CHAPTER 4

REGULARIZATION WITH APPROXIMATELY SPECIFIED OPERATORS

In this chapter we consider the problem of solving the operator equation $Tx = y$ approximately when the data $y, T$ are known only approximately. More precisely, we consider the regularization of $Tx = y$ with the help of the approximate data $y\delta, T_h$ where $\|y - y\delta\| \leq \delta$ and $\|T - T_h\| \leq \epsilon_h$, $\epsilon_h \to 0$ as $h \to 0$. The regularized equations and modified forms of the discrepancy principles (2.7), (3.6) and (3.13) are introduced in Section 4.1. The results corresponding to these discrepancy principles have been discussed in Sections 4.2, 4.3 and 4.4.

4.1. INTRODUCTION

We are concerned with the problem of solving the operator equation

$$(4.1) \quad Tx = y$$

approximately when the data $y, T$ are known only approximately. In reality there are two occasions, where one has to consider an approximately specified operator $T_h$ instead of $T$ (e.g., [6], [33], [36], [37], [44]). One such occasion arises from the modeling error and the other when one considers numerical approximation of $T$. 

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If \( y \) and \( T \) are not known exactly, but instead some approximations \( y_\delta \) and \( T_h \) are known, then a natural way to look for approximations to \( \hat{x} \), the minimal norm solution of (4.1), is to solve

\[
(T_h^*T_h + \alpha I)x_{\alpha,h} = T_h^*y_\delta
\]

instead of (2.2). Here \( \{T_h\}_{h>0} \) is a family of bounded linear operators between Hilbert spaces \( X \) and \( Y \). If \( \|y - y_\delta\| \leq \delta \) and \( \|T - T_h\| \leq \varepsilon_h \) with \( \varepsilon_h \geq 0 \) such that \( \varepsilon_h \to 0 \) as \( h \to 0 \), then one requires

\[
\|x - x_{\alpha,h}\| \to 0 \quad \text{as} \quad \alpha \to 0, \quad \delta \to 0 \quad \text{and} \quad h \to 0.
\]

But it can be shown that if \( R(T) \) is not closed and \( \varepsilon_h \to 0 \) as \( h \to 0 \), then for every \( h_0 > 0 \), \( \delta_0 > 0 \), the set

\[
\{ x_{\alpha,h} : \|y - y_\delta\| \leq \delta, \quad \|T - T_h\| \leq \varepsilon_h, \quad 0 < \delta \leq \delta_0, \quad 0 < h \leq h_0 \}
\]

is not bounded. Therefore it is important to choose the regularization parameter \( \alpha \) in dependence of the error level \( \delta \) and \( \varepsilon_h \) properly so as to satisfy (4.3). For this purpose we consider a class of discrepancy principles

\[
\|T_h x_{\alpha,h} - y_\delta\| = \frac{(\delta + \varepsilon_h)^p}{\alpha^q}, \quad p > 0, \quad q > 0
\]
to compute $α := α(δ, h)$. If $T_h = T$ and $ε_h = 0$, then the above discrepancy principle is reduced to (2.7) considered in Section 2.2.

Discrepancy principles with approximately specified operators have been considered in the literature (See [36], [45]). For example Neubauer [36] considered the discrepancy principle

$$(4.5) \quad α^3 \langle (T_h T_h^* + αI)^{-2} Q_m γ δ, Q_m γ δ \rangle = (d_1 δ + d_2 ε_h)^2,$$

where $T_h = Q_m T_h$ and $Q_m$ is the orthogonal projection onto a finite dimensional subspace $W_m$ of $Y$ such that $Q_m$ converges to $I$ pointwise and $T$ is a compact operator. Our procedure can also be put in this setting with some modifications in the proof. It can be seen that the square of the left hand side of the equation (4.4) is $α^2 \langle (T_h T_h^* + αI)^{-2} γ δ, γ δ \rangle$, so that the method (4.4) is simpler than the procedure of (4.5) of Neubauer [36]. Moreover the method (4.4) generalizes the procedure investigated in Section 2.2.

If $X = Y$ and the operator $T$ is a positive self-adjoint operator on $X$, then as in Chapter 3, we use different notations for the operator and the data and consider the solution $w^δ_{α, h}$ of the equation

$$(4.6) \quad (A_h + αI)w^δ_{α, h} = g^δ,$$

for obtaining approximations for $w$, the minimal norm solution of
the equation \( Aw = g \). Here \((A_h)_{h>0}\) is a family of self-adjoint operators on \( X \) with \( \| A - A_h \| \leq \varepsilon_h, \varepsilon_h \to 0 \) as \( h \to 0 \). In this case we consider the discrepancy principles

\[
\| A_h w^{\delta}_{\alpha, h} - g^{\delta} \| = \frac{(\delta + \varepsilon_h)^p}{\alpha^q}, \quad p > 0, \quad q > 0
\]

and

\[
\alpha^{2(p+1)}(A_h + \alpha I)^{-2(p+1)}Q_h g^{\delta}, Q_h g^{\delta} = (c\delta + d\varepsilon_h)^2, \quad p > 0,
\]

where \( c \) and \( d \) are properly chosen positive constants and \( Q_h \) is the orthogonal projection on to \( R(A_h) \), for obtaining convergence and error estimates.

