1.1. GENERAL INTRODUCTION

Many problems in mathematical physics and applied mathematics, particularly those involving remote sensing, indirect measurement, etc. have their mathematical formulation as an operator equation of the first kind,

\[ Tx = y, \]

where \( T : X \rightarrow Y \) is a bounded linear operator between Hilbert spaces \( X \) and \( Y \). The above equation is, in general, 'ill-posed', i.e., the existence of a unique solution which depends continuously on the data \( y \) is not guaranteed. In case there is no solution in the usual sense, one seeks the 'least-square solution of minimal norm', which in general does not depend continuously on the data \( y \). In fact, if \( R(T) \), the range of \( T \), is not closed, then the map which associates each \( y \in R(T) + R(T)^\perp \) to the least-square solution of minimal norm is not continuous. In such situation one has to regularize \( Tx = y \), often with an inexact data \( y^\delta \) with \( \| y - y^\delta \| \leq \delta \). The regularization which has been studied most extensively is the so called Tikhonov regularization, in which one considers \( x^\delta_\alpha \), the solution of the equation
\[(T^*T + \alpha I)x^\delta_\alpha = T^*y^\delta, \quad \alpha > 0,\]

for obtaining approximations for \( \hat{x} \), the least-square solution of minimal norm. The crucial problem here is to choose the regularization parameter \( \alpha \) depending on \( \delta \) and \( y^\delta \) such that we must have \( x^\delta_\alpha \to x^\delta \) as \( \delta \to 0 \) and obtain the 'optimal' estimate for the error \( \|\hat{x} - x^\delta_\alpha\| \). It is known [39] that if \( \hat{x} \in R((T^*T)^\nu), 0 < \nu \leq 1 \), then the optimal rate for \( \|\hat{x} - x^\delta_\alpha\| \) is \( O(\delta^{2\nu/(2\nu+1)}) \).

Morozov [31] and Arcangeli [1] had considered 'discrepancy principles', namely,

\[\|T x^\delta_\alpha - y^\delta\| = \delta \quad \text{and} \quad \|T x^\delta_\alpha - y^\delta\| = \frac{\delta}{\sqrt{\alpha}}\]

respectively, for choosing the parameter \( \alpha \) in Tikhonov regularization. For Morozov's method, the best possible rate for \( \|\hat{x} - x^\delta_\alpha\| \) is \( O(\delta^{1/2}) \) ([14]) and for Arcangeli's, the known rate was \( O(\delta^{1/3}) \) which is attained for \( \hat{x} \in R(T^*) \) ([18]). In an attempt to obtain optimal rate, i.e., \( O(\delta^{2\nu/(2\nu+1)}) \), Schock [38] considered the discrepancy principle

\[\|T x^\delta_\alpha - y^\delta\| = \frac{\delta^p}{\alpha^q}, \quad p > 0, \quad q > 0\]

and proved that the rate is arbitrarily close to the optimal rate for large values of \( q \). Later Nair [34], considered the above discrepancy principle and improved the result of Schock [38]. In
fact, the result in [34], shows that, the Arcangeli's method does give the best rate \( O(\delta^{2/3}) \) for \( \hat{x} \in R(T^*T) \).

In Chapter 2 we consider the discrepancy principle of Schock [38] and prove that if \( \hat{x} \in R((T^*T)^{\nu}), \) \( 1/2 \leq \nu \leq 1, \) then the optimal rate \( O(\delta^{2\nu/(2\nu+1)}) \) is achieved. Our result improves the result of Nair [34] for \( 0 < \nu < 1, \) and for \( \nu = 1 \) our result coincides with the result in [34]. In the final section of Chapter 2 we consider Schock's discrepancy principle for iterated Tikhonov regularization.

If \( Y = X \) and the operator under consideration is 'positive and self-adjoint', then one can consider a simpler regularization method, namely, the Simplified regularization. In this case we use the notation \( A \) for the operator \( T, \) and consider the equation \( Aw = g. \) In Simplified regularization of the equation

\[
Aw = g
\]

one takes the solution \( w_\alpha^{\delta} \) of the equation

\[
(A + \alpha I)w_\alpha^{\delta} = g_\delta
\]

for obtaining approximations for \( \hat{w}, \) the minimal norm solution of the equation \( Aw = g. \) Here \( g_\delta \) is such that \( \|g - g_\delta\| \leq \delta. \) For choosing the regularization parameter \( \alpha \) in Simplified
regularization Groetsch and Guacaneme [16] considered Arcangeli's method and proved the convergence of $\hat{w}_\alpha$ to $\hat{w}$. But in [16], no attempt has been made for obtaining the estimate for the error $\|\hat{w} - w_\alpha\|$. In Section 3.1, we consider a generalized Arcangeli's method, namely,

$$\|A w_\alpha^\delta - g^\delta\| = \frac{\delta^p}{q}, \quad p > 0, \quad q > 0,$$

