CHAPTER 5

REGULARIZED PROJECTION METHOD AND NUMERICAL APPROXIMATION

In practical implementation of regularization methods for obtaining approximations for the minimal norm least-square solution of the equation \( Tx = y \), one uses finite dimensional subspaces rather than the space \( X \) itself. This amounts to, for example, the projection methods for solving the regularized equations

\[
(T^* T + \alpha I)x_\alpha^\delta = T^* y_\delta \quad \text{and} \quad (A + \alpha I)\omega_\alpha^\delta = g_\delta.
\]

In Section 5.1, first we consider the projection method for Tikhonov regularization of the equation \( Tx = y \) with a modified form of the discrepancy principle (4.4) for choosing the regularization parameter. We show that this procedure leads to a generalization and modification of the Marti's method ([28], [29], [30]). In this case the results include, and in certain cases improve the conclusions of Engl and Neubauer [6] under weaker conditions. Then projection method is applied to Simplified regularization with corresponding modified form of the discrepancy principles (4.24) and (4.26) considered in Chapter 4. In Section 5.2 we present the Algorithms to implement the methods of Section 5.1. Finally in Section 5.3 we present results of some numerical experiments which confirm the Theoretical results presented in Section 5.1.
5.1. REGULARIZED PROJECTION METHOD

Let \( (P_h)_{h>0} \) be a family of orthogonal projections on \( X \). Our aim in this Section is to obtain an approximate solution for the equation

\[
Tx = y, \quad y \in R(T)
\]

in the finite dimensional space \( R(P_h) \). For the results that follow, we impose the conditions

\[
\eta_h := \| (I-P_h) \hat{x} \| \to 0 \quad \text{and} \quad \gamma_h := \| T(I-P_h) \| \to 0 \quad \text{as} \quad h \to \infty.
\]

on \( P_h \) and \( \hat{x} \), where \( \hat{x} \) is the minimal norm solution of (5.1). The above conditions are satisfied if, for example, \( P_h \to I \) pointwise and if \( T \) is a compact operator.

Projection Method for Tikhonov Regularization:

The projection method for the regularized equation

\[
(T^*T + \alpha I)x^\delta = T^*y^\delta,
\]

consists of solving the equation

\[
(5.2) \quad (P_hT^*TP_h + \alpha I)x^\delta_{\alpha,h} = P_hT^*y^\delta.
\]
where \( \gamma \delta \in D^\delta = \{ u \in Y : \| u - y^\delta \| \leq \delta \} \). The unique solution \( x^\delta_{\alpha, h} \) of the equation (5.2) can be interpreted as the unique element satisfying

\[
\langle (T^*T + \alpha I)x^\delta_{\alpha, h}, u \rangle = \langle T^*y^\delta, u \rangle \quad \text{for all } u \in R(P_h).
\]

In fact, equation (5.2) is a particular case of (4.2) obtained by taking \( T_h = TP_h \). Hereafter we use the notation \( T_h \) instead of \( TP_h \). It is proved in Groetsch [12] (Lemma 4.2.3) that

\[
\|x^\alpha_{\alpha, h} - x^\alpha\| \leq \sqrt{(1 + (\gamma^2/\alpha)) \| (I - P_h)x^\alpha\|},
\]

where \( x^\alpha = (T^*T + \alpha I)^{-1}T^*y^\alpha \) and \( x^\alpha_{\alpha, h} = x^0_{\alpha, h} \).

Note that

\[
\hat{x} - x^\delta_{\alpha, h} = \hat{x} - x^\alpha + x^\alpha - x^\alpha_{\alpha, h} + x^\alpha_{\alpha, h} - x^\delta_{\alpha, h},
\]

so that

\[
\|\hat{x} - x^\delta_{\alpha, h}\| \leq \|\hat{x} - x^\alpha\| + \|x^\alpha - x^\alpha_{\alpha, h}\| + \|x^\alpha_{\alpha, h} - x^\delta_{\alpha, h}\|.
\]

Now by (5.3) and the fact that

\[
\|x^\alpha_{\alpha, h} - x^\delta_{\alpha, h}\| = \|(T_h^*T_h + \alpha I)^{-1}T_h^*(y - y^\delta)\| \leq \frac{\delta}{\sqrt{\alpha}},
\]

we have

\[
\|\hat{x} - x^\delta_{\alpha, h}\| \leq \|\hat{x} - x^\alpha\| + \sqrt{(1 + (\gamma^2/\alpha)) \| (I - P_h)x^\alpha\| \| P_h \| + \frac{\delta}{\sqrt{\alpha}}}
\]

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\[
\|\tilde{x} - x\|_{\alpha} \leq \|\tilde{x} - x\| + \sqrt{(1 + \left(\frac{\gamma^2}{\alpha}\right)\|\Pi(I - P_h)(x - \tilde{x})\| + \|\Pi(I - P_h)\tilde{x}\|)} + \frac{\delta}{\sqrt{\alpha}}.
\]

Therefore

\[
(5.4) \quad \|\tilde{x} - x\|_{\alpha, h} \leq (2 + \frac{1}{\sqrt{\alpha}})\|\tilde{x} - x\|_{\alpha} + (1 + \frac{1}{\sqrt{\alpha}})\eta_h + \frac{\delta}{\sqrt{\alpha}}.
\]

From (4.11) and (4.12), we have

\[
\|T_h\tilde{x}\|_{\alpha, h} - \gamma\delta \| \leq \delta + \|\alpha(T_hT_h^* + \alpha I)^{-1}(T - T_h)(T^*T + \alpha I)^{-1}T^*y\|
\]

\[
+ \|\alpha(T_hT_h^* + \alpha I)^{-1}T_h(T^*T - T_h^*)(T^*T + \alpha I)^{-1}T^*y\|
\]

\[
+ \|\alpha(T^*T + \alpha I)^{-1}T\|.
\]

Note that

\[
T_h(T^* - T_h^*) = TP_h(T^* - P_hT^*) = 0
\]

so that

\[
\|T_h\tilde{x}\|_{\alpha, h} - \gamma\delta \| \leq \delta + \|\alpha(T_hT_h^* + \alpha I)^{-1}(T - T_h)(T^*T + \alpha I)^{-1}T^*y\|
\]

\[
+ \|\alpha(T^*T + \alpha I)^{-1}T\|,
\]

where

\[
\|\alpha(T_hT_h^* + \alpha I)^{-1}(T - T_h)(T^*T + \alpha I)^{-1}T^*y\| \leq \|T - T_h\|_{\alpha, h} \|T^*T + \alpha I\|^{-1} \|T^*y\|
\]

\[
\leq \|T(I - P_h)\|_{\alpha, h}.
\]
\[ \leq \| T(I-P_h)(x - \hat{x}) \| + \| T(I-P_h)\hat{x} \| \]

\[ \leq \| T(I-P_h)(x - \hat{x}) \| + \| T(I-P_h)(I-P_h)\hat{x} \| \]

\[ \leq \gamma_h(\| x - \hat{x} \| + \eta_h) . \]

Therefore

\[ \| T_{\alpha,h} x_{\alpha,h}^\delta - y \| \leq \delta + \gamma_h(\| x - \hat{x} \| + \eta_h) + \| R_{\alpha} \hat{x} \| \]

where \( R_{\alpha} := \alpha(T^*+\alpha I)^{-1}T \) satisfies \( \| R_{\alpha} \hat{x} \| \leq c\alpha^\omega \), with

\[ \omega = \min\{1, \nu+1/2\} \], whenever \( \hat{x} \in R((T^*T)^\nu) \), \( \nu > 0 \). For choosing the regularization parameter \( \alpha \), we consider a modified form of the discrepancy principle \( (4.4) \), namely,

\[ \| T_{\alpha,h} x_{\alpha,h}^\delta - y \| = \frac{(\delta + b_h)^p}{\alpha^q}, \]

where \( (b_h)_{h \in H} \) is a set of positive reals such that \( b_h \to 0 \) as \( h \to 0 \). Note that, since \( T_h = TP_h \) and \( x_{\alpha,h}^\delta \in R(P_h) \), the above equation can be written as

\[ (5.5) \quad \| T_{\alpha,h} x_{\alpha,h}^\delta - y \| = \frac{(\delta + b_h)^p}{\alpha^q}, \]

