CHAPTER - I

INTRODUCTION

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PART - I  SURJECTIVITY THEOREMS

1.1 In nonlinear analysis one of the classical methods is to define and generalize a degree of nonlinear maps and to find a fixed point or prove the surjectivity of those maps. The mapping \( f : X \rightarrow Y \) is said to be surjective, if every element in \( Y \) is the \( f \)-image of at least one element in \( X \). In this case, the range of \( f \) is equal to \( Y \), that is, \( f(X) = Y \).

In 1962, E. Bishop and R.R. Phelps \([6]\) used Zorn's lemma to prove that if \( C \) is a closed convex subset of a Banach space \( X \), then the support points of \( C \) are dense in the boundary of \( C \). Moreover, if \( C \) is a closed bounded convex set in a Banach space \( X \), then the collection of functionals that achieve their maximum on \( C \) is dense in \( X^* \). This could be obtained by partially ordering a complete subset of a normed linear space using a certain convex cone.

Phelps \([56]\) generalized the results proved in \([6]\) in a real Hausdorff topological vector

On the other hand, J. Caristi and W.A. Kirk provided a fixed point theorem [15, Theorem 2.1] which generalizes the Banach contraction principle.

1.2 SURJECTIVITY OF EXPANSIVE MAPS

In 1976, Browder [7] proved that if $T$ is a locally expansive, continuous open map of $E$ into $F$, then $T$ is a homeomorphism onto $F$. 
The following theorem is a generalized version of Browder's result presented by Kirk and Schöenberg [46].

**THEOREM:**

Let $X$ and $Y$ be complete metric spaces with $Y$ metrically convex, and $T : X \to Y$ an open map having closed graph. Suppose also that $T$ is locally expansive on $X$. Then $T(X) = Y$.

In this context, we proved in chapter II two generalized version of surjectivity theorems in Banach spaces by using closed graph condition over functions.

**DEFINITION:** 1

A metric space $Y$ is said to be metrically convex if for all $u,v$ in $Y$ with $u \neq v$ there exists $w \in Y$, distinct from $u$ and $v$, such that $d(u,v) = d(u,w) + d(w,v)$.

**DEFINITION:** 2

Let $P : D(P) \subset X \to Y$ be a nonlinear operator from a linear subspace $D(P)$ of $X$ to a Banach space $Y$, is said to have closed graph if $x_n \to x$, $x_n \in D(P)$ and $Px_n \to Y$ imply $x \in D(P)$ and $y = Px$.

**DEFINITION:** 3

Let $c : (0, \infty) \to (0, \infty)$ be a continuous non increasing function such that $\int_0^\infty c(ks)ds = \infty$ whenever $k > 1$. A nonlinear map $T$ from a subset $B$ of a
Banach space $X$ into a metric space $Y$ is said to be locally $c$-expansive if each $x \in B$ has a neighbourhood $N$ of $x$ in $B$ such that

$$c(||x||, ||h||) ||u-v|| \leq d(Tu, Tv)$$

for all $u, v \in N$, $o \neq h \in X$ and $||h|| \leq 1$.

We precisely proved the following:

**THEOREM 1.**

Let $X$ be a Banach space and $Y$ a complete metric space with metric convexity, and $B$ open in $X$. Let $T: B \rightarrow Y$ have a closed graph. If $T$ is locally $c$-expansive and open on $B$, then for $y \in Y$ the following are equivalent:

1. $y \in T(B)$.
2. There exists $x_0 \in B$ such that $d(Tx_0, y) = d(Tx, y)$ for all $x \in B$.

### 1.3 SURJECTIVITY OF WEIGHTED $\phi$-ACCRETIVE OPERATORS

In 1984, J.A. Park and S. Park used the definitions of strongly $\phi$-accretive maps as following:

Let $X$ and $Y$ be Banach spaces with $Y^*$ the dual of $Y$, and let $\phi: X \rightarrow Y$ be a map satisfying:

1. $\phi(X)$ is dense in $Y$,
2. for each $x \in X$ and each $C \geq 0$,

$$||\phi(x)|| = ||x||$$

$A$ map $P:X \rightarrow Y$ is said to be strongly...
\( \phi \)-accretive if there exists a constant \( c > 0 \) such that for all \( u, v \in X \),

\[
\langle Pu - Pn, \phi(u-v) \rangle \geq c||u-v||^2.
\]

The various types of generalized forms of locally \( \phi \)-accretive operators obtained by J.A. Park and S. Park [52] and further J.A. Park and S. Park [53] replaced the constant \( c \) in the definitions of (locally) strongly \( \phi \)-accretive maps by a certain functional value and also extended the known surjectivity results.