### 4.2. ON THE APPLICATION OF GENERALIZED ARCANGELI\'S METHOD FOR TIKHONOV REGULARIZATION

Let \( T \in BL(X,Y), \ y \in R(T) \) and let \( \hat{x} \) be the minimal norm solution of the equation (4.1). Let \( H \) be a bounded subset of positive reals such that zero is a limit point of \( H \). Let \( \{T_h\}_{h \in H} \) be a family of bounded linear operators between \( X \) and \( Y \), such that \( \|T - T_h\| \leq \varepsilon_h, \ v \in H \), where \( \{\varepsilon_h\}_{h \in H} \) is a set of non-negative real numbers satisfying \( \varepsilon_h \to 0 \) as \( h \to 0 \). For \( \delta > 0 \), let \( D^\delta \) be as in Section 2.1, i.e., \( D^\delta = \{u \in Y : \|u - y\| \leq \delta\} \).

In the following, \( x_{\alpha}^\delta \) is the solution of (4.2) with exact data \((y,T)\) in place of \((y^\delta, T_h)\) and \( x_{\alpha, h}^\delta \) is the solution of (4.2) for
Hereafter we assume that there exists \( c_0 > 0 \) such that \( \delta_h \leq c_0 \) for all \( h \in H \). This is the case when \( \{T_h\}_{h \in H} \) is a uniformly bounded family. Let \( \delta_0 \) be such that \( 0 < \delta_0 \leq \frac{\|y\|}{2} \).

**Theorem 4.2.1.** For a fixed pair \( p, q \) of positive reals, and for each \( \delta \in (0, \delta_0], h \in H, \) and \( y \delta \in D \), there exists a unique \( \alpha := \alpha(\delta, h) > 0 \) satisfying (4.4). Moreover,

(i) \( \alpha(\delta, h) : 0 < \delta \leq \delta_0, h \in H \) is a bounded set of reals,

(ii) \( \alpha(\delta, h) \leq c(\delta + \varepsilon_h)^{\frac{p}{q+1}} \) for some constant \( c > 0 \),

(iii) \( \frac{p}{q+1} < \frac{4q}{2q+1} \) and \( \varepsilon_h \to 0 \) as \( h \to 0 \), imply

\[
\|\hat{x} - x^\delta_{\alpha, h}\| \to 0 \quad \text{as} \quad \delta \to 0, h \to 0.
\]

**Proof:** The existence and uniqueness of \( \alpha := \alpha(\delta, h) \) satisfying (4.4) follows as in Proposition 2.1.4.

If the set \( \{\alpha(\delta, h) : 0 < \delta \leq \delta_0, h \in H\} \) is not bounded then there exist sequences \( (\delta_n) \) and \( (h_n) \) with \( 0 < \delta_n \leq \delta_0, h_n \in H \) such that

\[
\alpha_n := \alpha(\delta_n, h_n) \to \infty \quad \text{as} \quad n \to \infty.
\]
Now since

\[ (4.7) \quad \frac{\alpha_n {\parallel y \parallel}}{2(1 + M/\alpha_n)} \leq \alpha_n {\parallel T_n x \delta_n \parallel} - y \delta_n = (\delta_n + \epsilon_n)^p, \]

where \( M \geq (\epsilon_0 + \|T\|)^2 \), we have

\[ \alpha_n \leq \frac{2(\delta_n + \epsilon_n)^p(1 + M/\alpha_n)}{\parallel y \parallel}. \]

This leads to a contradiction. Thus (i) is proved.

Again from (4.7), by using (i) we have

\[ \alpha_n^{s+1} \leq \frac{2(\delta_n + \epsilon_n)^p(\alpha_n + M)}{\parallel y \parallel} \leq c(\delta + \epsilon_n)^p, \]

proving (ii).

If \( \epsilon_n \to 0 \) as \( h \to 0 \), then it follows from (ii) that \( \alpha(\delta, h) \to 0 \) as \( \delta \to 0, \ h \to 0 \). It can be seen as in Neubauer [36] that

\[ (4.8) \quad \|\hat{x} - x \delta_{\alpha, h}\| \leq c(\|\hat{x} - x \| + \delta + \epsilon_n). \]

Therefore to prove (iii) it is enough to show that

\[ (4.9) \quad \frac{\delta + \epsilon_n}{\sqrt{\alpha(\delta, h)}} \to 0 \text{ as } \delta \to 0, \ h \to 0 \]

and

\[ (4.10) \quad \|\hat{x} - x \| \to 0 \text{ as } \delta \to 0, \ h \to 0. \]
We note that

\[
\frac{(\delta + e_h)^p}{a(\delta, h)} = ||hT_h^x\delta_h - y\delta_h|| = ||h\alpha(T_hT_h^* + aI)^{-1}y\delta_h||
\]

\[
\leq ||h\alpha(T_hT_h^* + aI)^{-1}(y\delta - y)|| + ||h(T_hT_h^* + aI)^{-1}y||
\]

(4.11)

\[
\leq \delta + ||h\alpha(T_hT_h^* + aI)^{-1}y||
\]

where

(4.12) \[\alpha(T_hT_h^* + aI)^{-1}y = \alpha(T_hT_h^* + aI)^{-1}(TT^* - T_hT_h^*) (TT^* + aI)^{-1}y + \alpha(TT^* + aI)^{-1}y\]

\[= \alpha(T_hT_h^* + aI)^{-1}(T - T_h)(T^*T + aI)^{-1}y + \alpha(TT^* + aI)^{-1}y\]

\[+ \alpha(TT^* + aI)^{-1}T_h (T^* - T_h) (T^*T + aI)^{-1}y\]

\[+ \alpha(TT^* + aI)^{-1}\hat{T}x.\]