for obtaining the regularization parameter $\alpha$. We obtain the optimal rate $\alpha(\delta^{1/3})$ (see [39]) for the error $\|\hat{w} - w_\alpha^\delta\|$, whenever $\hat{w} \in R(\sigma A^\nu)$, $0 < \nu \leq 1$. As a particular case we prove that the Arcangeli's method considered in [16] gives the rate $\alpha(\delta^{1/3})$, and the best rate $\alpha(\delta^{1/2})$ is obtained when $\nu = 1$ by taking $\frac{p}{q+1} = \frac{1}{2}$. The result for the case when $\nu = 1$ has also been considered by Guacaneme [19]. In Section 3.2, we consider the discrepancy principle, namely,

$$\alpha^{(\rho+1)}((A + \alpha I)^{-\rho}g^\delta, Qg^\delta) = c\delta^2, \quad \rho > 0,$$

where $c > 1$ is a constant and $Q$ is the orthogonal projection onto the closure of the range of $A$. Result of this section includes a result of Guacaneme [21], which he proved when $A$ is compact and $\nu = 1$. In the final section of Chapter 3 we consider the discrepancy principles considered in Sections 3.1 and 3.2 for iterated Simplified regularization.
In reality there are two occasions, where one has to consider perturbed operators instead of the original operator. One such occasion arises from the modeling error and the other when one considers numerical approximation. Many authors (e.g., [31], [36], [37], [43]) considered the equation $Tx = y$ with a perturbed operator $T_h$ instead of $T$ with

$$
\|T - T_h\| \leq \varepsilon_h, \quad \varepsilon_h \to 0 \; \text{as} \; h \to 0.
$$

In Chapter 4 we consider Tikhonov regularization and simplified regularization with perturbed operators. Specifically, we modify the discrepancy principles of Chapter 2 and 3 so as to include the case of perturbed operators.

In Chapter 5 we consider projection method for the regularized equations

$$(T^* T + \alpha I)x_\delta^* = T^* y_\delta \quad \text{and} \quad (A + \alpha I)\omega_\delta = g_\delta.$$ 

The projection method for the equation

$$(T^* T + \alpha I)x_\delta^* = T^* y_\delta$$

is a special case of the method considered in Section 4.2 and under certain conditions this method leads to a better error estimate than the one obtained in Section 4.2. In order to
illustrate the theoretical results, some numerical experiments have been performed, and the results are reported in the last section of the thesis.

Now we formally define well-posed and ill-posed operator equations and discuss the peculiar problems associated with the solution of the ill-posed operator equations. Operator theoretic foundation for the sequel is laid by considering some preliminary results from Functional Analysis, which facilitates in discussing the concept of a generalized inverse and regularization methods.

**WELL-POSED AND ILL-POSED PROBLEMS**

Let $X$ and $Y$ be Hilbert spaces (over real or complex field) and $T: X \rightarrow Y$ be a linear operator. We consider the problem of solving the operator equation

\[(1.1) \quad Tx = y.\]

A typical example of equation $(1.1)$ is the Fredholm integral equation of the first kind

\[(1.2) \quad \int_a^b k(s,t)x(t)dt = y(s), \quad a \leq s \leq b\]

with non-degenerate kernel $k(s,t)$. Here $X = Y = L^2[a,b]$.

An important fact concerning the equation $(1.2)$ is that, the
associated operator \( T: L^2[a, b] \to L^2[a, b] \) defined by

\[
(Tx)(s) = \int_a^b k(s, t)x(t)dt, \quad a \leq s \leq b
\]

is a 'compact operator' of infinite rank, and therefore \( T \) can not have a continuous inverse (See, [26], Theorem 17.2 and 17.4). This observation is very important in view of its application, for this amounts to large deviations in the solutions corresponding to 'nearby' data. Therefore equation (1.2) is a typical example of the so called 'ill-posed problems'. Many inverse problems in physical sciences lead to the solution of the equation of the above type.

In the beginning of this century, Hadamard [22] specified the essential requirements for an equation to be well-posed. In our setting, the equation (1.1) is said to be well-posed if

(i) (1.1) has a solution \( x \), for all \( y \in Y \)

(ii) (1.1) can not have more than one solution,

(iii) the unique solution \( x \), if exists, depends continuously on the data \( Y \).

In operator theoretic language, (i), (ii), (iii) means that \( T \) is
bijective and $T^{-1} : Y \to X$ is a continuous operator. The equation (1.1) is said to be ill-posed if it is not well-posed. By the remark in the previous paragraph, if $T$ is a compact operator of infinite rank, then the equation (1.1) is ill-posed.

We now mention a few examples of inverse problems in physical sciences which lead to solution of an integral equation of the type (1.2). Detailed discussions on these can be found in Groetch [15].

THE VIBRATING STRING. The free vibration of a nonhomogeneous string of unit length and density distribution $\rho(x) > 0$, $0 < x < 1$, is modeled by the partial differential equation

(1.3) \[ \rho(x)U_{tt} = U_{xx}; \]

where $U(x,t)$ is the position of the particle 'x' at time $t$. Assume that the end of the string are fixed and $U(x,t)$ satisfies the boundary conditions

$U(0,t) = 0$, $U(1,t) = 0$.

