide wireless trading. The method of trading is new to the Indian Investor and still limited to a small population. However, it is still gaining pace and the security concerns need to be addressed as and when they arise providing alternate means in case of failure.

II.4.5 Algorithmic Trading and High Frequency Trading

Algorithmic trading is the use of computer programs and software to execute trades based on pre-defined criteria and without any human intervention. High Frequency trading is a subset of algorithmic trading which involves buying and selling of thousands of shares in fractions of seconds. Both algorithmic and high frequency trading exist in high volumes in India. The impact they impose on individual trade is rather very low. However, they are capable of causing large market movements. No real damage has been witnessed so far as a result of these swings, but a word of caution is required for the exchanges.

In March 2012, SEBI released broad guidelines to put a check on algorithmic trading programmes providing measures to check an excessive flood of orders and irregular price quotations. The regulator has advised exchanges to set up systems complying with latest guidelines for reining in algorithmic traders. The brokers would require prior exchange permission before offering algorithmic trading to their clients. Existing algorithmic traders would be vetted for risk management systems. To prevent order flooding and high number of orders as a proportion of actual executed trades, exchanges were asked to give economic disincentives. Algorithms would not be allowed to quote beyond a certain number of securities per order, or in violation of price
bands. A dummy filter would act as an early warning system to detect sudden surge in prices. Other such moves preserving market integrity have been included in SEBI guidelines which need to be incorporated by the Indian stock exchanges.

II.4.6 Smart Order Routing

In August 2010, SEBI allowed for introduction of Smart Order Routing allowing the brokers trading engines to systematically select the execution destination based on price, costs, speed, likelihood of execution and settlement, size, nature or some other consideration relevant to execution of the order. In 2011, smart order routing finally took off in India on the country’s two premier bourses after resolution of their long standing dispute over the audit trail of orders.

By using smart order routing technology, investors are able to obtain the best possible price while buying or selling shares, similar to what was being done manually by stock brokers. This technology makes this much faster to execute orders. Smart order routing determines which exchanges offer the best price at any given time. Speed is the key to the success of programme trading. If the price feed is not fast enough, the software will be unable to capitalise on the opportunities that last for a second or less. It helps in better price discovery and induces increased electronic trading volume.

Lack of interoperability between India’s two securities clearing houses remains an unresolved issue for smart order routing in the equities market. Clarification was given by the regulator on permission of smart order routing for all types of orders following confusion among the market participants. High clearing charges have negated any improvements from smart order routing. Many front-end trading systems used by buy-side trading desks are unable to split an order that is executed across two venues for confirmation purposes. Competition and interoperability already exists between Central Depository Services (India) and National Security Depository.

Some market participants have suggested interoperability agreements such as those by central counterparties in Europe while others are in favour of a single-central clearing organisation. Clearing the interoperability would allow trading firms to use capital more effectively and move out from the current artificial inefficiency situation. Interoperability arrangement would have to account for risk management implications. A central mechanism would be needed for interoperability to monitor potential margin breaches across both exchanges.
II.4.7 Minimum Public Shareholding

Securities and Exchange Board of India (SEBI) has been seriously promoting the idea of Minimum Public Shareholding ever since 2001 when it amended the Clause 40A of the Listing Agreement to provide for mandatory non-promoter shareholding. The same are mandatory under the Securities Contracts (Regulation) Rules, 1957 and Securities Contracts (Regulations) Act, 1956. Minimum public shareholding has been considered beneficial from the perspective of an investor as it ensures increased liquidity, lesser price manipulations, price discovery, low volatility, increased access to capital, enhanced corporate governance and better endorsement of the brand value. The most recent amendment came in 2010 which raised the minimum public shareholding to a uniform of 25% for all companies, listed and seeking to list. For Government companies and public sector companies this percentage has been kept as 10%. A lower public shareholding is justified, as it would prevent any large scale disinvestment by government companies which may distort the market. From the international perspective, countries like Singapore, Taiwan, UK, China, Hong Kong and Brazil require a minimum public float of 12% to 25%. The common routes for increasing minimum public shareholding include (1) Issuance of shares through an Initial Public Offer or Follow on Public Offer; (2) sale of shares by promoters in secondary market; (3) the Institutional Placement Programme\(^\text{13}\) (IPP); and (4) Offer for Sale\(^\text{14}\) (OFS) of shares by promoters. Qualified institutional placement, preferential allotment or issue of depositary receipts is not a valid method of increasing public shareholding.