Now using the relations

\[||T_h^x|| + aI)^{-1}T_h\| \leq \frac{1}{2\gamma a'},\]

\[\|T^*T + aI)^{-1}T^*y\| \leq \|\hat{x}\|,\]

\[\|h\alpha(T_hT_h^* + aI)^{-1}\| \leq 1 \text{ and } \|h(TT^* + aI)^{-1}\| \leq \frac{\|\hat{x}\|}{2\gamma a'},\]

it follows that

(4.13) \[\|h\alpha(T_hT_h^* + aI)^{-1}y\| \leq \|\hat{x}\| e_h + h\alpha(TT^* + aI)^{-1}y\|.
\]
Now by Lemma 2.2.1, we have

\[(4.14) \quad \|a(\mathbf{T}^\ast + aI)^{-1}\mathbf{T}\| \leq c\omega\]

where \(\omega = \min(1, \nu+1/2)\) for \(x \in R((\mathbf{T}^\ast \mathbf{T})^\nu), 0 < \nu \leq 1\) so that (4.13) implies

\[(4.15) \quad \frac{(\delta+\epsilon_h)^p}{\alpha} \leq \max\{1, 2\|x\|\}(\delta+\epsilon_h) + \|a(\mathbf{T}^\ast + aI)^{-1}\mathbf{T}\|,\]

\[\leq c((\delta+\epsilon_h)^{\frac{p}{2(\nu+1)}})\]

Therefore

\[(4.16) \quad \left(\frac{\delta+\epsilon_h}{\sqrt{\alpha}}\right)^{2q} = (\delta+\epsilon_h)^{2q-p}\left(\frac{(\delta+\epsilon_h)^p}{\alpha}\right)\]

\[\leq c((\delta+\epsilon_h)^{2q-p+1} + (\delta+\epsilon_h)^{2q-p+\frac{p}{2(\nu+1)}})\]

Now the assumption \(\frac{p}{q+1} \left(\frac{4q}{2q+1}\right) < \min(1, \nu+1/2)\) implies \(2q-p+1 > 2q-p + \frac{p}{2(\nu+1)}\) \(\Rightarrow 0\), so that \(\frac{\delta+\epsilon_h}{\sqrt{\alpha}} \to 0\) as \(\delta \to 0\), proving (4.9) and (4.10) follows from (ii) and arguments used in the proof of Theorem 2.1.1(a).

**Theorem 4.2.2.** Let \(x \in R((\mathbf{T}^\ast \mathbf{T})^\nu), 0 < \nu \leq 1, \omega = \min(1, \nu+1/2)\). If \(\frac{p}{q+1} \leq \min\left(\frac{1}{\beta}, \frac{2}{1+(1-\omega)/q}\right)\) and \(\alpha := \alpha(\delta, h)\) is chosen according to (4.4) for \(0 < \delta \leq \delta_0, h \in H\). Then

\[(i) \quad \frac{\delta+\epsilon_h}{\sqrt{\alpha}} \leq c_1(\delta+\epsilon_h)^{\mu}\]

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(ii) \( \|\hat{x} - x_\delta^{\alpha,h} \| \leq c_2(\delta + \varepsilon_h)^r \),

where \( \mu = 1 - \frac{p}{2(q+1)}(1+\frac{1}{q}) \), \( r = \min(\mu, \frac{\nu}{q+1}) \).

In particular, if \( \frac{p}{q+1} = \frac{2}{2\nu+1+(1-\omega)/q} \), then

(iii) \( \|\hat{x} - x_\delta^{\alpha,h} \| \leq c_3(\delta + \varepsilon_h)^{\frac{\nu}{q+1}} \).

Proof: In view of (4.8), Theorem 4.2.1 (ii) and Theorem 2.1.1(i), the result in (ii) and (iii) will follow, once (i) is proved. The proof of (i) is a consequence of the relations in (4.14), (4.15), (4.16) and Theorem 4.2.1 (ii). 

Proof of the following Corollary is along the same lines as the proof of the Corollary 2.2.5.

Corollary 4.2.3 Let \( p, q \) be positive reals satisfying \( \frac{p}{q+1} \leq 1 \) and let \( \hat{x} \in \mathbb{R}((\mathbb{R}^{|T|^T})^\nu), 0 < \nu \leq 1, \omega = \min(1, \nu+1/2) \),

\( l = \min\left\{ \frac{1}{\nu}, \frac{2}{2\nu+1+(1-\omega)/q} \right\} \). If \( \alpha := \alpha(\delta,h) \) is chosen according to (4.4), for \( 0 < \delta \leq \delta_0, h \in H \), then

\( \|\hat{x} - x_\delta^{\alpha,h} \| \leq c(\delta + \varepsilon_h)^\tau \)

where

\[ \tau = \left\{ \begin{array}{ll} \frac{p}{q+1}, & \frac{p}{q+1} \leq l \\ 1 - \frac{p}{\nu(w+1)}, & \frac{p}{w+1} \geq l \end{array} \right. \]
Remark 4.2.4. We note that if \( \hat{x} \in R((T^*T)^\nu) \), \( 1/2 \leq \nu \leq 1 \), then Theorem 4.2.2, provides the order \( \mathcal{O}(\delta + \varepsilon_h) r \) with \( r = \min \left( 1 - \frac{p}{2(q+1)}, \frac{p\nu}{q+1} \right) \) for \( \frac{p}{q+1} \leq 1 \). Also if \( \hat{x} \in R((T^*T)^\nu) \), \( 0 < \nu \leq 1 \) and \( \nu_1 \) is any estimate for \( \nu \) such that \( \nu \leq \nu_1 \) and \( \nu_1 \geq 1/2 \), then by taking \( \frac{p}{q+1} = \frac{2}{2\nu_1+1} \), we obtain the rate \( \mathcal{O}(\delta + \varepsilon_h)^{2\nu_1+1} \). In particular for \( \frac{p}{q+1} = \frac{2}{3} \), i.e., with \( \nu_1 = 1 \), the rate \( \mathcal{O}(\delta + \varepsilon_h)^3 \) is guaranteed. This case includes the Arcangeli's type discrepancy principle, i.e.,

\[
\parallel \hat{\gamma}_{\alpha,h} - \gamma \parallel = \frac{\delta + \varepsilon_h}{\sqrt{\alpha}},
\]

for \( p = 1 \), \( q = 1/2 \).