Here and below, as in Section 4.2, \( H \) is a bounded subset of non-negative reals such that zero is a limit point of \( H \). Imitating the proof of Theorem 4.2.1 (i) and (ii), it can be seen that there exists a unique \( \alpha := \alpha(\delta,h) \) such that \( (5.5) \) is satisfied and that
\[\alpha(\delta, h) \leq c(\delta+b_h)^{\frac{p}{Q+1}}, \quad 0 < \delta \leq \delta_0, \ h \in H.\]

**Theorem 5.1.1.** Let \(\eta_h = \alpha(b_h^k)\) and \(\gamma_h = \alpha(b_h^\lambda)\) for some positive reals \(k\) and \(\lambda\). If \(\frac{p}{q+1} \leq \min\{2k, \frac{4q\lambda}{2q+1}\}\) and \(\alpha\) is chosen according to (5.5) then we have the following

(i) If \(\frac{p}{q+1} < \frac{4q}{2q+1}\), then \(\|\tilde{x} - x\|_{\alpha, h} \to 0\) as \(h \to 0, \ \delta \to 0\).

(ii) If \(\tilde{x} \in R((T^*T)^\nu), 0 < \nu \leq 1\) and \(\frac{p}{q+1} \leq \frac{1}{\nu}\), then

\[\|\tilde{x} - x\|_{\alpha, h} \leq c(\|\tilde{x} - x\| + \eta_h + (\delta+b_h)^\frac{\nu}{\tau})\]

where \(\omega = \min\{1, \nu+1/2\}, \ \nu = 1 - \frac{p}{2(q+1)}(1+(1-\omega)/q)\) and \(\tau = \min\{\frac{p\nu}{q+1}, 1\}\).

**Proof:** We recall from Theorem 2.1.1 (i) and Lemma 2.2.2 that for \(\tilde{x} \in R((T^*T)^\nu), 0 < \nu \leq 1\),

\[\|\tilde{x} - x\| \leq c\alpha^\nu\ \text{and} \ \|R\tilde{x}\| \leq c\omega.\]

Therefore using the assumption \(\frac{p}{q+1} \leq \frac{1}{\nu}\) it follows from (4.15) that

\[\frac{(\delta+b_h)^p}{\alpha^\nu} \leq c(\delta+b_h)^{\frac{p\nu}{\tau}},\]

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so that for any \( s > 0 \), we have

\[
\frac{(\delta+b_h)^s}{\sqrt[2/\alpha]{\alpha}} \leq (\delta+b_h)^{s-\frac{d}{2(1-w)}}<2^{q+1}\left(\frac{\delta+b_h}{\sqrt[2q+1]{\alpha}}\right)^{1/2q}
\]

\[
\leq c(\delta+b_h)^{s-\frac{d}{2q}+\frac{d}{2q+1}}
\]

From this we have

\[
\frac{I_d}{\sqrt[2/\alpha]{\alpha}} \leq \frac{(\delta+b_h)^\lambda}{\alpha} = \mathcal{O}(1) \quad \text{and} \quad \frac{\delta}{\sqrt[2/\alpha]{\alpha}} \leq \frac{\delta+b_h}{\sqrt[2/\alpha]{\alpha}} = \mathcal{O}(\delta+b_h^{1/2})
\]

where \( I = 1 - \frac{P}{2(q+1)}(1+(1-w)/q) \). Note that \( I \geq 1 - \frac{(2q+1)p}{4q(q+1)} \), so that by the assumption \( \frac{P}{q+1} < \frac{4q}{2q+1} \), we have \( \frac{\delta}{\sqrt[2/\alpha]{\alpha}} = \mathcal{O}(1) \). Now the result in (i) follows from (5.4) by using the arguments used in Theorem 2.2.1 and (ii) follows from (5.4).

Remarks 5.1.2. We note that if \( \nu \geq 1/2 \), then \( \hat{x} \in \mathbb{R}(T^*) \), so that \( \|\hat{I}-P_h\|\hat{x}\leq c\|\hat{I}(I-P_h)\| \). Thus we may take \( k \geq \lambda \). We consider two special cases.

Case (i) \( \hat{x} \in \mathbb{R}(T^*) \), i.e., \( \nu = 1/2 \) :- Let \( \frac{P}{q+1} = 1 \). Taking \( k \geq \lambda = \frac{2q+1}{4q} \) in Theorem 5.1.1 (ii), we obtain the rate

\[
\|\hat{x}-x\|_{\alpha,h} = \mathcal{O}(\delta+b_h^{1/2}).
\]

For obtaining the same result Engl and Neubauer [6] requires the condition \( \lambda = \frac{q+1}{2q} \), which is stronger than ours. As in [6], from Theorem 5.1.1 (ii), we also obtain the rate arbitrarily close to

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\( \alpha(\gamma_h) \) for large values of \( q \) provided

\[
 b_h \sim \frac{4q}{\sqrt{q+1}} \quad \text{and} \quad \delta = \delta_h = \alpha(b_h).
\]

Case (ii) \( \hat{x} \in R(T^*T) \), i.e., \( \nu = 1 \) - Let \( \frac{D}{q+1} = \frac{2}{3} \). Taking \( k \approx \lambda := \frac{2q+1}{6q} \), the rate in Theorem 5.1.1 (ii) becomes \( \Omega(\delta^2b_h^3) \)

with \( s = \min(2/3, k) \). In this case, if \( \delta = \delta_h \leq cb_h \sim \frac{4q}{\sqrt{q+1}} \)

then the rate is

\[
\|\hat{x} - x^\delta_{a,h}\| = \Omega(\gamma_h^t), \quad t = \frac{2q}{2q+1} \min\{2, 3k\}.
\]

Note that if \( q \approx 1/2 \), then \( t \approx 1 \), and if \( q > 1/2 \) and \( k \approx \lambda := \frac{2q+1}{6q} \) then \( t > 1 \). In particular if \( k \approx 2/3 \) then the rate

\[
\|\hat{x} - x^\delta_{a,h}\| = \Omega(\gamma_h^{\frac{4q}{2q+1}})
\]

is arbitrarily close to \( \alpha(\gamma_h^2) \) for large values of \( q \), where as the result in [6] can give only up to \( \alpha(\gamma_h) \). Since \( T^*T \) is self-

adoint, we have \( \|I(P_h)\hat{x}\| = \Omega(\|T^*T(I-P_h)\|) \), so that the condition \( k \approx 2/3 \) is satisfied if the operator \( T \) has the property

\( \|T^*T(I-P_h)\| = \Omega(\|T(I-P_h)\|^2) \) for then one can take \( k = 2\lambda = \frac{2}{3} + \frac{1}{3q} \).

Such cases do occur. For example, suppose that \( T \) is an injective compact operator with \( R(T) \) dense in \( Y \). Let \( \{\sigma_k\} \) be the set of singular values of \( T \) satisfying \( \sigma_1 > \sigma_2 \geq \ldots \), and \( \{u_k\} \) and \( \{v_k\} \) be orthogonal basis of \( X \) and \( Y \) respectively such that \( Tu_k = \sigma_k u_k, \quad T^*v_k = \sigma_k v_k \) for \( k = 1, 2, \ldots \). If \( h = 1/n, n = 1, 2, \ldots \). \( P_h \)
is the orthogonal projection of \( X \) onto \( V_n := \text{span} \{ u_1, \ldots, u_n \} \) then it can be seen that \( \| T^* T (I - P_h) \| = \| T (I - P_h) \|^2 = \sigma_{m1}^2 \).