In chapter-III we established the following types of weighted forms of locally \( \phi \)-accretive operators.

**RESULTS:** Let \( c : (0, \infty) \rightarrow (0, \infty) \) be a continuous non-increasing function. The weighted (locally) \( \phi \)-accretive maps considered as following:

\[
\langle Pu - Pn, \phi(u-v) \rangle \geq c(||u-v|| ||h||^{-1}) ||u-v||^2.
\]

\[
\langle Pu - Px, \phi(u-x) \rangle \geq c(||u-x|| ||h||^{-1}) ||u-x||^2.
\]

\[
\langle Pu - Pn, \phi(u-v) \rangle \geq c(\max \{ ||u||, ||v|| \} ||h||^{-1}) ||u-v||^2.
\]
(1.3.7) For any $x \in X$, there exists an $\varepsilon > 0$ such that for any $u \in B(x; \varepsilon)$,
\[ \langle Pu-Px, \phi (u-x) \rangle \geq c(||x||, ||h||^{-1}) ||u-x||^2. \]
Moreover we stated the following types of weighted (locally) $\phi$-accretive maps:

(1.3.8) For each $y \in Y$ and $r > 0$ there exists a non-increasing function $c: [0, \infty) \to (0, \infty)$ such that if $||Px-y|| \leq r$ then for all $u, v \in X$ sufficiently near to $x$
\[ \langle Pu-Pv, \phi (u-v) \rangle \geq c(||u-v||, ||h||^{-1}) ||u-v||^2. \]

(1.3.9) For each $y \in Y$ and $r > 0$ there exists a non-increasing function $c: [0, \infty) \to (0, \infty)$ such that if $||Px-y|| \leq r$ then for all $u \in X$ sufficiently near to $x$,
\[ \langle Pu-Px, \phi (u-x) \rangle \geq c(||x||, ||h||^{-1}) ||u-x||^2. \]

From the above results it is obvious that
(1.3.4) $\Rightarrow$ (1.3.5) $\Rightarrow$ (1.3.8) $\Rightarrow$ (1.3.9) and
(1.3.6) $\Rightarrow$ (1.3.8) $\Rightarrow$ (1.3.9).

We precisely proved the following surjectivity theorem:

THEOREM: 2

Let $X$ and $Y$ be Banach spaces and $P: X \to Y$ be a locally Lipschitzian map satisfying condition (1.3.9). If the duality map $J$ of $X$ is l.s.c and $P(X)$ is closed, then $P(X) = Y$. 
1.4 SURJECTIVITY OF WEAK DIRECTIONAL CONTRACTOR

In chapter IV we have established the concept of weak-directional contractor as following:

Let \( P : D(P) \subseteq X \rightarrow Y \) be a nonlinear operator from a linear subspace \( D(P) \) of \( X \) to a Banach space \( Y \), and \( \overline{\langle x \rangle} : Y \rightarrow D(P) \) a bounded linear operator associated with \( x \in D(P) \). There exists a positive number \( q = q(P) < 1 \) such that for any \( x \in D(P) \) and \( y \in Y \), \( o \neq M \in X \) and there exists \( o < \in (x, y) \leq 1 \) satisfying

\[
(1.4.1) \quad ||P(x + \epsilon \overline{\langle x \rangle} y - Px - \epsilon y)|| ||M|| \leq q \epsilon ||y||
\]

where \( M = M(q) \) such that \( ||M|| \leq 1 \).

Then \( \overline{x} \) is a weak-directional contractor for \( P \) at \( x \in D(P) \) and \( \overline{\langle \cdot \rangle} : D(P) \subseteq X \rightarrow L(Y, X) \) is called a directional contractor for \( P \), where \( L(Y, X) \) denotes the set of all linear continuous map of \( Y \) into \( X \). In particular, if there exists a constant \( B(>0) \) such that \( ||\overline{\langle x \rangle}|| \leq B \) for all \( x \in D(P) \), then \( \overline{\langle \cdot \rangle} \) is called weak-directional contractor for \( P \).

In fact, we proved the following surjectivity theorem for some non-linear operators by using the concept of weak-directional contractor.
THEOREM: 3

A non-linear map \( P : D(P) \subset X \rightarrow Y \) which has closed graph and bounded weak directional contractor is surjective.

The results obtained by the above theorem have been applied to obtain the solution of certain functional equations.

DEFINITION: 1:

If there exists continuous increasing function \( B : (0, \infty) \rightarrow (0, \infty) \) such that

\[
\| \int \{x\} \| \leq B(\|x\|) \quad \text{for all } x \in D(P),
\]

then \( P \) is said to have a point wise bounded directional contractor \( r \).