In Chapter 5, we consider a special case of the operator \( T_h \), namely, \( T_h = TP_h \), where \( \{P_h\}_{h \in H} \) is a sequence of orthogonal projections on \( X \). This case, under certain conditions, leads to improved accuracy.

4.3. ON THE APPLICATION OF GENERALIZED ARCANGELI'S METHOD FOR SIMPLIFIED REGULARIZATION

Let \( A \in \text{BL}(X) \), \( g \in R(A) \) and let \( \hat{w} \) be the minimal norm solution of the operator equation

(4.17) \[ Aw = g. \]

Let \( H \) be as in Section 4.2 and \( \{A_h\}_{h \in H} \) be a family of self-
adjoint operators on $X$ satisfying

\[(4.18) \quad \|A - A_h\| \leq \varepsilon_h, \quad h \in H,\]

where $(\varepsilon_h)_{h \in H}$ is a set of non-negative real numbers satisfying $\varepsilon_h \to 0$ as $h \to 0$.

In case $A_h$ is not positive, then one may consider the operator $B_h = A_h + \varepsilon_h I$ (See [45]) and $2\varepsilon_h$ in place of $A_h$ and $\varepsilon_h$ respectively. Then $B_h$ is a positive self-adjoint operator satisfying $\|B_h - A\| \leq 2\varepsilon_h$. This is seen as follows. From (4.18) it is clear that $\|B_h - A\| \leq 2\varepsilon_h$. Now using the fact that $A$ is positive self-adjoint operator, we have

\[
\langle A_h x, x \rangle \geq \langle A x, x \rangle - \langle \varepsilon_h x, x \rangle
\]

\[
\geq -\langle \varepsilon_h x, x \rangle,
\]

so that

\[
\langle B_h x, x \rangle = \langle A_h x, x \rangle + \langle \varepsilon_h x, x \rangle
\]

\[
\geq -\langle \varepsilon_h x, x \rangle + \langle \varepsilon_h x, x \rangle
\]

\[
\geq 0.
\]

Thus without loss of generality we may assume that $(A_h)_{h \in H}$ is a family of positive self-adjoint operators on $X$. 

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For $\delta > 0$, let $F^\delta := \{ u \in X : \|u-g\| \leq \delta \}$. Let $w_\alpha$ be the solution of

\[(4.19)\quad (A + \alpha I)w_\alpha = g,\]

and $w^\delta_{\alpha,h}$ be the solution of the equation

\[(4.20)\quad (A_h + \alpha I)w^\delta_{\alpha,h} = g^\delta, \quad g^\delta \in F^\delta.\]

We assume that there exists $\varepsilon_0 > 0$ such that $\varepsilon_h \leq \varepsilon_0$ for all $h \in H$. Let $\delta_0$ be such that $\delta < \delta_0 \leq \|g\|/2$.

*Theorem 4.3.1* Let $\hat{w} \in R(A^\nu)$, $0 < \nu \leq 1$ and $w^\delta_{\alpha,h}$ be defined as in (4.20). Then

\[\|\hat{w} - w^\delta_{\alpha,h}\| \leq c_1 \frac{\delta + \varepsilon_h}{\alpha} + c_2 \alpha^\nu\]

where $c_1$ and $c_2$ are positive constants.

*Proof:* Using triangle inequality, we have

\[\|\hat{w} - w^\delta_{\alpha,h}\| \leq \|\hat{w} - w_\alpha\| + \|w_\alpha - w_{\alpha,h}\| + \|w_{\alpha,h} - w^\delta_{\alpha,h}\|\]

where $w_{\alpha,h} := w^0_{\alpha,h}$. Thus by the definition of $w_\alpha$, $w_{\alpha,h}$ and the fact that $g = Aw$, it follows that
The result follows from the above relations together with estimate (3.3), i.e., \( \| w - w_n \| = O(\alpha^\nu) \).

From the above Theorem it is clear that if \( \alpha := c(\delta + \varepsilon_h)^{-1/\nu} \) for some constant \( c > 0 \), then

\[
(4.23) \quad \| \hat{w} - w_\alpha \| = O((\delta + \varepsilon_h)^{\nu})
\]

and this order is 'optimal' (See. (3.4)) in the sense that in general it cannot be improved.

In order to obtain the convergence of \( w_\alpha \) to \( \hat{w} \) and to obtain the order in (4.23) we suggest the discrepancy principle

\[
(4.24) \quad \| A_{\mu_\nu} w_\alpha - g \delta \| = \frac{(\delta + \varepsilon_h)^p}{\alpha^q}, \quad p > 0, \quad q > 0
\]

and \( \delta \in (0, \delta_0] \).