Marti (See [28], [29], [30]), used an algorithm to compute approximate solution for the equation (5.1). In this method, a sequence of finite dimensional subspaces \( V_1 \subset V_2 \subset \ldots \) of \( X \) with

\[
\bigcup_{n \in \mathbb{N}} V_n = X
\]

is used to obtain an approximate solution \( x_n \) of (5.1). More precisely, let for \( n \in \mathbb{N} \)

\[
a_n = \inf \{ \| T x - y \| : x \in V_n \},
\]

\( P_h, \ h = 1/n, \) be the orthogonal projection of \( X \) onto \( V_n \), and \( b_h > 0 \) be chosen such that

\[
\lim_{h \to 0} \frac{\| P_h x - x \|}{b_h} = 0, \quad \lim_{h \to 0} b_h = 0.
\]

Then \( x_n \) is defined by

\[
x_n \in V_n,
\]

\[
\| T x_n - y \|^2 \leq a_n^2 + b_n^2,
\]

\[
\| x_n \| = \inf \{ \| x \| : x \in V_n \text{ and satisfies (5.6)} \}
\]

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In [12, Section 4.3], Groetsch has reformulated Marti's method as solving

$$\alpha x_n + T_h^* T_h x_n = T_h^* y,$$

for $x_n$ with the regularization parameter $\alpha$ is determined by

$$\|T x_n - y\|^2 = a_n^2 + b_h^2.$$  

Thus the method (5.2), (5.5) is a generalization and modification of Marti's method.

**Projection Method for Simplified Regularization:**

Here we consider the case when the operator under consideration is a positive self adjoint operator. More precisely we consider the operator equation

$$Aw = g, \quad g \in \mathbb{R}(A),$$  

where $A$ is a positive self adjoint operator on $X$. As earlier, let $(P_h)_{h>0}$ is a family of finite rank orthogonal projection on $X$. In this case the projection method for the equation

$$(A + \alpha I)w_{\alpha}^\delta = g^\delta,$$

will take the form.
\[(5.9) \quad (P_h AP_h + \alpha I)u_\delta^{\alpha, h} = P_h g^{\delta},\]

where \(g^{\delta} \in F^{\delta} = \{u \in X : \|u - g\| \leq \delta\} \).

Note that

\[u^{\delta}_{\alpha, h} = P_h w^{\delta}_{\alpha, h},\]

where \(w^{\delta}_{\alpha, h}\) is the solution of the equation

\[(5.10) \quad (P_h AP_h + \alpha I)w^{\delta}_{\alpha, h} = g^{\delta}.\]

In fact the equation (5.10) is a particular case of (4.21) obtained by taking \(A_h = P_h AP_h\). Hereafter we use the notation \(A_h\) instead of \(P_h AP_h\). Let \(\|A - AP_h\| = o(b_h)\), where \((b_h)_{h>0}\) is a family of positive reals such that \(b_h \to 0\) as \(h \to 0\).

The following Theorem is a companion result of Theorem 4.3.1.

**Theorem 5.1.3.** Let \(u^{\delta}_{\alpha, h}\) be defined as in (5.9) and \(\hat{w}\) be the minimal norm solution of the equation (5.8). If \(\hat{w} \in R(A^\nu)\), \(0 < \nu \leq 1\), then

\[\|\hat{w} - u^{\delta}_{\alpha, h}\| \leq c_1 \frac{\delta + b_h}{\alpha} + c_2 \alpha^\nu\]

where \(c_1\) and \(c_2\) are positive constants.

**Proof:** Note that
Now from (5.9) and (5.10), we have

\[
\|w - u^\delta\|_{\alpha,h} \leq \|w - w^\delta\|_{\alpha,h} + \|w^\delta - u^\delta\|_{\alpha,h}.
\]

(5.11)

Then from (5.11), we have

\[
\alpha \|w - w^\delta\|_{\alpha,h} = \|P_h - I\| g^\delta
\]

\[
\leq \|P_h - I\| (g^\delta - g) + \|P_h - I\| g
\]

\[
\leq \|g - g\| + \|P_h - I\| \hat{w}
\]

\[
\leq \delta + \|P_h - I\| (A - P_h A) \hat{w}
\]

\[
\leq \delta + \|A - P_h A\| \hat{w}
\]

Since \(\|A - A P_h\| = \|A - P_h A\|\), we obtain

\[
\alpha \|w - w^\delta\|_{\alpha,h} \leq \delta + b_h \|\hat{w}\|.
\]

Thus from (5.11), we have

(5.12)

\[
\|w - u^\delta\|_{\alpha,h} \leq \|w - w^\delta\|_{\alpha,h} + \max(1, \|\hat{w}\|) \frac{\delta + b_h}{\alpha}
\]

Now the result follows from Theorem 4.3.1, with \(A_h = P_h A P_h\) and \(\varepsilon_h = \alpha(b_h)\).

For choosing the regularization parameter \(\alpha\) in (5.9), we first consider the discrepancy principle.
which is a modified form of (4.24). Imitating the proof of Theorem 4.3.2 (i) and (ii), it can be seen that there exists a unique $a := a(\delta, h)$ such that (5.13) is satisfied and that

$$\alpha(\delta, h) \leq c(\delta + b_h)^{\frac{p}{q+1}}, \quad 0 < \delta \leq \delta_0, \quad h \in \mathcal{H}. \quad (5.14)$$

Theorem 5.1.4. Let $a = a(\delta, h)$ be chosen according to (5.13). Then

(i) If $p < q+1$ and $b_h \to 0$ as $h \to 0$, then

$$\|u_{a, h}^\delta\| \to 0 \quad \text{as} \quad \delta \to 0, \quad h \to 0.$$ 

(ii) $\|u_{a, h}^\delta\| = O((\delta + b_h)^{\frac{p}{q+1}})$

where $r = \min(\frac{p}{q+1}, 1 - \frac{p}{q+1})$.

In particular if $\frac{p}{q+1} = \frac{1}{q+1}$, then

(iii) $\|u_{a, h}^\delta\| = O((\delta + b_h)^{\frac{p}{q+1}})$. 

Proof: Note that

$$\frac{(\delta + b_h)^p}{a^q} = \|P_h u_{a, h}^\delta - P_h \varrho^\delta\|$$

(5.15)
\[ \|a(P_hA+aI)^{-1}P_h\delta\| \]
\[ \leq \|a(P_hA+aI)^{-1}P_h(g\delta-g)\| + \|\alpha(P_h|A+\alpha I) - I\| \]
\[ \leq \delta + \|\alpha[(P_hA+aI)^{-1}P_h - (A+aI)^{-1}]g\| \]
\[ + \|\alpha(A+aI)^{-1}g\| \]
\[ \leq \delta + \|\alpha(P_hA+aI)^{-1}[(P_h(A+\alpha I) - (P_hA+aI))] \]
\[ (A+aI)^{-1}]g\| + \|\alpha(A+aI)^{-1}g\| \]
\[ \leq \delta + \|\alpha^2(A_h+aI)^{-1}(P_h-I)(A+aI)^{-1}g\| \]
\[ + \|\alpha(A+aI)^{-1}g\| \]