Further, the algebraic property of nonlinear operators derived from the following:

THEOREM: 4

If a nonlinear operator \( P : X \rightarrow Y \) has a closed graph and a point wise bounded weak-directional contractor \( r \) with \( B(s) \), then \( P(B(o,k)) \) contains \( B(P(o), (1-q) \int_0^k B(s) \ ds) \) for any \( k > 0 \).

1.5. SURJECTIVITY OF A PAIR OF OPERATORS

A new method of proving surjectivity of an operator \( f \) in a Banach space was proposed by Bratislava [73]. For this the condition of coercivity of
\textbf{f} in the form
\[
\lim_{|x| \to \infty} |f(x)| = \infty
\]

was applied.

In this context, we have established in chapter-V the surjectivity of a pair of operators \(f\) and \(g\) in Euclidean space \(\mathbb{R}^n\) which generalizes the recent result of Bratislava \([73]\). The condition of coercivity for a pair of operators \(f\) and \(g\) takes the form
\[
\lim_{|x| \to \infty} |f(x)| = \lim_{|x| \to \infty} |g(x)| = \infty.
\]

We proved the following surjectivity theorem:

**THEOREM: 5**

Let \(f, g: \mathbb{R}^n \to \mathbb{R}\) satisfy the conditions

\begin{align*}
(1.5.1) \quad & f, g \text{ are continuous;} \\
(1.5.2) \quad & \lim_{|x| \to \infty} |f(x)| = \lim_{|x| \to \infty} |g(x)| = \infty; \\
(1.5.3) \quad & \text{the common fixed points set of } f \text{ and } g \text{ i.e.} \\
& F(f, g) \neq \emptyset; \quad \text{and one of the conditions; either} \\
(1.5.4) \quad & \text{there is an } x_0 \in F(f, g) \text{ such that} \\
& f(x) - x_0 = k(g(x) - x_0) \text{ implies } k \geq 0 \\
& \text{for each } x \in \mathbb{R}^n, \ x \neq x_0 \\
& \text{or} \\
(1.5.5) \quad & \text{there is an } x_0 \in F(f, g) \text{ such that} \\
& f(x) - x_0 = k (g(x) - x_0) \text{ implies } k \leq 0 \\
& \text{for each } x \in \mathbb{R}^n, \ x \neq x_0
\end{align*}
or

(1.5.6) there is an \( x_0 \in F \) \( f, g \) and an \( r > 0 \) such that the scalar product satisfies

\[
( f(x) - x_0, g(x) - x_0 ) \geq 0 \text{ for all } x \in \mathbb{R}^n, \\
|x - x_0| \geq r,
\]

or

(1.5.7) there is an \( x_0 \in F \) \( f, g \) and an \( r > 0 \) such that

\[
( f(x) - x_0, g(x) - x_0 ) \leq 0 \text{ for all } x \in \mathbb{R}^n, \\
|x - x_0| \geq r.
\]

Then \( f(\mathbb{R}^n) = g(\mathbb{R}^n) = \mathbb{R}^n \).

PART-2: FIXED POINT THEOREMS

1.6 Let \( F \) be a mapping of a set \( X \) into itself. An element \( u \in X \) is said to be a fixed point of the mapping \( F \) if \( Fu = u \). By a fixed point theorem we mean a statement which asserts that under certain conditions (on the mapping \( F \) and on the space \( X \)) a mapping \( F \) of \( X \) into itself admits one or more fixed points. Historically first theorem of this type involves a space \( X \) which is a topologically simple subset of \( \mathbb{R}^n \) and a mapping of \( X \) into itself which is continuous. Brouwer's fixed point theorem asserts the existence of a fixed point whenever \( X \) is the unit ball in \( \mathbb{R}^n \) and \( F \) is continuous. In this theorem \( X \) was replaced by any homeomorphic thereof.

Schauder [67] extended Brouwer's theorem to the case where \( X \) is a compact convex subset of a normed
linear space. Further, this theorem was extended by Tychonoff for locally convex topological vector space. Banach [12] obtained a fixed point theorem for contraction mapping. Edelstein [29] considered contractive mapping and proved a fixed point theorem for such mappings.

Recently several generalizations of contraction mappings considered by Kannan [42,43], Husain and Sehgal [38] and Caristi [16].

1.7 COMMON FIXED POINTS IN METRIC SPACES

In 1976, G. Jungck [40] obtained a fixed point theorem in a complete metric space as following:

THEOREM. B:

Let \((X, d)\) be a complete metric space and \(I: X \to X\) be the identity mapping of \(X\) and \(f\) be a continuous self mapping of \((X, d)\). If there exists a mapping \(g: X \to X\) and a constant \(0 \leq \alpha < 1\) such that

\[
(1.7.1) \quad f(g(x)) = g(f(x)) \text{ for every } x \in X,
\]

\[
(1.7.2) \quad g(X) \subseteq f(X),
\]

\[
(1.7.3) \quad d(gx, gy) \leq \alpha d(fx, fy) \text{ for every } x, y \in X,
\]

then \(f\) and \(g\) have a unique common fixed point.