Theorem 4.3.2. For a fixed pair \( p, q \) of positive reals and for each \( \delta \in (0, \delta_0] \), \( h \in H \) and \( g \delta \in F^\delta \), there exist a unique
\[ \alpha := \alpha(\delta,h) \text{ satisfying (4.24). Moreover} \]

(i) \( \{\alpha(\delta,h) : 0 < \delta \leq \delta_0, h \in H\} \) is a bounded set of reals,

(ii) \( \alpha(\delta,h) \leq c_1(\delta+\varepsilon_h)^\frac{\rho}{\kappa} \quad \text{and} \quad \frac{\delta+\varepsilon_h}{\kappa(\delta,\varepsilon_h)} = c_2(\delta+\varepsilon_h)^{1-\frac{\rho}{\kappa}} \)

for some constants \( c_1 > 0 \) and \( c_2 > 0 \),

(iii) \( p < q+1 \) and \( \varepsilon_h \to 0 \) as \( h \to 0 \) imply

\[ \|\tilde{w} - \omega^\delta_{\alpha,h}\| \to 0 \quad \text{as} \quad \delta \to 0, h \to 0. \]

Proof: The existence and uniqueness of \( \alpha := \alpha(\delta,h) \) satisfying (4.24) follows as in Lemma 3.1.2. The boundedness of the set \( \{\alpha(\delta,h) : 0 < \delta \leq \delta_0, h \in H\} \) and the estimate for \( \alpha(\delta,\varepsilon_h) \) follow by using similar arguments as in Theorem 4.2.1 ((i) and (ii)). To obtain estimate for \( \frac{\delta+\varepsilon_h}{\kappa(\delta,\varepsilon_h)} \) and the convergence of \( \omega^\delta_{\alpha,h} \) to \( \tilde{w} \), we first note that

\[ \frac{(\delta+\varepsilon_h)^\rho}{\alpha^\rho} = \|A_h\omega^\delta_{\alpha,h} - g^\delta\| \]

\[ = \|\alpha(A_h + \alpha I)^{-1}g^\delta\| \]

\[ \leq \|\alpha(A_h + \alpha I)^{-1}(g^\delta - g)\| + \|\alpha[(A_h + \alpha I)^{-1} - (A + \alpha I)^{-1}]g\| \]
\[ + \| \alpha(A + \alpha I)^{-1} g \| \]

where
\[ \| \alpha(A_h + \alpha I)^{-1}(g - g) \| \leq \delta, \]

and
\[ \| \alpha[A_h + \alpha I]^{-1} - (A + \alpha I)^{-1} g \| = \| \alpha(A_h + \alpha I)^{-1}(A - A_h)(A + \alpha I)^{-1} g \| \leq \epsilon_h \| \omega \| \]

Thus
\[ \frac{(\delta + \epsilon_h)^p}{\alpha} \leq c_1(\delta + \epsilon_h) + c_2 \alpha \]

where \( c_1 \) and \( c_2 \) are positive constants. Therefore by (ii), we have

\[
\frac{\delta + \epsilon_h}{\alpha} = (\delta + \epsilon_h)^{1 - \frac{p}{q}} \left( \frac{(\delta + \epsilon_h)^p}{\alpha^q} \right)^{\frac{1}{q}} \
\leq c_1(\delta + \epsilon_h)^{1 - \frac{p}{q}} + c_2(\delta + \epsilon_h)^{1 - \frac{p}{q}} + \frac{p}{\alpha^q}\]

Now since \( 1 - \frac{p}{q} + \frac{p}{\alpha^q(\delta + \epsilon_h)} = 1 - \frac{p}{\alpha^q} \) and \( p < q+1 \), we have

\[ \frac{\delta + \epsilon_h}{\alpha} = O((\delta + \epsilon_h)^{1 - \frac{p}{\alpha^q}}). \]

Now the convergence of \( \omega_{\alpha, h}^\delta \) to \( \hat{\omega} \) follows as in Theorem 3.1.3 (iii), once we prove that
\[ \| \hat{w} - w_{a,h}^\delta \| \leq c_1 \frac{\delta + \epsilon_n}{\alpha} + \| a(A + aI)^{-1}w \|. \]

But this is clear from (4.21), (4.22) and the inequality

\[ (4.25) \quad \| \hat{w} - w_{a,h}^\delta \| \leq \| \hat{w} - w_{a} \| + \| w_{a} - w_{a,h} \| + \| w_{a,h} - w_{a,h}^\delta \|. \]

This completes the proof. ■

**Theorem 4.3.3.** Let \( \hat{w} \in R(A^\nu) \), \( 0 < \nu \leq 1 \), \( q > 0 \), \( p < q+1 \) and \( \alpha := \alpha(\delta,h) \) be chosen according to (4.24) for \( 0 < \delta \leq \delta_0 \), \( h \in H \). Then for some constant \( c > 0 \), we have

(i) \[ \| \hat{w} - w_{a,h}^\delta \| \leq c(\delta + \epsilon_n)^r \]

\[ r = \min \left\{ \frac{\nu}{q+1}, 1- \frac{p}{q+1} \right\}. \]

In particular if \( \frac{p}{q+1} = \frac{1}{\nu+1} \), then,

(ii) \[ \| \hat{w} - w_{a,h}^\delta \| \leq c(\delta + \epsilon_n)^{\frac{\nu}{\nu+1}}. \]

**Proof:** Proof of (i) follows from Theorems 4.3.1 and 4.3.2 (ii), and proof of (ii) is a consequence of (i) and the fact that \( \frac{\nu}{q+1} = 1 - \frac{p}{q+1} \) if and only if \( \frac{p}{q+1} = \frac{1}{\nu+1} \), and in this case \( \frac{\nu}{q+1} = \frac{\nu}{\nu+1} \). ■
4.4. ON THE APPLICATION OF MODIFIED GUACANEME'S METHOD FOR SIMPLIFIED REGULARIZATION