Assuming the solution $U(x,t)$ is of the form

$U(x,t) = y(x)r(t)$,

one observes that $y(x)$ satisfies the ordinary differential
with boundary conditions

\[ y(0) = 0, \quad y(1) = 0. \]

Suppose the value of \( y \) at certain frequency \( \omega \) is known, then by integrating equation (1.4) twice, first from zero to \( s \) and then from zero to one, we obtain

\[
\int_0^1 y'(s; \omega)\,ds - y'(0; \omega) + \omega^2 \int_0^1 \varphi(x)y(x; \omega)\,dx\,ds = 0.
\]

or

\[
\int_0^1 (1-s)y(s; \omega)\varphi(s)\,ds = \frac{y'(0; \omega)}{\omega^2}.
\]

The inverse problem here is to determine the variable density \( \varphi \) of the string, satisfying (1.6) for all allowable frequencies \( \omega \).

**THERMAL ARCHAEOLOGY.** Consider a uniform bar of length \( \pi \) which is insulated on its lateral surface so that heat is constrained to flow in only one direction. With certain normalizations and scaling the temperature \( U(x,t) \) satisfies the partial differential equation

\[
U_t = U_{xx}, \quad 0 < x < \pi.
\]
We assume that the ends of the bar are kept at temperature zero, i.e.,

$$U(0,t) = 0 \text{ and } U(\pi,t) = 0.$$  

If $f(x) = U(x,0)$, $0 \leq x \leq \pi$, is the initial temperature distribution, then the temperature distribution at a later time, say at time $t = 1$, is given by

$$g(x) = U(x,1) = \sum_{n=1}^{\infty} a_n \sin nx,$$

where

$$a_n = \frac{2}{\pi} \int_{0}^{\pi} f(u) \sin nu \, du \cdot e^{-n^2}.$$

The inverse problem associated with the above consideration is to determine the initial temperature distribution $f(x)$, knowing a later temperature $g(x)$. From (1.7) and (1.8), the problem, then, is to solve the integral equation of the first kind,

$$\int_{0}^{\pi} k(x,u)f(u)du = g(x)$$

where

$$k(x,u) = \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-n^2} \sin nx \cdot \sin nu.$$

GEOLOGICAL PROSPECTING. Here the problem is to determine the location, shape and constitution of subterranean bodies from measurements at the earth's surface. Consider a variable
distribution of mass along a parallel line below one unit of the earth's surface. Suppose that a horizontal line measurement is made of the vertical component of the gravitational force due to the mass. If the variable mass density \( x(t) \) is distributed along the horizontal axis for \( 0 \leq t \leq 1 \) and one measures the vertical component of the force \( y(s) \), then a small mass element \( x(t) \Delta t \) at position \( t \) gives rise to a vertical force \( \Delta y(s) \) at \( s \), given by

\[
\Delta y(s) = y(x(t)\Delta t/((s-t)^2 + 1))\cos \theta
\]

\[
= y\left( x(t)\Delta t/((s-t)^2+1)^{3/2}\right)
\]

where \( y \) is the gravitational constant. Now the Fredholm integral equation

\[
y_0\int_0^1((s-t)^2+1)^{-3/2}x(t)\,dt = y(s)
\]

gives the relation between the vertical force \( y(s) \) at \( s \) and the density distribution \( x(t) \).

SIMPLIFIED TOMOGRAPHY. Consider a two dimensional object contained within a circle of radius \( R \). The object is illuminated with a radiation of intensity \( I_0 \). As the radiation beams passes through the object it absorbs some radiation. Assume that the radiation absorption coefficient \( f(x,y) \), of the object varies from point to point of the object. The absorption coefficient satisfies
the law
\[ \frac{dI}{dy} = -fI \]

where \( I \) is the intensity of the radiation. By taking the above equation as the definition of the absorption coefficient, we have

\[ I_x = I_0 \exp(-\int f(x,y)dy) \]

where \( y = \sqrt{R^2-x^2} \). Let \( p(x) = \ln(I_0/I_x) \), i.e.,

\[ p(x) = \int_{-y(x)}^{y(x)} f(x,y)dy \]

Suppose that \( f \) is circularly symmetric, i.e., \( f(x,y) = f(r) \) with \( r = \sqrt{x^2+y^2} \), then

\[ p(x) = \int_{-\sqrt{x^2+y^2}}^{\sqrt{x^2+y^2}} (2\pi/\sqrt{r^2-x^2})f(r)dr \]

(1.9)

The inverse problem is to find the absorption coefficient \( f \) satisfying the equation (1.9).

**BLACK BODY RADIATION.** When a black body is heated, it emits thermal radiation from its surface at various frequencies. The distribution of thermal power, per unit area of radiating surface, over the various frequencies is known as the power spectrum of the
black body. The relation between the power radiation by a unit area of surface at a given frequency $\nu$ and absolute temperature $T$ of the surface is given by the relation

$$ P(\nu) = \frac{2h\nu^2}{c^2} \frac{1}{\exp(h\nu/kT-1)} $$

where $c$ is the speed of light, $h$ is Planck's constant and $k$ is Boltzmann's constant.

Suppose that different patches of the surface of the black body are at different temperatures. Let $a(T)$ represents the area of the surface which is at temperature $T$, i.e., $a(.)$ is the area-temperature distribution of the radiating surface. Then the total radiated power at frequency $\nu$, $W(\nu)$, is given by

(1.10) \[ W(\nu) = \frac{2h\nu^3}{c^2} \int_0^\infty (\exp(h\nu/(kT-1))) a(T) dT. \]

The inverse problem is to find the area-temperature distribution $a(.)$ that can account for an observed power spectrum $W(.)$, i.e., to solve the integral equation (1.10).