The well-known Banach contraction principle can be obtained by taking \(f = I\). In recent years many
authors presented several generalizations of above theorem.

**DEFINITION 1:**

Let \((X,d)\) be a metric space and let \(S\) and \(I\) be mapping of \(X\) into itself. We define the pair \((S,I)\) said to be weak ** commutating, if \(S(X) \subseteq I(X)\) and

\[
d(S^2I^1x, I^2S^2x) \leq d(S^2Ix, IS^2x) \leq d(SI^2x, I^2Sx) \\
\leq d(SIx, ISx) \leq d(S^2x, I^2x)
\]

for all \(x\) in \(X\).

It is shown by an example that two commuting mappings are weak** commuting but two weak** commuting mappings do not necessarily commute.

**DEFINITION 2:**

A map \(T:X \rightarrow X\) is called idempotent if \(T^2 = T\).

**DEFINITION 3:**

The map \(T\) is called rotative w.r.t.\(I\),

if \(d(Tx, I^2x) \leq d(Ix, T^2x)\)

for all \(x\) in \(X\),

In this context, we have established in chapter-VI a common fixed point theorem for three self maps of a complete metric space satisfying a rational inequality which is a generalization of the results of Diviccaro, Sessa and Fisher [21] and Fisher [30].
THEOREM 6:

Let $S,T$ and $I$ be three mappings of a complete metric space $(X,d)$ such that for all $x,y$ in $X$ either

\[(1.7.4) \ d(S^2x, T^2y) \leq ad(I^2x, S^2x) d(I^2y, T^2y) + bd(I^2x, T^2y) \leq d(I^2x, S^2x) + d(I^2y, T^2y)\]

if $d(I^2x, S^2x) + d(I^2y, T^2y) \neq 0$,

where $1 < a < 2$ and $b > 0$, or

\[(1.7.5) \ d(S^2x, T^2y) = 0 \text{ if } d(I^2x, S^2x) + d(I^2y, T^2y) = 0\]

Suppose that the range of $I$ contains the range of $S$ and $T$. If

\[(1.7.6) \text{ either } I^2 \text{ is continuous, } I \text{ is weak** commuting with } S \text{ and } T \text{ is rotative w.r.t. } I, \text{ or}\]

\[(1.7.7) \ I^2 \text{ is continuous, } I \text{ is weak** commuting with } T \text{ and } S \text{ is rotative w.r.t. } I, \text{ or}\]

\[(1.7.8) \ S^2 \text{ is continuous, } S \text{ is weak** commuting with } I \text{ and } T \text{ is rotative w.r.t. } S, \text{ or}\]

\[(1.7.9) \ T^2 \text{ is continuous, } T \text{ is weak** commuting with } I \text{ and } S \text{ is rotative w.r.t. } T, \text{ then } S, T \text{ and } I \text{ have a unique common fixed point } z. \text{ Further } z \text{ is the unique common fixed point of } S \text{ and } I \text{ and } T \text{ and } I.\]

A pair of maps $(S,T)$ of metric space $(X,d)$ into itself is called $(1,m)$ linearly weak, if

\[(1-1) \ d(S^kx, T^mx) + (m-1) \ d(S^kx, T^mx) \leq (1+m-2) \ d(S^kx, T^mx)\]
for all \( x \) in \( X \), where \( l, m \) are positive integers such that at least one of \( l, m > 1 \).

We proved the following fixed point theorem for \((l, m)\)-linearly weak maps which is a generalization of the recent result of Fisher [29].

**THEOREM 7:**

Let \( S \) and \( T \) be \((l, m)\)-linearly weak maps of a complete metric space \((X, d)\) into itself satisfying the following inequality

\[
(1.7.10) \quad d((S^l T^m)^x, (T^m S^l)^y) \leq c \max \{ d((S^l T^m)^x, (T^m S^l)^y), d(S^l (T^m S^l)^y, T^m (S^l T^m)^x), d((S^l T^m)^x, T^m (S^l T^m)^x), d(S^l (T^m S^l)^y, (T^m S^l)^y) : 0 \leq c < 1, \ l, m \text{ and } p \text{ are fixed positive integers and } S^l, T^m \text{ are continuous maps.} \]

Then \( S \) and \( T \) have a unique common fixed point \( z \).