In this section we study the analogue of the parameter choice strategy considered in Section 3.2 for simplified regularization with approximately specified operator. Specifically, for a fixed real number \( p > 0 \), we consider the discrepancy principle

\[
\alpha^{(p+1)}((A_h + \alpha I)^{-2(p+1)})Q_h g_\delta, Q_h g_\delta) = (c\delta + d\epsilon_h)^2, \quad \alpha > 0,
\]

where \( c \) and \( d \) are constants and \( Q_h \) is the orthogonal projection onto \( \mathcal{R}(A_h) \) for choosing the parameter \( \alpha \) in (4.20). Here also \( g_\delta \in \mathcal{F}_\delta \) and \( A_h, h \in H \) are as in Section 4.3. If in addition \( g_\delta \) satisfies \( \|Q_h g_\delta\| \geq c\delta + d\epsilon_h \), then as in Lemma 3.2.2 one can prove that there exists a unique \( \alpha := \alpha(\delta, h) \) satisfying (4.26). The following result is used to prove our main result of this section.

**Lemma 4.4.1.** Let \( \alpha := \alpha(\delta, h) \) be the solution of (4.26) with \( c > 1 \) and \( d > e = (2+p)\|\hat{w}\| \), where \( \hat{w} \) is the minimal norm solution of (4.17). Then

\[
[(c-1)\delta + (d-e)\epsilon_h]^2 \leq \alpha^{(p+1)}((A + \alpha I)^{-2(p+1)})g, g \leq [(c+1)\delta + (d+e)\epsilon_h]^2.
\]

**Proof:** We note that...
\[
\alpha^{2(\rho^1)}((A + \alpha I)^{-2(\rho^1)}g, g) = \|\alpha^{\rho^1}(A + \alpha I)^{-\rho^1}g\|^2.
\]

Also
\[
(A + \alpha I)^{-\rho^1}g = (A_h + \alpha I,)^{-\rho^1}Q_h g\delta + (A_h + \alpha I)^{-\rho^1}Q_h (g - g\delta)
\]
\[+ (A_h + \alpha I)^{-\rho^1}(I - Q_h)g\]
\[+ [(A + \alpha I)^{-\rho^1} - (A_h + \alpha I)^{-\rho^1}]g.
\]

Therefore
\[
\|\alpha^{\rho^1}(A + \alpha I)^{-\rho^1}g\| \leq \|\alpha^{\rho^1}(A_h + \alpha I)^{-\rho^1}Q_h g\delta\|
\]
\[+ \|\alpha^{\rho^1}(A_h + \alpha I)^{-\rho^1}Q_h (g - g\delta)\|
\]
\[+ \|\alpha^{\rho^1}(A_h + \alpha I)^{-\rho^1}(I - Q_h)g\|
\]
\[+ \|\alpha^{\rho^1}[(A + \alpha I)^{-\rho^1} - (A_h + \alpha I)^{-\rho^1}]g\|,
\]

and
\[
\|\alpha^{\rho^1}(A_h + \alpha I)^{-\rho^1}g\| \geq \|\alpha^{\rho^1}(A_h + \alpha I)^{-\rho^1}Q_h g\delta\|
\]
\[-\|\alpha^{\rho^1}(A_h + \alpha I)^{-\rho^1}Q_h (g - g\delta)\|
\]
\[-\|\alpha^{\rho^1}(A_h + \alpha I)^{-\rho^1}(I - Q_h)g\|
\]
\[-\|\alpha^{\rho^1}[(A + \alpha I)^{-\rho^1} - (A_h + \alpha I)^{-\rho^1}]g\|.
\]

Now,
\[
\|\alpha^{\rho^1}(A_h + \alpha I)^{-\rho^1}Q_h (g - g\delta)\| = [\alpha^{2(\rho^1)}((A_h
\]
\]

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Therefore the Lemma will follow once we prove
\[
\|\alpha^{p+1}(A_h + \alpha I)^{(p+1)}Q_h(g - g\delta)\| \leq \delta
\]
and
\[
\|\alpha^{p+1}(A_h + \alpha I)^{(p+1)}(I-Q_h)g\| \leq \|\alpha^{p+1}(A_h + \alpha I)^{(p+1)}(I-Q_h)(A-A_h)\tilde{w}\|
\]
\[
\leq \|\alpha^{p+1}(A_h + \alpha I)^{(p+1)}(I-Q_h)(A-A_h)\tilde{w}\|
\]
\[
\leq \|(I-Q_h)(A-A_h)\tilde{w}\|
\]
\[
\leq \|I-Q_h\|\|(A-A_h)\tilde{w}\|
\]
\[
\leq \epsilon_h\|\tilde{w}\|.
\]

Therefore the Lemma will follow once we prove
\[
\|\alpha^{p+1}f(\alpha, h)\| \leq (p+1)\epsilon_h\|\tilde{w}\|
\]
where
\[
f(\alpha, h) = [(A + \alpha I)^{(p+1)} - (A_h + \alpha I)^{(p+1)}]g.
\]
To prove this first we note that
\[
f(\alpha, h) = (A_h + \alpha I)^{(p+1)}[(A_h + \alpha I)^{(p+1)}]
\]
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\[-(A + \alpha I)^{(p+1)}(A + \alpha I)^{-1}(p+1)g\]

\[= (A_h + \alpha I)^{-(p+1)}[(A_h + \alpha I)(A_h + \alpha I)^p - (A + \alpha I)^p](A + \alpha I)^{-(p+1)}g\]