Note that
\[ \|\alpha^2(A_h+aI)^{-1}(P_h-I)(A+aI)^{-1}g\| = \|\alpha^2(A_h+aI)^{-1}(P_h-I)A(A+aI)^{-1}\| \]
\[ \leq b_h\|\| \]

and
\[ \|\alpha(A+aI)^{-1}g\| \leq a\|\| \]

Therefore from (5.15) it follows that

\[ \frac{(\delta+b_h)^p}{\alpha^q} \leq c_1(\delta+b_h) + c_2, \]
where $c_1$ and $c_2$ are positive constants. Now from (5.14) and
the fact that $p < q+1$, we have

$$\frac{(\delta+b_h)^p}{\alpha} = O((\delta+b_h)^{p/q}).$$

Therefore

$$\frac{\delta+b_h}{\alpha} = (\delta+b_h)^{1 - \frac{p}{q}} \cdot O((\delta+b_h)^{p/q}).$$

i.e.,

(5.16)  

$$\frac{\delta+b_h}{\alpha} = O((\delta+b_h)^{1 - \frac{p}{q}}).$$

Therefore $\frac{\delta+b_h}{\alpha} \to 0$ as $\delta \to 0$, $h \to 0$. Also by Theorem 4.3.2 (iii) we have $\|w-w^\delta\|_{\alpha,h} \to 0$. Thus by (5.12), $\|w-w^\delta\|_{\alpha,h} \to 0$ as $\delta \to 0$, $h \to 0$. Now (ii) follows by applying the estimates in (5.14), (5.16) to the estimate in Theorem 5.1.3. The proof of (iii) is a consequence of (ii) and the fact that $\frac{p\gamma}{q+1} = 1 - \frac{p}{q+1}$ if and only if $\frac{p}{q+1} = \frac{1}{p+1}$, and in that case $\frac{p}{q+1} = \frac{\nu}{\nu+1}$.

Remark 5.1.5. As in Remark 3.1.5, the above method for simplified regularization can be used for Tikhono regularization also by taking $A = T^*T$, $g = T^*y$ and $g^\delta = T^*y^\delta$, $\|y-y^\delta\| \leq \frac{\delta}{c}$ where $c \geq \|T^*\|$. In this case the estimate $\alpha = O((\delta+b_h)^{p/q})$ of (5.14) can be used to obtain the estimate.
\[ \frac{\delta + b h}{\sqrt{\alpha}} = \mathcal{O}\left((\delta + b h)^{1-\frac{\rho}{2m}}\right). \]

Therefore if \( p < 2(q+1) \) and \( \hat{x} \in R((T*I)^{\nu}),\ 0 < \nu \leq 1 \), then we have

\[ \|\hat{x} - x^\delta\| = \mathcal{O}\left((\delta + b h)^m\right) \]

where \( m = \min\left\{ \frac{\rho \nu}{m}, 1 - \frac{\rho}{2m} \right\} \).

This, in particular, gives the optimal estimate \( \mathcal{O}\left((\delta + b h)^{\frac{2\nu}{2m+1}}\right) \) for \( \frac{\rho}{m} = \frac{2}{2m+1} \).

Next we use a modified form of the discrepancy principle (4.26), namely,

(5.17) \[ \alpha^2(\rho+1)(p_h A P_h + \alpha I)^{-\frac{\rho}{2}} P_h g^\delta, P_h g^\delta = (c \delta + db_h)^2, \]

where \( c \) and \( d \) are positive constants, for choosing the regularization parameter \( \alpha \) in (5.9). Before proving the existence and uniqueness of \( \alpha \) satisfying (5.17) we prove the following result.

Proposition 5.1.6. Let \( g \neq 0 \) and \( g^\delta \in F_\delta \). Then there exists \( \delta_0 > 0, \ h_0 > 0 \) such that

\[ \|(I-Q_h)P_h g^\delta\| \leq c \delta + db_h \leq \|P_h g^\delta\| \]

for all \( \delta \leq \delta_0 \) and \( h \leq h_0 \), where \( c > 1, \ d \) are constants.
and $Q_h$ is the orthogonal projection onto $\mathbb{R}(P_hAP_h^*)$.

Proof: Since $P_h \to I$ pointwise as $h \to 0$ and $g^\delta \to g$ as $\delta \to 0$ we have

$$\|P_h g^\delta\| \to \|g\| \neq 0 \quad \text{and} \quad c\delta + db_h \to 0 \quad \text{as} \quad \delta \to 0, \quad h \to 0.$$ 

Therefore there exists $\delta_0 > 0$ and $h_0 > 0$ such that

$$c\delta + db_h \leq \|P_h g^\delta\|, \quad \text{for all} \quad \delta < \delta_0 \quad \text{and} \quad h < h_0.$$ 

Also, since $P_h$ and $Q_h$ are orthogonal projections and $g = \hat{A} \hat{w}$,

$$\| (I-Q_h)P_h g^\delta \| \leq \| (I-Q_h)P_h (g^\delta-g) \| + \| (I-Q_h)P_h g \|$$

$$\leq \delta + \| (I-Q_h)P_h \hat{A} \hat{w} \|$$

But $\| (I-Q_h)P_h AP_h \| = 0$, so that

$$\| (I-Q_h)P_h g^\delta \| \leq \delta + \| (I-Q_h)P_h (A - AP_h) \hat{w} \|$$

$$\leq \delta + b_h \| \hat{w} \|$$

for all $\delta > 0$ and $h > 0$. This completes the proof of the Proposition. \qed

Lemma 5.1.7. Let $\delta_0 > 0$ and $h_0 > 0$ be as in Proposition 5.1.6. Then for $\delta \leq \delta_0$, $h \leq h_0$, there exists a unique $\alpha := \alpha(\delta, h)$ satisfying (5.17).
Proof: For fixed $0 < \delta \leq \delta_0$, $h \leq h_0$, let

$$\phi(\alpha) = \alpha^{2(\rho+1)}((p_1\mathcal{A}p_1^*aI)^{-2(\rho+1)}p_h\delta, p_h\delta).$$

Then as in Lemma 3.2.1,

$$\phi(\alpha) = \int \left( \frac{\alpha}{\alpha + \lambda} \right)^2(\rho+1)d\langle E_\lambda p_h\delta, p_h\delta \rangle.$$  

where $(E_\lambda)$ is the spectral family of the operator $p_1\mathcal{A}p_1^*$.

Now the map

$$\alpha \mapsto f_\rho(\alpha, \lambda) = \left( \frac{\alpha}{\alpha + \lambda} \right)^2(\rho+1)$$

is strictly increasing for each $\lambda > 0$, and satisfies

$$f_\rho(\alpha, \lambda) \to 0 \text{ as } \alpha \to 0$$

and

$$f_\rho(\alpha, \lambda) \to 1 \text{ as } \alpha \to \infty.$$  

Therefore by Dominated Convergence Theorem we have

(5.18) \hspace{1cm} \phi(\alpha) \to \|E_0 p_h\delta\|^2 \text{ as } \alpha \to 0

where $E_0$ is the projection on to $\mathcal{R}(p_1\mathcal{A}p_1^*)$ and

(5.19) \hspace{1cm} \phi(\alpha) \to \|p_h\delta\|^2.
Now since
\[ E_0 P_h \delta = E_0 Q_h P_h \delta + E_0 (I-Q_h) P_h \delta \]
\[ = E_0 (I-Q_h) P_h \delta. \]
Thus
\[ \|E_0 P_h \delta\| = \|E_0 (I-Q_h) P_h \delta\| \]
\[ \leq \|I-Q_h\| P_h \delta. \]

This together with Proposition 5.1.6, gives
\[ \|E_0 P_h \delta\| \leq c\delta + d \leq \|P_h \delta\| \]
for all \( \delta \leq \delta_0, h \leq h_0 \). Now the Lemma follows by Intermediate value Theorem by using (5.18) and (5.19).