Further, we proved some new results on fixed point theorems on expansion mappings.

In 1984, Wang, Li, Gao and Iseki [74] established fixed point theorems for certain expansion mappings.

Rhoades [63] proved the following theorem for a pair of surjective mappings in a complete metric spaces.
THEOREM C:

Let $f, g$ be surjective self maps of a complete metric space $(X, d)$. Suppose there exists a constant $a > 1$ such that

$$\text{(1.7.11)} \quad d(fx, gy) \geq ad(x, y)$$

for each $x, y$ in $X$. Then $f$ and $g$ have a unique common fixed point.

A distinct result from that of Wang, Li, Gao and Iseki [74] and Rhoades [63] was established by Dubey and Pathak [24] as following:

THEOREM D:

Let $f$ and $g$ be surjective continuous self maps of a complete metric space $(X, d)$. Suppose there exists a constant $a > 1$ such that

$$\text{(1.7.12)} \quad [d(fx, gy)]^2 \geq a[d(x, fx) d(y, gy)]$$

for each $x, y$ in $X$, then $f$ and $g$ have a unique common fixed point.

In this context, we proved the following theorem by unifying the contraction principles of Dubey and Pathak [24] and Rhoades[63].

THEOREM 8:

Let $f$ and $g$ be surjective continuous self maps of a complete metric space $(X, d)$. Suppose there exists a constant $a > 1_{1/2}$ such that

$$\text{(1.7.13)} \quad d(fx, gy) \geq a[d(x, y) + \{d(x, fx), d(y, gy)\}]^{1/2}$$
for each $x, y$ in $X$, then $f$ and $g$ have a unique common fixed point.

For a sequence of surjective continuous self maps we proved the following:

**THEOREM. 9:**

Let $\{ f_n \}$ $(n=1, 2, \ldots)$ be a sequence of surjective continuous self maps of a complete metric space $X$. If there exists a real number $a > 1/2$ such that for all $n \in \mathbb{N}$

\[(1.7.14) \quad d(f_n x, f_n y) \geq \min \{ \left( d(x, y) + (d(x, f_0 x) \\
\quad d(y, f_0 y) \right)^{1/2} \}, d(x, f_0 x), d(y, f_0 y) \}\]

for each $x, y \in X$, then there exists a common fixed point of $f_n$ in $X$.

Khan, Swaleh and Sessa [49] proved fixed point theorems for self maps on complete metric spaces by altering the distances between the points with the use of a function $\Psi: \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfying the following properties:

\[(1.7.15) \quad \Psi \text{ is continuous and increasing in } \mathbb{R}^+; \]

\[(1.7.16) \quad \Psi(t) = 0. \]

The set of above function $\Psi$ further denoted by $\Phi$.

In [49] the following theorem was proved.

**THEOREM. E:**

Let $(X, d)$ be a complete metric space, $T$ a self map of $X$, and $\Psi: \mathbb{R} \rightarrow \mathbb{R}$ an increasing continuous function satisfying property (1.7.16).
Further more, let $a, b, c$ be three decreasing function from $\mathbb{R}^+ \setminus \{0\}$ into $[0,1]$ such that $a(t) + 2b(t) + c(t) < 1$ for every $t>0$. Suppose that $T$ satisfies the following condition:

\[
\Psi(d(Tx,Ty)) \leq a(d(x,y)) \Psi(d(x,y)) + b(d(x,y)).
\]

I. $\{\Psi(d(x,Tx)) + \Psi(d(y,Ty))\} + c(d(x,y))$

\[
\min \{\Psi(d(x,Ty)), \Psi(d(y,Tx))\}
\]

where $x, y \in X$ and $x \neq y$. Then $T$ has a unique fixed point.

Further more, our main theorem is an improvement upon some fixed point theorems of Skof [70], Rakotch [65], Reich [66], Khan, Swaleh and Sessa [49] and results of Fisher [34] and Edelstein [28] in compact metric spaces.

A result in compact metric spaces proved by Fisher [34] as following:

THEOREM.F:

Let $T$ be a continuous self map of a compact metric space $(X,d)$ such that

II. $d(Tx,Ty) < 1/2 \{d(x,Tx) + d(y,Ty)\}$

for all distinct $x, y$ in $X$. Then $T$ has a unique fixed point.

In fact, we obtained the generalized version of above Theorem F as the following:
THEOREM 1.10:

Let $T$ be a continuous self map of a metric space $(X, d)$ such that for some $x_0 \in X$ the sequence $\{T^n x_0\}$ has a cluster point $z \in X$. Let there exist a continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the property (1.7.16).