1.2. NOTATIONS AND SOME BASIC RESULTS FROM FUNCTIONAL ANALYSIS.

Throughout this thesis $X$ and $Y$ denote Hilbert spaces over real or complex field and $BL(X,Y)$ denotes the space of all
bounded linear operators from $X$ to $Y$. If $Y = X$, then we denote $BL(X, X)$ by $BL(X)$. We will use the symbol $\langle \cdot, \cdot \rangle$ to denote the inner product and $\| \cdot \|$ to denote the corresponding norm for the spaces under consideration. The results quoted in this section with no references can be found in any text book on functional analysis, for example, [26] or [13].

For a subspace $S$ of $X$, its closure is denoted by $\overline{S}$, and its annihilator is denoted by $S^\perp$, i.e.,

$$S^\perp = \{ u \in X : \langle x, u \rangle = 0 \text{ for all } x \in S \}.$$ 

If $T \in BL(X, Y)$, then its adjoint, denoted by $T^*$, is a bounded linear operator from $Y$ to $X$ defined by

$$\langle Tx, y \rangle = \langle x, f^*y \rangle$$

for all $x \in X$ and $y \in Y$. Denoting the range and null space of $T$ by $R(T)$ and $N(T)$ respectively, i.e.,

$$R(T) = \{ Tx : x \in X \}$$

and

$$N(T) = \{ x \in X : Tx = 0 \},$$

we have the following.
Theorem 1.2.1. If $T \in BL(X,Y)$, then $R(T) = N(T^*)$, $N(T^*) = R(T)$, $R(T^*) = N(T)$ and $N(T^*) = R(T)$. 

The spectrum and the spectral radius of an operator $T \in BL(X)$ are denoted by $\sigma(T)$ and $r_T(T)$ respectively, i.e.,

$$\sigma(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ does not have bounded inverse} \},$$

where $I$ is the identity operator on $X$, and

$$r_T(T) = \sup \{ |\lambda| : \lambda \in \sigma(T) \}.$$

It is known that $r_T(T) \leq \|T\|$, and $\sigma(T)$ is a compact subset of the scalar field. If $T$ is a nonzero self-adjoint operator, i.e., $T = T^*$, then $\sigma(T)$ is a nonempty set of real numbers, and

$$(1.11) \quad r_T(T) = \|T\|.$$  

If $T$ is a positive self-adjoint operator, i.e., $T = T^*$ and $(Tx,x) \geq 0$, $x \in X$, then $\sigma(T)$ is a subset of the set of non-negative reals. If $T \in BL(X)$ is compact, i.e., closure of $(Tx : x \in X, \|x\| \leq 1)$ is compact, then $\sigma(T)$ is a countable set.
with zero as the only possible limit point. In fact we have the following result.

**Theorem 1.2.2.** Let $T \in BL(X)$ be a non-zero compact self-adjoint operator. Then there is a finite or infinite sequence of non-zero real number's $(\lambda_n)$ with $|\lambda_1| \geq |\lambda_2| \geq \ldots$, and a corresponding sequence $(u_n)$ of orthonormal vectors in $X$ such that for all $x \in X$,

$$Tx = \sum \lambda_n \langle x, u_n \rangle u_n,$$

where $\lambda_n \to 0$ whenever the sequence $(\lambda_n)$ is infinite. Here $\lambda_n$'s are eigenvalues of $T$ with corresponding eigenvectors $u_n$.

If $T \in BL(X,Y)$ is a non-zero compact operator, then $T^*T$ is a positive, compact and self-adjoint operator on $X$. Then by Theorem 1.2.2, and by the observation that $\sigma(T^*T)$ consists of non-negative reals, there exist a sequence $(s_n)$ of positive reals with $s_1 \geq s_2 \geq \ldots$ and a corresponding sequence of orthonormal vectors $(v_n)$ in $X$ satisfying.

$$T^*Tx = \sum s_n \langle x, v_n \rangle v_n, \text{ for all } x \in X$$

and $T^*Tv_n = s_nv_n$, $n = 1, 2, \ldots$. Let $\lambda_n$ be the positive square root of $s_n$, $\mu_n = 1/\lambda_n$ and $u_n = \mu_n v_n$. Then $(u_n)$ is
complete orthonormal sequence in $Y$ and $\mu_n T^* u_n = v_n$. Using Theorem 1.2.2, it can be seen (See, [12]) that $(u_n)$ is a complete orthonormal set for $R(T) = N(T^*)^\perp$ and $(v_n)$ is a complete orthonormal set for $R(T^*) = N(T)^\perp$. The sequence $(u_n, v_n, \mu_n)$ is called a singular system for $T$.

In order to define functions of operators on a Hilbert space, we require the spectral theorem for a self-adjoint operator which is a generalization of Theorem 1.2.2.