We note that
\[ \|A-P_h A P_h\| \leq \|A-P_h A\| + \|P_h A P_h\| \]
\[ \leq 2 \|A-P_h A\| \]
\[ \leq 2b_h. \]

Therefore, if \( \alpha = \alpha(\delta,h) \) satisfies (5.17), then Lemma 4.4.1 and Lemma 4.4.3 holds with \( 2b_h \) in place of \( \epsilon_h \). Thus in view of Theorem 5.1.3, we have the following result which is same as Theorem
4.4.4 with $u_{\alpha,h}$, in place of $w_{\alpha,h}$.

Theorem 5.1.8. Let $q_\delta \in F_\delta$, $h \in H$ and let $\alpha := \alpha(\delta,h)$ be the unique solution of (5.17) with $c > 1$ and $d > e = 2(2+\rho)\|\hat{w}\|$. Let $\hat{w} \in R(\lambda P)$, $0 < \nu \leq 1$. Then

$$\|\hat{w} - u_{\alpha,h}\| = \begin{cases} \alpha((\delta+b_h)^{\nu}, & 0 < \nu < 1 \\ \alpha((\delta+b_h)^{\nu}, & \nu = 1. \end{cases}$$

If $0 < \nu \leq 1$ and $\nu < \rho$, then

$$\|\hat{w} - u_{\alpha,h}\| = \alpha((\delta+b_h)^{\nu}).$$

In particular if $\rho = 1$ in (5.17), then

$$\|\hat{w} - u_{\alpha,h}\| = \begin{cases} \alpha((\delta+b_h)^{\nu}, & 0 < \nu < 1 \\ \alpha((\delta+b_h)^{\nu}, & \nu = 1. \end{cases}$$

5.2. ALGORITHMS

In this Section we give algorithms for implementing the methods considered in Sections 5.1. Let $(V_n)$ be a sequence of finite dimensional subspaces of $X$ and $P_n$ denote the orthogonal
projection on $X$ with $R(P_n) = V_n$. We assume that $\dim V_n = n$, and
\begin{equation}
(5.20) \quad \|P_n x - x\| \to 0 \quad \text{as} \quad n \to 0,
\end{equation}
for all $x \in X$. Let $(v_1, \ldots, v_n)$ be a basis of $V_n$, $n = 1, 2, \ldots$.

Algorithm 5.2.1. Let $T \in BL(X,Y)$ be a compact operator and let $T_h = TP_n$ where $h = \frac{1}{n}$. Now by assumption (5.20) and the fact that $T^*$ is compact, we have $\|T - T_h\| = \|T - TP_n\| = \|(I - P_n)T^*\| \to 0$ as $n \to \infty$.

With the above notation (5.2) takes the form
\begin{equation}
(5.21) \quad (P_n T^* TP_n + \alpha I)x^\delta = P_n T^* y^\delta.
\end{equation}

From (5.21) it follows that
\begin{equation}
x^\delta_{\alpha, h} = \frac{1}{\alpha} (P_n T^* y^\delta - P_n T^* TP_n x^\delta_{\alpha, h})
= \frac{1}{\alpha} (T^* y^\delta - T^* TP_n x^\delta_{\alpha, h}) \in V_n.
\end{equation}

Thus $x^\delta_{\alpha, h}$ is of the form \( \sum_{i=1}^{n} \lambda_i v_i \) for some scalars \( \lambda_1, \ldots, \lambda_n \).

It can be seen that $x^\delta_{\alpha, h} = \sum_{i=1}^{n} \lambda_i v_i$ is the solution of (5.21) if and only if \( \mathbf{\lambda} = (\lambda_1, \ldots, \lambda_n)^T \) is the unique solution of
\begin{equation}
(5.22) \quad (M_n + \alpha B_n)\mathbf{\lambda} = \mathbf{w}_n
\end{equation}

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where

\[ M_n = (\langle Tv_i, Tv_j \rangle), \ i, j = 1, \ldots, n, \]
\[ B_n = (\langle v_i, v_j \rangle), \ i, j = 1, \ldots, n, \]
and
\[ W_n = (\langle y^\delta, Tv_1 \rangle, \ldots, \langle y^\delta, Tv_n \rangle)^T. \]

Here and below \((\beta_1, \ldots, \beta_n)^T\) denotes the transpose of \((\beta_1, \ldots, \beta_n)\).

Note that (5.22) is uniquely solvable because \(M_n\) is a positive definite matrix (i.e., \(x^T M_n x > 0\) for all non-zero vector \(x\)) and \(B_n\) is an invertible matrix. The parameter \(\alpha\) in (5.22) is chosen according to (5.5), which is same as

\[
(5.23) \quad \|T x^\delta - y^\delta\| = \frac{(\delta+b_h)^p}{\alpha^q}.
\]

This is equivalent to solving the non linear equation

\[
(5.24) \quad f(\alpha) := \alpha^2 [ -\alpha C(\alpha) B_n x(\alpha)^T - W_n^T \lambda(\alpha) + y^\delta, y^\delta] - (\delta+b_h)^2 \alpha = 0
\]

where \(\lambda(\alpha)\) is the solution of (5.22). The parameter \(\alpha = \alpha(\delta, b_h)\) satisfying (5.24) can be found as follows.

Step 1 For some initial (good) approximation \(\alpha_0 > 0\) find \(\lambda(\alpha_0)\) satisfying (5.22). Specifically we use cholesky decomposition to compute \(\lambda(\alpha_0)\).
Step 2 (Newton's method) Using $\lambda(\alpha_0)$ of Step 1 compute

$$\alpha_1 = \alpha_0 - \frac{f(\alpha_0)}{f'(\alpha_0)}$$

where

$$f(\alpha) = a^2 \left[ -\alpha \lambda(\alpha)^T B_n \lambda(\alpha) - \Omega_n^T \lambda(\alpha) + \langle y_\delta, y_\delta \rangle \right] - (\delta + b_n)^2$$

and

$$f'(\alpha) = 2a a^{2q-1} \left[ -\alpha \lambda(\alpha)^T B_n \lambda(\alpha) - \Omega_n^T \lambda(\alpha) + \langle y_\delta, y_\delta \rangle \right]$$

$$+ a^2 \left[ -\lambda(\alpha)^T B_n \lambda(\alpha) - 2\alpha \lambda(\alpha)^T B_n \lambda(\alpha) - \Omega_n^T \lambda(\alpha) \right].$$

Repeat Step 1 with $\alpha_1$ in place of $\alpha_0$ and Step 2 with $\lambda(\alpha_1)$ in place of $\lambda(\alpha_0)$ and so on. In the $2m$th step, we obtain

$$\alpha_m = \alpha_{m-1} - \frac{f(\alpha_{m-1})}{f'(\alpha_{m-1})}.$$  

(5.25) For sufficiently good initial approximation, the iterates in (5.25) converges to $\alpha(\delta, h)$, the zero of the function $f(\alpha)$.