Further more, for all distinct $x, y$ in $X$ the inequality:

$$\psi(d(Tx, Ty)) < c \left\{ \psi(d(x, y)) + \alpha \left( \psi(d(x, Ty)) + \psi(d(y, Tx)) \right) \right\}^{1/2} + (1-c)/2 \left\{ \psi(d(x, Tx) + \psi(d(y, Ty)) \right\}$$

holds, where $0 \leq c < 1$ and $\alpha \geq 0$. then $z$ is the unique fixed point of $T$.

Assuming $\alpha = 0$ in condition III and if $\psi(t) = t$ for any $t > 0$ and $c = 1$, Theorem 10 becomes a well known result of Edelstein [28].

1.8 COMMON FIXED POINTS IN BANACH SPACES

In 1986, Pathak [54] presented the following:

DEFINITION 1:

Two self mappings $T$ and $I$ of a metric space $(X, d)$ are said to be weak commuting if

$$d(TIx, ITx) \leq d(T^2 x, I^2 x)$$

for all $x$ in $X$. Two commuting mappings are weak commuting but converse is not generally true.

Two self mappings $T$ and $I$ of a closed convex subset $C$ of a Banach space $X$ are said to be weak commuting
pair if
\[(1.8.1) \quad ||T^2x - T^2x|| \leq ||T^2Ix - T^2x|| \leq ||TI^2x - IT^2x|| \]
\[\leq ||TIx - ITx|| \leq ||T^2x - I^2x|| \]
for any \(x\) in \(X\) and satisfy the inequality
\[(1.8.2) \quad ||T^2x - T^2y|| \leq a \quad ||I^2x - I^2y|| + (1-a) \max\{||T^3x - I^2x||, ||T^2y - I^2y||\} \]
for any \(x, y\) in \(C\), where \(0 < a < 1\). If \(I\) is linear and nonexpansive in \(C\) such that \(I^2C\) contains \(T^2C\), then \(T\) and \(I\) have a unique common fixed point in \(C\).

In 1986, Fisher and Sessa [35] proved the following:

**Theorem G:**

Let \(T\) and \(I\) be two weakly commuting mappings of \(C\) into itself satisfying the inequality
\[(1.8.3) \quad ||Tx - Ty|| \leq a \quad ||Ix - Iy|| + (1-a) \max\{||Tx-Ix||, \quad ||Ty-Iy||\} \]
for all \(x, y\) in \(C\), where \(0 < a < 1\). If \(I\) is linear, nonexpansive in \(C\) and such that \(IC\) contains \(TC\), then \(T\) and \(I\) have a unique common fixed point in \(C\).

In this context, we have established in chapter VII a generalized version of above Theorem G as following:

**Theorem II:**

Let \(T\) and \(I\) be two weak** commuting mappings of \(C\) into itself satisfying the inequality.
(1.8.4) \[ \|T^2x - T^2y\| \leq a \|T^2x - I^2y\| + (1-a) \]
\[
\max \left\{ \|T^2x - I^2x\|, \|T^2y - I^2y\| \right\},
\]
for all \( x, y \in C \), where \( 0 < a < 1 \). If \( I \) is linear, nonexpansive in \( C \) and such that \( I^2C \) contains \( T^2C \), then \( T \) and \( I \) have a unique common fixed point in \( C \).

Further, we proved a common fixed point theorem for three involution maps.

In 1986, Pathak [53] established the following fixed point theorem for commuting mapping in closed convex subset of Banach space.

**THEOREM.**

Let \( M \) be a non-empty closed convex subset of a Banach space \( B \). Let \( F: M \rightarrow M \) and \( G: M \rightarrow M \) satisfy the following conditions:

(1.8.5) \( F \) and \( G \) commutes.

(1.8.6) \( F^2 = I \), \( G^2 = I \) where \( I \) denotes the identity mapping.

(1.8.7) \[ \|Fx-Fy\|^2 \leq q \max \left\{ \|Gx - Fx\|, \|Gy - Fy\|, \|Gx - Fy\| \right\} \]
\[ \|Gy - Fx\|, \|Gx - Fy\|, \|Gy - Fx\| \]
for all \( x, y \in M \), where \( q \in (0, 1) \). Let \( x_1 \in M \) be arbitrary, \( t \in (0, 1) \) and

\[ Gx_{n+1} = (1-t) Gx_n + t Fx_n \]
for each integer \( n \geq 1 \). If the sequence \( \{Gx_n\} \)
converges to a point of \( F \) and \( G \).

Further, the above fixed point theorem improved for three mappings by Dubey [23]. We precisely proved the following theorem in a closed convex subset of a Banach space which is a slight
variant of the result of Pathak [55].