**Theorem 1.2.3.** Let $T \in BL(X)$ be self-adjoint and let $a = \inf \sigma(T), \ b = \sup \sigma(T)$. Then there exists a family $(E_\lambda)_{\lambda \in [a, b]}$ of projection operators on $X$ such that

(i) $\lambda_1 < \lambda_2$ implies $\langle E_{\lambda_1} x, x \rangle \leq \langle E_{\lambda_2} x, x \rangle$ for all $x \in X$.

(ii) $E_a = 0, E_b = I$, where $I$ is the identity operator on $X$.

(iii) $T = \int_a^b \lambda dE_\lambda$.

The integral in (iii) is understood in the Riemann-Stieltjes sense. The family $(E_\lambda)_{\lambda \in [a, b]}$ is called the spectral family of the operator $T$. If $f$ is a continuous real valued function on $[a,b]$, then $f(T) \in BL(X)$ is defined by
\[ f(T) = \int_a^b f(\lambda) dE_\lambda. \]

Then

\[ \sigma(f(T)) = \{ f(\lambda) : \lambda \in \sigma(T) \}. \]

Now by (1.11) we have

\[ \|f(T)\| = r_\sigma(f(T)) = \sup \{ |f(\lambda)| : \lambda \in \sigma(T) \}. \]

For real valued functions \( f \) and \( g \), we use the notation

\[ f(\alpha) = o(g(\alpha)) \text{ as } \alpha \to 0 \]

to denote the relation

\[ \frac{|f(\alpha)|}{g(\alpha)} \leq M \text{ as } \alpha \to 0, \]

where \( M > 0 \) is a constant independent of \( \alpha \), and

\[ f(\alpha) = o(g(\alpha)) \text{ as } \alpha \to 0 \]

to denote

\[ \lim_{\alpha \to 0} \frac{f(\alpha)}{g(\alpha)} = 0. \]

1.3. GENERALIZED INVERSE.

If the operator equation (1.1) has no solution in the usual sense, i.e., if \( y \) does not belong to the range of \( T \), then one
may broaden the notion of a solution in a meaningful sense. This can be done using the concept of a least-square solution.

For $T \in BL(X,Y)$ and $y \in Y$, we say that $u \in X$ is a least square solution of the equation (1.1), $Tx = y$, if

$$\|Tu-y\| = \inf(\|Tx-y\| : x \in X).$$

It is to be remarked that if $T$ is not one-one, then a least-square solution $u$, if it exists, is not unique, since $u+v$ is also a least-square solution for every $v \in N(T)$. The following theorem provides characterizations of least-square solutions.

**Theorem 1.3.1.** (Groetsch \[12\], Theorem 1.3.1). For $T \in BL(X,Y)$ and $y \in Y$, the following are equivalent.

(i) $\|Tu-y\| = \inf(\|Tx-y\| : x \in X)$

(ii) $T^*Tu = T^*y$

(iii) $Tu = Py$

where $P : Y \rightarrow Y$ is the orthogonal projection onto $\overline{R(T)}$.

From (iii) it is clear that (1.1) has a least-square solution if and only if $Py \in R(T)$, i.e., if and only if $y'$ belongs to the
dense subspace $R(T) + R(T)^\perp$ of $Y$. Any of (i)-(iii) in Theorem 1.3.1 shows that the set of all least-square solutions is a closed convex set, and therefore, by Theorem 1.1.4 in [11], there is a unique least-square solution of smallest norm. For $y \in R(T) + R(T)^\perp$, the unique least-square solution of minimal norm of (1.1) is called the \textit{generalized solution} or \textit{pseudo solution} of (1.1). It can be easily seen that the generalized solution belongs to the subspace $N(T)^\perp$ of $X$. For $T \in BL(X,Y)$, the map $T^\dagger$ which associates each $y \in D(T^\dagger) := R(T) + R(T)^\perp$, the generalized solution of (1.1) is called the \textit{generalized inverse} of $T$. We note that if $y \in R(T)$ and $T$ is injective, then the generalized solution of (1.1) is the solution of (1.1). If $T$ is bijective, then it follows that $T^\dagger = T^{-1}$.

\textbf{Theorem 1.3.2.} (Groetch [11], [13]). Let $T \in BL(X,Y)$. Then $T^\dagger : D(T^\dagger) \to X$ is a closed densely defined linear operator, and $T^\dagger$ is bounded if and only if $R(T)$ is closed.

If equation (1.1) is ill-posed then one would like to obtain the generalized solution of (1.1). But Theorem 1.3.2 shows that the problem of finding the generalized solution of (1.1) is also ill-posed, i.e., $T^\dagger$ is discontinuous, if $R(T)$ is not a closed subspace of $Y$. Recall that if $T \in BL(X,Y)$ is a compact operator of infinite rank, then $R(T)$ is not closed. This observation is important since a wide class of operators of practical interest, as we have seen in Section 1.2, are compact operators of infinite rank.
rank. In application, the data $y$ may not be available exactly. So, one has to work with an approximation, say $\tilde{y}$, of $y$. If $T^+$ is discontinuous, then for $\tilde{y}$ close to $y$, the generalized solution $T^+\tilde{y}$, even when it is defined, need not be close to $T^+y$. Therefore some regularization procedures have to be employed, to obtain approximations for $T^+y$, for $y \in D(T^+)$. 