\[= (A_h + \alpha I)^{-(p+1)}[A_h(A_h + \alpha I)^p - A(A + \alpha I)^p](A + \alpha I)^{-(p+1)}g\]

\[+ \alpha(A_h + \alpha I)^{-(p+1)}[(A_h + \alpha I)^p - (A + \alpha I)^p](A + \alpha I)^{-(p+1)}g\]

\[= (A_h + \alpha I)^{-(p+1)}A_h[(A_h + \alpha I)^p - (A + \alpha I)^p](A + \alpha I)^{-(p+1)}g\]

\[+ (A_h + \alpha I)^{-(p+1)}(A_h - A)(A + \alpha I)^{-1}g\]

\[+ \alpha(A_h + \alpha I)^{-(p+1)}[(A_h + \alpha I)^p - (A + \alpha I)^p](A + \alpha I)^{-(p+1)}g\]

\[= (A_h + \alpha I)^{-p}[(A_h + \alpha I)^p - (A + \alpha I)^p](A + \alpha I)^{-1}g\]

\[+ (A_h + \alpha I)^{-(p+1)}(A_h - A)(A + \alpha I)^{-1}g.\]

Thus

\[\|\alpha^{p+1}f(\alpha, h)\| \leq \|U(\alpha, h)\|\]

\[+ \|\alpha^{p+1}(A_h + \alpha I)^{-(p+1)}(A_h - A)(A + \alpha I)^{-1}g\|\]

where

\[U(\alpha, h) = \alpha^{p+1}(A_h + \alpha I)^{-p}[(A_h + \alpha I)^p - (A + \alpha I)^p](A + \alpha I)^{-1}g.\]
Now since
\[ \| \alpha^{p+1}(A_h + \alpha I)^{-p}(A_h - A)(A + \alpha I)^{-1} g \| \leq \varepsilon_h \| \hat{w} \|, \]
we have
(4.27) \[ \| \alpha^{p+1} f(\alpha, h) \| \leq \| U(\alpha, h) \| + \varepsilon_h \| \hat{w} \|. \]

We note that if \( p = 1 \), then
\[ \| U(\alpha, h) \| \leq \| \alpha^2(A_h + \alpha I)^{-1}(A_h - A)(A + \alpha I)^{-2} g \| \]
\[ \leq \varepsilon_h \| \hat{w} \|, \]
so that in this case
\[ \| \alpha^{p+1} f(\alpha, h) \| \leq 2\varepsilon_h \| \hat{w} \|. \]

Now consider the case when \( 0 < p < 1 \). In this case,
(4.28) \[ U(\alpha, h) = \alpha^p(A_h + \alpha I)^{-p}(A + \alpha I)^{-1}(A_h + \alpha I)^{p-1} \]
\[ - (A + \alpha I)^{p})(\alpha^p(A + \alpha I)^{-p+1} g), \]
so that
\[ \| U(\alpha, h) \| \leq \| \alpha^p(A_h + \alpha I)^{-p}\| \| \alpha^{1-p}[(A_h + \alpha I)^p \]
\[ - (A + \alpha I)^p](\alpha^p(A + \alpha I)^{-p+1} g) \| \]
Now recall the formula ([25] page 287),

\[ B^p_x = \frac{\sinnx}{n} \int_{\lambda = 0}^{\infty} \frac{1}{\lambda^2} \left[ (\lambda^2 + B^2)^{-1} x - \frac{\Theta(\lambda)}{\lambda} x + \cdots + (\lambda^n - 1) B^{-1} x \right] d\lambda \]

\[ + \frac{\sinnx}{n} \left[ \frac{x}{z - z} + \cdots + (\lambda^n - 1) B^{-1} x \right], \quad x \in X \]

where

\[ \Theta(\lambda) = \begin{cases} 0 & \text{if } 0 < \lambda \leq 1 \\ 1 & \text{if } 1 < \lambda < \omega \end{cases} \]

for any positive self-adjoint operator \( B \) and for complex number \( z \) such that \( 0 < \text{Re} z < n \). Taking \( z = \rho \), \( 0 < \rho < 1 \), we have

\[ B^p_x = \frac{\sinnx}{n} \left[ \frac{x}{\rho} + \int_{\lambda = 0}^{\infty} \frac{\rho}{\lambda^2} (\lambda^2 + B^2)^{-1} x d\lambda + \int_{\lambda = 1}^{\infty} \frac{\rho}{\lambda - \rho} x d\lambda \right] \]

Using the above formula, taking

\[ \xi = \alpha^p (A + \alpha I)^{-1} \rho \]

and

\[ Q_{\alpha, h} = (A_h + \alpha I)^{\rho} - (A + \alpha I)^{\rho}, \]

we obtain

\[ Q_{\alpha, h} \xi = \frac{\sinnx}{n} \int_{t = 0}^{\infty} \rho \left[ (A_h + (t+\alpha I)^{-1} - (A + (t+\alpha I)^{-1}) \xi dt. \right] \]
\[
\begin{align*}
\alpha^{-1} Q_{\alpha, h} \xi &= \frac{\sin \rho}{\pi} \int_0^\infty t^\rho \left( A_h + (t+\alpha)I \right)^{-1} (A + (t+\alpha)I)^{-1} \xi dt.
\end{align*}
\]

Thus
\[
\|Q_{\alpha, h}\|_{\xi} \leq \frac{\sin \rho}{\pi} \int_0^\infty t^\rho \left( A_h + (t+\alpha)I \right)^{-1} (A - A_h) (A + (t+\alpha)I)^{-1} \xi \xi dt.
\]

Now since
\[
\|A_h + (t+\alpha)I\|^{-1} \leq \frac{1}{\alpha + t} \quad \text{and} \quad \|A + (t+\alpha)I\|^{-1} \leq \frac{1}{\alpha + t},
\]
we have
\[
\|A_h + (t+\alpha)I\|^{-1} \left\|A - A_h\right\| \|\xi\| \leq \alpha^{-1} \rho \frac{\sin \rho}{\pi} \int_0^\infty \frac{t^\rho}{(\alpha + t)^2} dt \left\|A_h - A\right\| \|\xi\|
\]
(4.30)

where \( \beta = \frac{t}{\alpha} \).