Algorithm 5.2.2. Let $A \in BL(X)$ be a compact positive self-adjoint operator and let $A_n = P_n AP_n$ where $h = \frac{1}{n}$. In this case we consider the equation (5.9), i.e.,

$$P_n AP_n u_{\delta, h}^{\alpha, h} + \alpha u_{\delta, h}^{\alpha, h} = P_n g_{\delta}.$$  

(5.26) As in Algorithms 5.2.1, it can be seen that $u_{\alpha, h}^{\delta, h} \in V_n$. Thus
$u_{\alpha, h}^\delta$ is of the form $\sum_{i=1}^{n} \mu_i v_i$ for some scalars $\mu_1, \ldots, \mu_n$. We note that $u_{\alpha, h}^\delta = \sum_{i=1}^{n} \mu_i v_i$ is the solution (5.26) if and only if $\bar{\mu} = (\mu_1, \ldots, \mu_n)^T$ is the solution of

$$\text{(5.27) } (A_n + \alpha B_n) \bar{\mu} = Y_n$$

where

$$A_n = (\langle Av_i, v_j \rangle), \quad i, j = 1, \ldots, n,$$

$$B_n = (\langle v_i, v_j \rangle), \quad i, j = 1, \ldots, n,$$

and

$$Y_n = (\langle y^\delta, v_1 \rangle, \ldots, \langle y^\delta, v_n \rangle)^T.$$ 

The parameter $\alpha$ in (5.27) is chosen according to (5.13), i.e.,

$$\text{(5.28) } \|P_n A u_{\alpha, h}^\delta - P_n g^{\delta} \| = \frac{(\delta + b_h)^p}{\alpha^q}.$$ 

This is equivalent to solving the non linear equation

$$\text{(5.29) } g(\alpha) := \alpha^{2q+2} (\mu(\alpha)^T B_{\alpha} \mu(\alpha)) - (\delta + b_h)^{2p} = 0.$$

The parameter $\alpha = \alpha(\delta, b_h)$ satisfying (5.29) can be found as follows.

**Step 1** For some initial approximation $\alpha_0 > 0$ find $\mu(\alpha_0)$ satisfying (5.27).
Step 2 Using \( \mu(\alpha_0) \) of step 1 compute

\[
\alpha_1 = \alpha_0 - \frac{g(\alpha_0)}{g'(\alpha_0)}
\]

where

\[
g(\alpha) = \alpha^{2q+2} \frac{3}{\mu(\alpha)^T B_n \mu(\alpha)} - (\delta + b_h)^2
\]

and

\[
g'(\alpha) = 2(q+1)\alpha^{2q+1} \frac{\mu(\alpha)^T}{\mu(\alpha)^T B_n \mu(\alpha)} + 2\alpha^{2q+2} \mu(\alpha)^T B_n \mu(\alpha)
\]

As in Algorithm 5.2.1, in the 2mth step we have

\[
(5.30) \quad \alpha_m = \alpha_{m-1} - \frac{g(\alpha_{m-1})}{g'(\alpha_{m-1})}.
\]

For sufficiently good initial approximation, the iterates (5.30) converges to \( \alpha(\delta, h) \), the zero of the function \( g(\alpha) \). We note that the procedure in the above Algorithm is similar to the one given in Engl and Neubauer [6] with \( A = T^* T \) and \( g^\delta = T^* y^\delta \).

Algorithm 5.2.3. Let \( A \) be as in Algorithm 5.2.2. We choose the regularization parameter according to the discrepancy principle (5.17), i.e.,

\[
(5.31) \quad \alpha^{X^{P+1}} \langle (A + \alpha I)^{-2(P+1)} P_n g^\delta, P_n g^\delta \rangle = (c\delta + db_h)^2.
\]

We consider only two values of \( \rho \), namely, \( \rho = 1/2 \) and \( \rho = 1 \).

Case 1 Let \( \rho = 1/2 \). Then (5.31) takes the form
In this case, choosing the parameter $\alpha$ satisfying (5.32) is equivalent to solving the non linear equation

$$\alpha^3 ((A + \alpha I)^{-3} P_n g^\delta \cdot P_n g^\delta) = (c\delta + db_h)^2.$$  

where $\alpha$ is the solution of the equation (5.27) and $\zeta(\alpha) := (\zeta_1(\alpha),...,\zeta_n(\alpha))$ is the solution of

$$(A_n + \alpha B_n)\zeta(\alpha) = \alpha (\zeta_1(\alpha),...,\zeta_n(\alpha)).$$

Now the parameter $\alpha := \alpha(\delta, b_n)$ satisfying (5.33) can be found as follows.

**Step 1** For $\alpha_0 > 0$ find $\mu(\alpha_0)$ satisfying (5.27). We use-cholosky decomposition for computing $\mu(\alpha_0)$.

**Step 2** Using $\mu(\alpha_0)$ compute, $\zeta(\alpha_0)$ satisfying (5.34). Here also we use cholosky decomposition.

**Step 3** Using $\mu(\alpha_0)$ and $\zeta(\alpha_0)$ compute

$$\alpha_1 = \alpha_0 - \frac{h(\alpha_0)}{h'(\alpha_0)}$$

where

$$h(\alpha) = \alpha^3 \zeta(\alpha)^T B_n \mu(\alpha) - (c\delta + db_h)^2.$$
and

\[ h''(\alpha) = 3\alpha^2 \zeta(\alpha)^T B_{n}(\alpha) + \alpha^3 \zeta(\alpha)^T B_{n}(\alpha) + \zeta(\alpha)^T B_{n}(\alpha)'. \]

Repeat the process with \( \alpha_1 \) to compute \( \alpha_2 \) and so on. In the 3\textsuperscript{rd} step we have

\[ \alpha_m = \alpha_{m-1} - \frac{h(\alpha_{m-1})}{h'(\alpha_{m-1})}. \]

For sufficiently good initial approximation, the iterate in (5.35) converges to the zero of \( h(\alpha) \).

**Case 2** Let \( p = 1 \). In this case (5.31) becomes

\[ \alpha^4 (A + \alpha I)^{-4} p \sigma \delta, p \sigma \delta = (c\delta + db_h)^2. \]

Now choosing the parameter \( \alpha \) satisfying (5.36) is equivalent to solving the equation

\[ k(\alpha) = \alpha^4 \zeta(\alpha)^T B_{n}(\alpha) - (c\delta + db_h)^2 = 0 \]

where \( \zeta(\alpha) \) is the solution of the equation (5.34).

The parameter \( \alpha := \alpha(\delta, b_h) \) satisfying (5.37) can be found as follows.

**Step 1** For \( \alpha_0 > 0 \) find \( \mu(\alpha_0) \) satisfying (5.27).
Step 2 Using cholosky decomposition and \( \mu(\alpha_0) \) of step 1, compute \( \zeta(\alpha_0) \) satisfying (5.34).

Step 3 Using \( \zeta(\alpha_0) \) compute

\[
\alpha_1 = \alpha_0 - \frac{k(\alpha)}{k'(\alpha_0)}
\]

where

\[
k(\alpha) = \alpha^4 \zeta' \zeta B_n \zeta - (c_0 + d_0)^2
\]

and

\[
k'(\alpha) = 4 \alpha^3 \zeta' \zeta B_n \zeta + 2 \alpha^4 \zeta^2 B_n \zeta
\]

Now as in Case 1, in the 3\textsuperscript{rd} step we have

\[
(5.38) \quad \alpha_m = \alpha_{m-1} - \frac{k(\alpha_{m-1})}{k'(\alpha_{m-1})}.
\]

Here also, for sufficiently good initial approximation, the iterates in (5.38) converges to the zero of \( k(\alpha) \).