1.4. THE REGULARIZATION PRINCIPLE AND THE TIKHONOV REGULARIZATION.

Here onwards we are concerned with the problem of finding the generalized solution of (1.1) where $T \in BL(X,Y)$ and $y \in D(T^+) = R(T) + R(T)^\perp$. For $\delta > 0$, let $\tilde{y} \in Y$ be an inexact data such that $\|y - \tilde{y}\| \leq \delta$. By regularization of the equation (1.1) with $\tilde{y}$ in place of $y$, we mean a procedure of obtaining a family $(\tilde{x}_\alpha)$ of vectors in $X$ such that each $\tilde{x}_\alpha$, $\alpha > 0$, is a solution of a well-posed equation satisfying $\tilde{x}_\alpha \to T^+y$ as $\alpha \to 0$ and $\delta \to 0$.

A regularization method which has been studied most extensively is the so called Tikhonov regularization ([43], [44]) introduced in the early sixties, where $\tilde{x}_\alpha$ is taken as the minimizer of the functional

$$x \mapsto F_\alpha(x) = \|Tx - \tilde{y}\|^2 + \alpha \|x\|^2, \quad x \in X, \alpha > 0.$$ 

The fact that $\tilde{x}_\alpha$ is the unique solution of the well-posed equation, namely,
(1.13) \((T^*T + \alpha I)x_\alpha = T^*y\),

is included in the following well known result, the proof of which is included for the sake of completion.

\[ (T^*T + \alpha I)x_\alpha = T^*y, \]

Theorem 1.4.1. (See [35]) Let \( T \in BL(X;Y) \) and \( y \in Y \). For each \( \alpha > 0 \) there exists a unique \( x_\alpha \in X \) which minimizes the function

(1.14) \( x \mapsto F_\alpha(x) = \|Tx-y\|^2 + \alpha\|x\|^2, x \in X \).

More over, the map \( y \mapsto x_\alpha \) is continuous for each \( \alpha > 0 \), and

\[ x_\alpha = (T^*T + \alpha I)^{-1}T^*y. \]

Proof: First we prove that there exists a unique \( x_\alpha \) which minimizes the function (1.14). Consider the product space \( X \times Y \) with the usual innerproduct defined by

\[ \langle (x_1,y_1),(x_2,y_2) \rangle = \langle x_1,x_2 \rangle + \langle y_1,y_2 \rangle, \quad x_1, x_2 \in X; \quad y_1, y_2 \in Y. \]

It is seen that, with respect to this innerproduct, \( X \times Y \) is a Hilbert space. For \( \alpha > 0 \), consider the function

\[ F_\alpha : X \to X \times Y, \quad F_\alpha(x) = (\langle x,Tx \rangle,y), x \in X. \]
Since $T \in \text{BL}(X, Y)$, the graph of $T$,

$$G(T) = \{(x, Tx) : x \in X\},$$

is a closed subspace of $X \times Y$, so that the range $R(F_\alpha)$ of $F_\alpha$ is closed in $X \times Y$. Thus by Theorem 1.3.2, the generalized inverse $F_\alpha^+$ is a bounded linear operator from $X \times Y$ into $X$. Let $x_\alpha = F_\alpha^+(0, y)$. Since $F_\alpha$ is one-one, it is clear from the definition of the generalized inverse that $x_\alpha$ is the unique element in $X$ satisfying

$$\|F_\alpha(x_\alpha) - (0, y)\| = \inf \{\|F_\alpha(x) - (0, y)\| : x \in X\}$$

i.e.,

$$\|T x_\alpha - y\|^2 + \alpha \|x_\alpha\|^2 = \inf \{\|Tx - y\|^2 + \alpha \|x\|^2 : x \in X\}.$$

Now since the function $J : Y \rightarrow X \times Y$ defined by $J(y) = (0, y)$, $y \in Y$, is continuous, the function $y \mapsto x_\alpha := F_\alpha^+(0, y)$ is also continuous.

Now to prove that $x_\alpha$ is given by $x_\alpha = (T^*T + \alpha I)^{-1}T^*y$, first we note that $T^*T$ is a positive self adjoint operator and hence $-\alpha \notin \sigma(T^*T)$, if $\alpha > 0$. Thus for $\alpha > 0$, $(T^*T + \alpha I)^{-1}$ exist and is a bounded linear operator on $X$. Let $u_\alpha = (T^*T + \alpha I)^{-1}T^*y$, $\alpha > 0$, then

$$\|T(u_\alpha + v) - y\|^2 + \alpha \|u_\alpha + v\|^2 = \|T u_\alpha - y\|^2 + \alpha \|u_\alpha\|^2 + \langle (T^*T + \alpha I)v, v \rangle,$$
for all \( v \in X \). Now since \( \langle (T^*T + \alpha I)v, v \rangle \geq 0 \), for all \( v \in X \), it follows that

\[
\|Tu_\alpha - y\|^2 + \alpha \|u_\alpha\|^2 \leq \|Tx - y\|^2 + \alpha \|x\|^2, \text{ for all } x \in X.
\]

This, together with the fact that \( x_\alpha = F_\alpha(0, y) \) is the unique element in \( X \) such that

\[
\|Tx_\alpha - y\|^2 + \alpha \|x_\alpha\|^2 = \inf (\|Tx - y\|^2 + \alpha \|x\|^2 : x \in X)
\]

shows that \( x_\alpha = u_\alpha = (T^*T + \alpha I)^{-1}T^*y \).