A large number of companies who have not been able to fulfil these requirements are also planning to delist themselves. However, the delisting norms are quite harsh (expensive and cumbersome) on the company and its promoters and is being considered only as the last option. Despite all the effort by the regulator and amendments year after year (in 1993, 2001, 2005 and 2010), out of 4977 listed companies only 3525 have been able to comply with the regulations for minimum public shareholding. As many as 1259 listed companies have not submitted the shareholding information to the exchanges and the regulator.

Existing norms of Not more than 50% of the net offer to the public for Qualified Institutional Buyers\(^\text{15}\) (QIBs), Not less than 15% of the offer for Non-Institutional Bidders\(^\text{16}\) and Not less than 35% of the offer size for Retail Individual Bidders\(^\text{17}\) ensures greater depth and breadth in the securities market. These limits provide the upper bounds for institutional and non-institutional stock holding affecting
portfolio composition. However, it also limits the amount of investment a particular investor group can make.

The two issues which arise from the preceding discussion is that on one hand we have the regulator pushing listed companies to increase their public shareholding while on the other hand a large number of promoters of listed companies not willing to part with their stock holding. The issue becomes extremely important because if these 1259 companies actually comply with the 2013 deadline then it may result in supply of a large number of shares on NSE and BSE. Excess supply may result in fall in share prices of a large number of stocks. Another possible scenario could be that the deadline is extended. This would certainly dampen the spirits of those companies who have willingly complied with the SEBI regulations as the inefficient would be rewarded by this extension. The situation becomes complex in a scenario where a large number of investors are shying away from Indian Capital Markets in the light of downgrades by credit rating agencies, poor economic performance and policy bottlenecks at the national level. Even the two new measures introduced by SEBI of IPP and OFS have not yield the desired results. The emerging issues and challenges posed by the requirement of minimum public shareholding require an in depth research and analysis and would certainly affect the returns and volatility of Indian Capital Markets in future.

II.5 Summary and Conclusions

An attempt has been made to introduce the theoretical framework of Mean-Variance efficient portfolio selection and Quadratic Programming technique which can be used to accommodate plethora of constraints an investor faces in the real life situation. A case for relevance of quadratic programming technique for portfolio selection decisions has been discussed. Existing frameworks related to mean-variance portfolio optimisation using quadratic programming including the recent developments in the area has been presented. A general quadratic programming model which may be used to minimise risk and maximise return has been explained. The theoretical framework explained in this chapter will enable a reader to better understand the development and empirical testing of quadratic programming based mean-variance efficient portfolio selection model attempted in a constrained setting. Discussion of the Kuhn-Tucker conditions providing basis for recognising candidates for the optimal solution to the non linear problem and the Lagrange multiplier clarifies these concepts.
The section on emerging policy issues and challenges related to portfolio selection, raises questions related to risk management, disclosures and accounting standards, wireless trading, co-location, programming interface, investor protection and education, algorithmic trading and high frequency trading and smart order routing. Incorporation of changes in the risk management and margin system of the cash and derivatives segment to reduce the transmission of risk has been emphasised. Coordination between Securities and Exchange Board of India and Institute of Chartered Accountants of India by constituting study teams for providing inputs to the Accounting Standard Board (ASB) for evolving new accounting standards and reviewing the existing ones has been suggested for improving the disclosure and accounting standards.

The development of a method to assess the impact of investor education initiatives is stressed. A need for clearing the interoperability issues between the two exchanges is recommended. Incorporation of moves to preserve market integrity in case of algorithmic trading as suggested by the regulator must be done by the exchanges to promote algorithmic trading. In case of mobile trading, measures for user identification, authentication and access control are important and require frequent upgradation in the ever changing computing environment. The issue of mandatory minimum public shareholding of twenty five percent by the deadline in 2013 might lead to excess supply of securities in the secondary market or an extension of time to the 1259 non-complying companies. In both the scenarios, the image of Indian capital markets needs to be protected against falling investor confidence and downgrades by the credit rating agencies.

Low levels of Gross Domestic Product (GDP) growth, declining index of industrial production (IIP), rising inflation, fiscal management, management of external sector transactions and depreciation of Indian rupee are some important issues affecting Indian Capital markets. The effects are more pronounced on India’s financial markets-particularly securities market than on real economy. The resources mobilised through public and rights issue in the primary market, market-capitalisation to GDP ratio, average daily market turnover, Net Foreign Institutional Investor’s investment in debt and equity, all have fallen. The reduction of transaction costs, simplification and transparency in the systems and procedures of legal framework, quick and efficient handling of investor grievances and a strong regulatory framework especially for intermediaries and mutual funds are a few important areas requiring immediate attention.