5.3. NUMERICAL EXAMPLES

In order to illustrate the methods considered in Section 5.1, we consider the space \( X = Y = L^2[0,1] \) and consider the Fredholm integral equations of the first kind

\[
(5.39) \quad \int_0^1 k(s,t)x(t)dt = y(s)
\]
with \( k(s,t) \) defined by

\[
(5.40) \quad k(s,t) = \begin{cases} 
  s(1-t), & s \leq t \\
  t(1-s), & s > t
\end{cases}
\]

We apply the algorithms in Section 5.2 by choosing \( V_n \) as the space of linear splines in a uniform grid of \( n+1 \) points in \([0,1]\). Specifically for fixed \( n \) we consider \( t_i = \frac{i-1}{n}, \ i = 1, 2, \ldots, n+1 \) as the grid points. We take the basis function \( v_j, i = 1, \ldots, n+1 \) of \( V_n \) as follows:

\[
 v_j(t) = \begin{cases} 
  \frac{t_2-t}{t_2} & \text{if } 0 \leq t \leq t \leq t_2 \\
  0 & \text{if } t_2 \leq t \leq t_{n+1} = 1
\end{cases}
\]

for \( j = 2, \ldots, n \),

\[
 v_j(t) = \begin{cases} 
  0 & \text{if } 0 \leq t \leq t_2 \\
  \frac{t-t_{j-1}}{t_j-t_{j-1}} & \text{if } t_{j-1} \leq t \leq t_j \\
  \frac{t_{j+1}-t}{t_{j+1}-t_j} & \text{if } t_j \leq t \leq t_{j+1} \\
  0 & \text{if } t_{j+1} \leq t \leq t_{n+1} = 1
\end{cases}
\]

and

\[
 v_{n+1}(t) = \begin{cases} 
  0 & \text{if } 0 \leq t \leq t_n \\
  \frac{t-t_n}{t_{n+1}-t_n} & \text{if } t_n \leq t \leq t_{n+1} = 1
\end{cases}
\]

Let \( P_n \) be the orthogonal projection onto \( V_n \). We note that for \( x \in C[0,1] \).
\[ \| P_n x - x \|_2 = \text{dist}(x, R(P_n)) \]
\[ \leq \| P_n x - x \|_2 \]
\[ \leq \| P_n x - x \|_\infty \]

where \( P_n \) is the (piecewise linear) interpolatory projection onto \( V_n \). It is known [27] that \( \| P_n x - x \|_\infty \to 0 \) as \( n \to \infty \). Therefore using the fact that \( C[0,1] \) is dense in \( L^2[0,1] \), it follows that

\[ \| P_n x - x \|_2 \to 0 \]

for all \( x \in L^2[0,1] \).

The elements \( T v_i, \ i = 1, \ldots, n+1 \) and the entries of the matrix \( B_n \) needed in the Algorithms are computed explicitly. Finally the scalar product, \( \langle T v_i, T v_j \rangle \) and \( \langle y^\delta, T v_j \rangle \), \( i, j = 1, \ldots, n+1 \) are computed by trapezoidal rule. For the operator \( T \) defined by (5.39) and (5.40), \( y_n = \| T - TP_n \| = O(n^{-2}) \) (see [17]). We take \( y^\delta(s) = y(s) + \delta, 0 \leq s \leq 1 \). The iterations in the algorithms have been stopped as soon as \( |\alpha_n - \alpha_{n-1}| \leq 10^{-7} \).

In the tables in Examples 5.3.1 and 5.3.2, \( e = \| x - x_{\alpha,h} \| \), \( \bar{e} = \| x - x^\delta_{\alpha,h} \| \) and the last column shows that we obtain the expected convergence rates.

Example 5.3.1 Here we use Algorithm 5.2.1.
a) Let \( y(s) = \frac{1}{24}(s - 2s^3 + s^4) \). Then the exact solution is 
\[ \hat{x} = T^t y(t) = \frac{1}{2}(t - t^2) \in R(T^t), \] since \( \hat{x} = T^t 1 \) (See [6]). In this example we take \( p = 2, \ q = 1 \) and \( b_h = 10^{-1/2}n^{-2} \) where \( h = 1/n \). According to Remark 5.1.2 (i) we should obtain the rate \( O(n^{-110^{-1/4}}) \). The computational result are as follows.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \alpha )</th>
<th>( e )</th>
<th>( e \cdot n \cdot 10^{1/4} )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>64</td>
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</tr>
</tbody>
</table>

b) We take \( y, \ p, \ q, \ b_h \) are as in (a) and \( \delta = n^{-210^{-1/2}}\|y\| \). According to Remark 5.1.2 we should obtain the rate \( O(n^{-110^{-1/4}}) \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \alpha )</th>
<th>( \delta )</th>
<th>( \delta \cdot n \cdot 10^{1/4} )</th>
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</tr>
</tbody>
</table>

c) Let \( y(s) = \frac{1}{30}(3s - 5s^3 + 3s^5 - s^8) \). Then 
\[ \hat{x} = T^t y(t) = (t - 2t^3 + t^4) \in R(T^t) \] (See [6]). Here we take \( p = 1, \ q = 1/2 \) (i.e., Arcangeli's method) and \( b_h = 10^{-1}n^{-2} \). In this case we
should get the rate $O(n^{-4/3}10^{-2/3})$.

<table>
<thead>
<tr>
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<th>$e$</th>
<th>$e \cdot n^{4/3} \cdot 10^{2/3}$</th>
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</table>

d) Here $\gamma$, $\rho$, $q$ are as in (c) and $b_h = (n^{-2})^\varphi$. Then by Remark 5.1.2 we should obtain the rate $O(\gamma_h) = O(n^{-2})$.

<table>
<thead>
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</tr>
</tbody>
</table>

e) We take $\gamma$, $\rho$, $q$ are as in (c), $b_h = 10^{-3}n^{-2}$ and $\delta = 10^{-1}n^{-2}y_\|$. According to Remark 5.1.2 we should get the rate $O(n^{-4/3}10^{-2/3})$.  

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f) Let $y, p, q$ and $b_h$ be as in (d) and $\delta = n^{-3/2}y_{\|}$. Again by Remark 5.1.2 we should obtain the rate $O(y_h) = O(n^{-2})$.

\[
\begin{array}{cccc}
 n & \alpha & \delta & \delta.n^{3/2}  \\
 4 & 9.764295E-02 & 2.006114E-01 & 5.912469  \\
 8 & 1.460831E-02 & 1.301518E-01 & 9.665779  \\
 16 & 4.005746E-03 & 6.221194E-02 & 11.642170  \\
 32 & 1.389137E-03 & 9.623386E-02 & 12.465050  \\
 64 & 5.236521E-04 & 1.079201E-02 & 12.823570  \\
\end{array}
\]

$g)$ Let $\gamma(s) = \frac{1}{6}(s - s^3)$. Then $\hat{x} = T^\dagger \gamma(t) = t \in R((T^T T)^\nu)$ for all $\nu < \frac{1}{8}$ (See [36]). Here we take $p = 1, q = \frac{1}{2}, b_h = 10^{-1/2}n^{-2}$ and $\nu = \frac{1}{8}$. According to Theorem 4.2.2 (ii), we should obtain the rate $O(n^{-1/2}10^{-1/12})$.
h) Let $y$, $p$, $q$ and $b_n$ be as in (g) and $\delta = 10^{-1/2}n^{-2}\|y\|$. 

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\alpha$</th>
<th>$e$</th>
<th>$e \cdot n^{1/6} \cdot 10^{1/12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2.032875E-01</td>
<td>6.933492E-01</td>
<td>1.058348</td>
</tr>
<tr>
<td>8</td>
<td>2.345335E-02</td>
<td>6.264760E-01</td>
<td>1.073378</td>
</tr>
<tr>
<td>16</td>
<td>5.452919E-03</td>
<td>5.571122E-01</td>
<td>1.071427</td>
</tr>
<tr>
<td>32</td>
<td>1.624781E-03</td>
<td>5.220351E-01</td>
<td>1.126915</td>
</tr>
<tr>
<td>64</td>
<td>4.782124E-04</td>
<td>4.951615E-01</td>
<td>1.199804</td>
</tr>
</tbody>
</table>

Now we illustrate the use of Algorithms 5.2.2 and 5.2.3 by considering the operator equation 

$$T^*Tx = T^*y$$

where $T: L^2[0,1] \rightarrow L^2[0,1]$ is given by 

$$(Tx)(s) = \int_0^1 k(s,t)x(t)dt.$$
Note that, the Simplified regularization of the above equation is the Tikhonov regularization of the equation $Tx = y$. With this observation we have the following Example.