It can be seen that
\[
\int_0^\infty \frac{\beta^\rho}{(1+\beta)^2} d\beta = \rho \int_0^\infty \frac{ds}{s^{1+\rho}(1+s)}
\]
and
\[
\int_0^\infty \frac{ds}{s^{1+\rho}(1+s)} = \frac{\pi}{\sin \rho},
\]
so that
\[
\int_0^\infty \frac{\beta^\rho}{(1+\beta)^2} d\beta = \frac{\pi \rho}{\sin \rho}.
\]

Therefore, from (4.30), we have
(4.31) \[ \| \alpha^{1-p}[(A_h + \alpha I)^p - (A + \alpha I)^p] \xi \| \leq \rho \varepsilon_h \| \xi \|. \]

where
\[ \| \xi \| = \| \alpha^p(A + \alpha I)^{-p+1}g \| = \| \alpha^p(A + \alpha I)^{-p+1} A \hat{w} \| \leq \| \hat{w} \|. \]

Thus from (4.27), (4.29) and (4.30), we have
\[ \| \alpha^{p+1}[(A + \alpha I)^{-p+1} - (A_h + \alpha I)^{-p+1}]g \| \leq (p+1)\varepsilon_h \| \hat{w} \|. \]

This completes the proof. \[ \] 

Theorem 4.4.2. Let \( g^\delta \epsilon F^\delta, \ h \epsilon H \) and let \( \alpha = \alpha(\delta, h) \) is chosen according to (4.26) with \( c \) and \( d \) are as in Lemma 4.4.1. Then
\[ w^\delta_{\alpha, h} \rightharpoonup \hat{w} \quad \text{as} \quad \delta \rightarrow 0, \ h \rightarrow 0. \]

Proof: Note that
\[ \| \hat{w} - w^\delta_{\alpha, h} \| \leq \| \hat{w} - w_{\alpha} \| + \| w_{\alpha} - w_{\alpha, h} \| + \| w_{\alpha, h} - w^\delta_{\alpha, h} \|. \]

Thus from (4.21) and (4.22), we have
\[ \| \hat{w} - w^\delta_{\alpha, h} \| \leq \max(1, \| \hat{w} \|) \frac{\delta + \rho h}{\alpha} + \| \hat{w} - w_{\alpha} \|. \]

From this we obtain the result by using the arguments used in the proof of Theorem 3.2.4. \[ \]
Lemma 4.4.3 Let \( g^\delta \in F^\delta, h \in H \) and let \( \alpha := \alpha(\delta, h) \) is the unique solution of (4.26) with \( c \) and \( d \) are as in Lemma 4.4.1. Let \( \hat{w} \in R(A^\nu), 0 < \nu \leq 1 \). Then we have the following

(i) \( \alpha(\delta, h) = O((\delta + \epsilon_h)^{1/\nu}) \)

(ii) \( \frac{\delta + \epsilon_h}{\alpha(\delta, h)} = O((\delta + \epsilon_h)^{\nu}), \) if \( 0 < \nu \leq 1 \) and \( \nu \leq \rho \).

(iii) \( \frac{\delta + \epsilon_h}{\alpha(\delta, h)} = O((\delta + \epsilon_h)^{\nu}), \) if \( 0 < \nu < 1 \) and \( \nu < \rho \).

Proof: Proof of the Lemma follows in the line of the proof of Lemma 3.2.5 with \( ((c+1)\delta + (d+e)\epsilon_h)^2 \) in place of \( c_2\delta^2 \) and \( ((c-1)\delta + (d-e)\epsilon_h)^2 \) in place of \( c_1\delta^2 \).

Theorem 4.4.4. Let \( g^0 \in F^0, h \in H \) and let \( \alpha := \alpha(\delta, h) \) be the unique solution of (4.26) with \( c \) and \( d \) are as in Lemma 4.4.1. Let \( \hat{w} \in R(A^\nu), 0 < \nu \leq 1 \). Then

(i) \( \|\hat{w} - w^\delta_{\alpha, h}\| = \begin{cases} O((\delta + \epsilon_h)^{\nu}), & \nu \leq \rho \\ O((\delta + \epsilon_h)^{\rho}), & \nu \geq \rho \end{cases} \)

If \( 0 < \nu < 1 \) and \( \nu < \rho \), then

(ii) \( \|\hat{w} - w^\delta_{\alpha, h}\| = O(\delta + \epsilon_h)^{\nu}) \).
In particular if \( p = 1 \) in (4.26) then

\[
(iii) \quad \| \tilde{w} - w_{\alpha,h}^\delta \| = \begin{cases} 
O((\delta + \epsilon_h)^{\frac{\nu}{2}}), & 0 < \nu < 1 \\
O((\delta + \epsilon_h)^{\frac{1}{2}}), & \nu = 1
\end{cases}
\]

Proof: The proof of the Theorem follows from Theorem 4.3.1 and Lemma 4.4.3 by using the arguments used in the proof of Theorem 3.2.6. \( \blacksquare \)