If \( Y = X \) and \( T \) is a positive self-adjoint operator on \( X \), then one may consider (see Bakushinskii [2]) a simpler regularization method to solve equation (1.1), where the family of vectors \( \tilde{w}_\alpha, \alpha > 0 \), satisfying

\[
(T + \alpha I)\tilde{w}_\alpha = \tilde{y},
\]

is considered to obtain approximations for \( T^*y \). Note that for a positive self-adjoint operator \( T \), the ordinary Tikhonov regularization applied to (1.1) results in a more complicated equation \( (T^2 + \alpha I)x_\alpha = Ty \) than (1.15). Moreover it is known (see Schock [40]) that the approximation obtained by regularization procedure (1.15) has better convergence properties than the approximation obtained by Tikhonov regularization. As in Groetsch
and Guacaneme, [16], we call the above regularization procedure which gives the family of vectors \( \tilde{w}_\alpha \) in (1.15), the **simplified regularization** of (1.1).

One of the prime concerns of regularization methods is the convergence of \( \tilde{x}_\alpha (\tilde{w}_\alpha \) in the case of simplified regularization) to \( T^\dagger y \), as \( \alpha \to 0 \) and \( \delta \to 0 \). It is known ([12], Theorem 2.3.5) that, if \( R(T) \) is not closed, then there exist sequences \( (\delta_n) \) and \( (\alpha_n) : = (\alpha(\delta_n)) \) such that \( \delta_n \to 0 \) and \( \alpha_n \to 0 \) as \( n \to \infty \) but the sequence \( (\tilde{x}_{\alpha_n}) \) is divergent as \( \delta_n \to 0 \). Therefore it is important to choose the regularization parameter \( \alpha \) depending on the error level \( \delta \) and also possibly on \( y \), say \( \alpha = \alpha(\delta, y) \), such that \( \alpha(\delta, y) \to 0 \) and \( \tilde{x}_\alpha \to T^\dagger y \) as \( \delta \to 0 \). We shall see later (Section 2.1) that in the case of Tikhonov regularization, if we take \( \alpha = \delta \) a priorily then \( \tilde{x}_\alpha \to T^\dagger y \) as \( \delta \to 0 \). Practical considerations suggest that, it is desirable to choose the regularization parameter \( \alpha \) at the time of solving \( \tilde{x}_\alpha \), using a so-called *a posteriori* method which depends on \( y \) as well as \( \delta \).

1.5. THE CHOICE OF REGULARIZATION PARAMETER BY DISCREPANCY PRINCIPLES.

For choosing the regularization parameter a posteriorily, 'discrepancy principles' have been used extensively in the literature (e.g., [4], [6], [7], [10], [32], [38]). This idea was
first enunciated by Morozov[31]. The method is based on the reasonable view that the quality of the results of a computation must be comparable to the quality of the input data. To be more precise the magnitude of the error must be in agreement with the accuracy of the assignment of the input data (See, Morozov [31] or Groetsch [12]). The practical difficulty here is that even an asymptotic bound for the quantity \( \|x_\alpha - T^\dagger y\| \) usually requires information on the data \( y \). Therefore one has to consider an 'optimal order' (optimal in the sense that, in general, the order can not be improved) of the quantity \( \|x_\alpha - T^\dagger y\| \), based on the available information of the data. Now the crucial problem is to find the value of the regularization parameter \( \alpha \) which gives the optimal order of the quantity \( \|x_\alpha - T^\dagger y\| \).

The subject matter of this thesis is to provide optimal error bounds for the existing discrepancy principles for Tikhonov regularization and simplified regularization, and also to generalize a discrepancy principle for simplified regularization considered by Guacaneme [21]. Computational results are given in the last section of the thesis which confirm the theoretical results.

1.6 SUMMARY OF THE THESIS

In Chapter 2 we consider Tikhonov regularization for approximately solving the ill-posed operator equation \( Tx = y \),
where $T : X \to Y$ is a bounded linear operator between Hilbert spaces $X$ and $Y$ and $y \in R(T) + R(T)^\perp$, i.e., the problem of minimizing the functional

$$x \mapsto \|Tx - y\|^2 + \alpha \|x\|^2, \quad \alpha > 0.$$  

When only an approximation of the data $y$ is known, say $y^\delta$, with $\|y - y^\delta\| \leq \delta$, then the problem of choosing the regularization parameter $\alpha$ depending on $\delta$ and $y^\delta$ is important. For this purpose many discrepancy principles are known in the literature (e.g., [4], [10], [38]). In Section 2.2 we consider the discrepancy principle

$$\|Tx^\delta - y^\delta\|_\alpha = \frac{\delta^p}{\alpha^q}, \quad p > 0, \quad q > 0,$$