Example 5.3.2. In the following cases, (a)-(c) and (d), we take $y$ as the corresponding $y$ in (a)-(c) and (e) of Example 5.3.1. We use Algorithm 5.2.2 to compute the regularization parameter $\alpha$ in (5.27).

a) We take $p = 2$, $q = 1$ and $b_h = 10^{-1} n^{-2}$ so that in view of Remark 5.1.5 we should obtain the rate $O(n^{-1}10^{-1/2})$. The following table gives the numerical results.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\alpha$</th>
<th>$e$</th>
<th>$e_{\text{n.}}10^{1/2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>5.022955E-02</td>
<td>7.582424E-02</td>
<td>9.591092E-01</td>
</tr>
<tr>
<td>8</td>
<td>6.638567E-03</td>
<td>3.598772E-02</td>
<td>9.104252E-01</td>
</tr>
<tr>
<td>16</td>
<td>1.377367E-03</td>
<td>1.125443E-02</td>
<td>5.694289E-01</td>
</tr>
<tr>
<td>32</td>
<td>3.284572E-04</td>
<td>3.808466E-03</td>
<td>3.853897E-01</td>
</tr>
<tr>
<td>64</td>
<td>8.114147E-05</td>
<td>1.702159E-03</td>
<td>3.444927E-01</td>
</tr>
</tbody>
</table>

b) We take $p = 2$, $q = 1$, $b_h = 10^{-1} n^{-2}$ and $\delta = 10^{-1} n^{-2} \|y\|$. Again in view of Remark 5.1.5 we should obtain the rate $O(n^{-1}10^{-1/2})$. 

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c) In this case $p, q$ are as in (a) and $b_h = 10^{-3/4} n^{-2}$ so that we should obtain the rate $\alpha(n^{-2} 10^{-3/4})$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\alpha$</th>
<th>$\delta$</th>
<th>$\alpha.n.10^{1/2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>5.379350E-02</td>
<td>7.481332E-02</td>
<td>9.463220E-01</td>
</tr>
<tr>
<td>8</td>
<td>7.374059E-03</td>
<td>3.651361E-02</td>
<td>9.237294E-01</td>
</tr>
<tr>
<td>16</td>
<td>1.525748E-03</td>
<td>1.154376E-02</td>
<td>5.840729E-01</td>
</tr>
<tr>
<td>32</td>
<td>3.632755E-04</td>
<td>3.827736E-03</td>
<td>3.873397E-01</td>
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<tr>
<td>64</td>
<td>8.970418E-05</td>
<td>1.688976E-03</td>
<td>3.418248E-01</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\alpha$</th>
<th>$\delta$</th>
<th>$\alpha.n^{10^{3/4}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>6.306100E-02</td>
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</tr>
<tr>
<td>8</td>
<td>7.833364E-03</td>
<td>9.601407E-02</td>
<td>34.555320</td>
</tr>
<tr>
<td>16</td>
<td>1.584625E-03</td>
<td>2.967627E-02</td>
<td>42.721780</td>
</tr>
<tr>
<td>32</td>
<td>3.754008E-04</td>
<td>7.853163E-03</td>
<td>45.221460</td>
</tr>
<tr>
<td>64</td>
<td>9.259419E-05</td>
<td>2.237586E-03</td>
<td>51.539440</td>
</tr>
</tbody>
</table>

d) We take $p = 2, q = 1, b_h = 10^{-3/4} n^{-2}$ and $\delta = 10^{-3/4} n^{-2} \|y\|$. Here we should get the rate $\alpha(n^{-2} 10^{-3/4})$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\alpha$</th>
<th>$\delta$</th>
<th>$\alpha.n^{10^{3/4}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>6.377255E-02</td>
<td>1.907804E-01</td>
<td>17.165390</td>
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<tr>
<td>8</td>
<td>7.932854E-03</td>
<td>9.639260E-02</td>
<td>34.691550</td>
</tr>
<tr>
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<td>1.603083E-03</td>
<td>2.985455E-02</td>
<td>42.978430</td>
</tr>
<tr>
<td>32</td>
<td>3.796421E-04</td>
<td>7.902579E-03</td>
<td>45.506020</td>
</tr>
<tr>
<td>64</td>
<td>9.363182E-05</td>
<td>2.235775E-03</td>
<td>51.654350</td>
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</tbody>
</table>
Example 5.3.3. The kernel $k(s,t)$ and $y$ in (a)-(d) are as that of corresponding part of Example 5.3.2. We choose the regularization parameter $\alpha$ in (5.27) according to Algorithm 5.2.3. In the tables below $e = \|\hat{w} - u_{\alpha,h}\|$ and $\bar{e} = \|\hat{w} - u_{\delta_{\alpha,h}}\|$.

a) Here we take $\rho = 1/2$, $b_h = 10^{-2}n^{-2}$ and $d = 1.5$. According to Theorem 4.4.3 (i), we should get the rate $\mathcal{O}(n^{10})^{-2/3}$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\alpha$</th>
<th>$e$</th>
<th>$e.(n.10)^{2/3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
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<td>8</td>
<td>5.457997E-03</td>
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<td>5.913156E-01</td>
</tr>
<tr>
<td>16</td>
<td>1.638970E-03</td>
<td>1.296893E-02</td>
<td>3.822236E-01</td>
</tr>
<tr>
<td>32</td>
<td>5.931827E-04</td>
<td>5.758975E-03</td>
<td>2.694296E-01</td>
</tr>
<tr>
<td>64</td>
<td>2.271117E-04</td>
<td>3.032624E-03</td>
<td>2.252191E-01</td>
</tr>
</tbody>
</table>

b) We take $\gamma$, $\rho$, $b_h$ are as in (a) and $\delta = 10^{-2}n^{-2}$. Let $c = 1.5$ and $d = 0.5$. By Theorem 5.1.5 (i), we should obtain the rate $\mathcal{O}(n.10)^{-2/3}$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\alpha$</th>
<th>$e$</th>
<th>$e.(n.10)^{2/3}$</th>
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</thead>
<tbody>
<tr>
<td>4</td>
<td>1.550571E-01</td>
<td>8.530082E-02</td>
<td>9.376845E-01</td>
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<tr>
<td>8</td>
<td>6.247536E-03</td>
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<td>6.280424E-01</td>
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<tr>
<td>16</td>
<td>1.827964E-03</td>
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<td>4.086806E-01</td>
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<td>32</td>
<td>6.562740E-04</td>
<td>6.124841E-03</td>
<td>2.865463E-01</td>
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<td>2.505496E-04</td>
<td>3.186567E-03</td>
<td>2.366518E-01</td>
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</tbody>
</table>
c) In this case $p = 1$, $b_h$ and $c$ are as in (a). According to theory we should get the rate $\mathcal{O}(n^{-1}10^{-1})$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\alpha$</th>
<th>$e$</th>
<th>$e \times n \times 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
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<td>$3.215186E-02$</td>
<td>5.144298</td>
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<td>$3.859191E-04$</td>
<td>$8.075576E-03$</td>
<td>5.168369</td>
</tr>
</tbody>
</table>

d) Let $p = 1$, $b_h$, $c$, $d$ be as in (b) and $\delta = 10^{-2}n^{-2}$. Here also we should get the rate $\mathcal{O}(n^{-1}10^{-1})$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\alpha$</th>
<th>$e$</th>
<th>$e \times n \times 10$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>$1.361748E-01$</td>
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<tr>
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