considered by Schock [38] and later by Nair [34] and prove that this discrepancy principle gives the optimal estimate $c(\delta^{2\nu/(2\nu+1)})$, $1/2 \leq \nu \leq 1$, for the error $\|\hat{x} - x^\delta\|$ whenever $\hat{x}$ belongs to $R((T^*T)^\nu)$. The result of this section improves the result of Schock [38], and also it improves the result of Nair [34], except for the case $\nu = 1$. A particular case of the result, as proved in [34], shows that the Arcangeli's method does give the optimal rate $c(\delta^{2/3})$. In Section 2.3 we show that one can use the discrepancy principle considered above for iterated Tikhonov regularization also.
Chapter 3 is concerned with the problem of approximately solving ill-posed operator equation $Aw = g$, where $A : X \rightarrow X$ is a positive self-adjoint operator on a Hilbert space $X$ and $g \in R(A)$, the range of $A$. Here we consider the Simplified regularization, where the solution $w_\alpha$ of the equation

$$(A + \alpha I)w_\alpha = g$$

is taken as an approximation for the minimal norm solution $\hat{w}$ of the equation $Aw = g$. If the data $g$ is known only approximately, say $g^\delta$, with $\|g - g^\delta\| \leq \delta$, then we consider the solution $w_\alpha^\delta$ of the equation

$$(A + \alpha I)w_\alpha^\delta = g^\delta$$

for obtaining approximations for $\hat{w}$. In this case, for choosing the parameter $\alpha$, Groetsch and Guacaneme [16] considered the discrepancy principle

$$\|Aw_\alpha^\delta - g^\delta\| = \frac{\delta}{\sqrt{\alpha}}$$

and proved that $w_\alpha^\delta \rightarrow \hat{w}$ as $\delta \rightarrow 0$, but no attempt has been made for obtaining an estimate for the error $\|\hat{w} - w_\alpha^\delta\|$. In Section 3.1 we consider a general class of discrepancy principle, namely,

$$\|Aw_\alpha^\delta - g^\delta\| = \frac{\delta^p}{\alpha^q}, \quad p > 0, \quad q > 0,$
which includes the one considered by Groetsch and Guacaneme [16], and obtain the optimal estimate for the error \( \| \hat{w} - w^\delta \| \). In Section 3.2 we consider a generalized form of a discrepancy principle considered by Guacaneme [21], namely,

\[
\alpha^{2(p+1)} \langle (A + \alpha I)^{-2(p+1)} Q g \delta, Q g \delta \rangle = c \delta^2, \quad \rho > 0
\]

where \( c > 1 \) is a constant and \( Q \) is the orthogonal projection onto \( \overline{R(T)} \), the closure of the range of \( A \). Results of this section include a result of Guacaneme [21], which he proved when \( A \) is, in addition, compact and \( \hat{w} \in R(A) \). In the last section of Chapter 3, we consider the discrepancy principles considered in Sections 3.1 and 3.2 for iterated simplified regularization.

Chapter 4 is devoted to the study of Tikhonov regularization and simplified regularization in the presence of modeling and data error, i.e., both the operator and the data are known only approximately. Knowing a family of operators \( T_h, h > 0 \), with

\[
\| T - T_h \| \leq \varepsilon h, \quad \varepsilon h \to 0 \quad \text{as} \quad h \to 0,
\]

we consider the solution \( x^\delta_{a,h} \) of the equation

\[
(T_h^* T_h + \alpha I) x^\delta_{a,h} = T_h^* y^\delta,
\]

as an approximation for \( \hat{x} \), the minimal norm solution of the
equation \( Tx = y \). In this case we consider the discrepancy principle

\[
\|T_h x_\alpha^\delta - y_\delta\| = \frac{(\delta + \epsilon_h)^p}{\alpha^q}, \quad p > 0, \quad q > 0,
\]

and obtain the optimal rate \( \alpha((\delta + \epsilon_h)^{2\nu/(2\nu + 1)}) \), \( 1/2 \leq \nu \leq 1 \) for \( \| \hat{x} - x_\alpha^\delta \| \) under the assumption \( \hat{x} \in R((T^* T)^\nu) \). In Sections 4.3 and 4.4 we consider a family of self-adjoint operators \( A_h \) with

\[
\|A - A_h\| \leq \epsilon_h, \quad \epsilon_h \to 0 \text{ as } h \to 0.
\]

For choosing the parameter in the case of Simplified regularization of \( Aw = g \), we consider the discrepancy principles

\[
\|A_h w_\alpha^\delta - g_\delta\| = \frac{(\delta + \epsilon_h)^p}{\alpha^q}, \quad p > 0, \quad q > 0
\]

and

\[
\alpha^2(\rho + 1)((A_h + \alpha I)^{-2}(\rho + 1)_{Q_h g\delta, Q_h g\delta}) = (c\delta + d\epsilon_h)^2, \quad p > 0,
\]

where \( c \) and \( d \) are properly chosen constants and \( Q_h \) is the orthogonal projection onto \( R(A_h) \).

In Chapter 5 we consider projection method for the regularized equations

\[
(T^* T + \alpha I)x_\alpha^\delta = T^* y_\delta \quad \text{and} \quad (A + \alpha I)w_\alpha^\delta = g_\delta.
\]
For the first equation, the method is a special case of the procedure considered in Section 4.2 and is a generalization and modification of Marti's method. Also in this case the regularized projection method improves the result of Section 4.2 under certain conditions. In order to illustrate the theoretical results, some numerical experiments have been performed, and the results are reported in the last section of the thesis.