**THEOREM.12:**

Let \( M \) be a non-empty closed convex subset of a Banach space \( B \). Let \( F, G \) and \( H : M \to M \) satisfy the following conditions:

(1.8.8) \( F, G \) and \( H \) commutes.

(1.8.9) \( F^2 = I, \ G^2 = I, \ H^2 = I \) where \( I \) denotes the identity mapping.

(1.8.10) \[ \| Fx - Fy \| \leq q \max \{ \| GHx - Fx \|, \| GHx - Fy \|, \| GHy - Fx \|, \| GHy - Fy \| \} \]

for all \( x, y \in M \), where \( q \in (0,1) \). Let \( x_1 \in M \) arbitrary, \( t_n \in (0,1) \) where \( t_n \to 1 \) as \( n \to \infty \) and

\[ GHx_{n+1} = (1-t_n) \cdot GHx_n + t_n Fx_n \]

for each integer \( n \geq 1 \). If the sequence converges to a point \( u \in M \), then \( u \) is the unique common fixed point of \( F, G, \) and \( H \).

Further, the results obtained have been applied to solve non-linear equations in the following:

**THEOREM.13:**

Let \( \{ f_n \} \) be a sequence of elements in a Banach space \( B \). Let \( \omega_n \) be the unique solution of the equation \( u - FGHu = f_n \), where \( F, G \) and \( H \) are mappings of \( B \) into itself satisfy all the conditions of
Theorem 12. If \( \| f_n \| \to 0 \) as \( n \to \infty \) then the sequence \( \{ w_n \} \) converges to the solution of the equation \( u = Fu = Gu = Hu \).

1.9 FIXED POINTS IN NORMED SPACES

In 1983, Naimpally and Singh \([51]\) extended the corresponding results of Rhoades \([64]\), and Hicks and Kubicek \([71]\) and obtained that for mapping \( T \) which satisfy following conditions, if the sequence of Ishikawa iterates converges, it converges to the fixed point of \( T \).

Let \( X \) be a Banach space and \( C \) be a non-empty subset of \( X \). Let \( T: C \to C \) be a mapping. The Ishikawa scheme was defined as follows:

(i) \( x_0 \in C \),
(ii) \( y_n = \beta_n T x_n + (1 - \beta_n) x_n \), \( n \geq 0 \)
(iii) \( x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T y_n \), \( n \geq 0 \)

In the Ishikawa scheme, \( \{ \alpha_n \} \), \( \{ \beta_n \} \) satisfy \( 0 < \alpha_n \leq \beta_n \leq 1 \) for all \( n \),

\[ \lim_{n \to \infty} \beta_n = 0 \]

and \( \sum \alpha_n \beta_n = \infty \).

The following assumptions were considered:

(iv) \( 0 \leq \alpha_n \), \( \beta_n \leq 1 \) for all \( n \),
(v) \( \lim_{n \to \infty} \alpha_n = \alpha > 0 \),
(vi) \( \lim_{n \to \infty} \beta_n = \beta < 1 \).

The two contractive conditions used as following:
There exists a constant $k$, $0<k<1$ such that for all $x, y \in X$,

\begin{equation}
\|Tx-y\| \leq k \max \{\|x-y\|, \|x-Tx\|, \|y-Ty\|, \|x-Ty\| + \|y-Tx\|\}.
\end{equation}

(1.9.2) At least one of the following conditions holds:

(i) For each $x, y \in X$,

\[ \|x-Tx\| + \|y-Ty\| \leq a \|x-y\|, \quad 1 \leq a < 2; \]

(ii) For each $x, y \in X$,

\[ \|x-Tx\| + \|y-Ty\| \leq b \{\|x-Ty\| + \|y-Tx\| + \|x-y\|\}, \quad 1 \leq b < 2; \]

(iii) For each $x, y \in X$,

\[ \|x-Tx\| + \|y-Ty\| + \|T_x-T_y\| \leq c \{\|x-Ty\| + \|y-Tx\|\}, \quad 1 \leq c < 2; \]

(iv) For each $x, y \in X$,

\[ \|T_x-T_y\| \leq k \max \{\|x-y\|, \|x-Tx\|, \|y-Ty\|, \|x-Ty\| + \|y-Tx\|\}/2\}, \quad 0 \leq k < 1. \]

In chapter VIII we extended the contractive conditions obtained by Naim-pally and Singh [51], for a pair of maps by using Mann-iteration scheme as following:

**THEOREM 14:**

Let $X$ be a closed, convex, bounded subset of a normed space $X$ and let $T_1$ and $T_2$ be self mappings satisfying any one of the following:

For all $x, y$ in $X$ and $p$ is any integer,

(i) $\|T_1x-T_2y\|^p \leq q \max \{c \{\|x-y\|^p, \|x-T_1x\|^p, \|y-T_2y\|^p, \|x-T_2y\|^p + \|y-T_1x\|^p\}, \quad 0 \leq q < 1.$
(ii) \[ \| x - T_1 x \|^p + \| y - T_2 y \|^p \leq a \| x - y \|^p, \quad 1 \leq a < 2. \]

(iii) \[ \| x - T_1 x \|^p + \| y - T_1 y \|^p \leq b (\| x - T_2 y \|^p + \| y - T_1 x \|^p), \quad 1 \leq b < 2 \cdot 3. \]

(iv) \[ \| x - T_1 x \|^p + \| y - T_2 y \|^p + \| T_1 x - T_2 y \|^p \leq c (\| x - T_2 y \|^p + \| y - T_1 x \|^p), \quad 1 \leq c < 3 \cdot 2 \cdot 6. \]

(v) \[ \| T_1 x - T_2 y \|^p \leq k \max \{ c \| x - y \|^p, \| x - T_1 x \|^p, \| y - T_2 y \|^p, \| x - T_2 y \|^p + \| y - T_1 x \|^p \}, \quad 0 \leq k < 1. \]

Let the sequence \( \langle x_n \rangle \) be defined in accordance with Mann iteration process associated with two mappings \( T_1 \) and \( T_2 \) as follows:

(vi) \[ x_{2n+1} = (1 - c_{2n}) x_{2n} + c_{2n} T_1 x_{2n}. \]

(vii) \[ x_{2n+1} = (1 - c_{2n+1}) x_{2n+1} + c_{2n+1} T_2 x_{2n+1}. \]
for $n \geq 0$ where $c_0 = 1, 0 < c_n < 1$ for $n > 0$ and $\lim c_n = h > 0$.

If $\langle x_n \rangle$ converges to $z$ in $X$ then $z$ is a common fixed point of $T_1$ and $T_2$.

1.10: FIXED POINTS IN HAUSDORFF SPACES

In Chapter IX, we obtained some new results on fixed point theorem for certain contractive mappings on Hausdorff spaces.

Definition 1:

Let $(X, d)$ be a metric space. A mapping $T$ of $X$ into itself is said to be contractive if

$$d(Tx, Ty) < d(x, y) ; \forall x \neq y \in X.$$  

In fact, we proved the following:

THEOREM 1.5:

Let $T$ be a continuous mapping of a Hausdorff spaces $X$ into itself and let $f$ be a continuous mapping of $X \times X$ into the non negative reals such that

(1.10.1) $f(x, y) \neq 0, x \neq y,$

(1.10.2) $\frac{f(Tx, Ty)}{f(x, Tx)} \leq \frac{d(x, y)}{f(x, Ty)}$

\[ f(y, Ty) \geq f(x, y) + \beta \{f(x, y)\}^2 \]
for all \( x \neq y ; \alpha , \beta \in \mathbb{R}_+ \) and \( \alpha + \beta < 1 \).

\[
(1.10.3) \quad \{ f(x,y) \}^3 \geq \{ f(x,y) \}^2 f(y,y), \quad x \neq y \in X.
\]

If for some \( x_0 \in X \) the sequence \( x_n = \{ T^n x_0 \} \) has a convergent subsequence, then \( T \) has a unique fixed point.

Further, we extended the above theorem for a pair of mappings. Precisely, we proved the following:

THEOREM 16:

Let \( T_1 \) and \( T_2 \) be continuous mappings of a Hausdorff space \( X \) into itself and let \( f \) be a continuous mapping of \( X \times X \) into non-negative reals such that

\[
(1.10.4) \quad f(x,y) = f(y,x) ; \quad \forall x,y \in X.
\]

\[
(1.10.5) \quad f(x,y) = 0 , \quad \forall x,y \in X.
\]

\[
(1.10.6) \quad \{ f(T_1 x, T_2 y) \}^2 \leq \alpha \{ f(x,T_1 x) \}^2 \quad f(y,T_2 y)/f(x,y) + \beta \{ f(x,y) \}^2
\]

for all \( x \neq y \in X ; \alpha , \beta \in \mathbb{R}_+ \) and \( \alpha + \beta < 1 \).

\[
(1.10.7) \quad \{ f(x,y) \}^3 \geq \{ f(x,x) \}^2 f(y,y) ; \quad \forall x,y \in X.
\]
If some \( x_0 \in X \), the sequence \( \{ x_n \} \) where \( T_1 x_{2n} = x_{2n+1} \) and \( T_2 x_{2n+1} = x_{2n+2} \) for \( n = 0,1,2 \ldots \) has a convergent subsequence of the type \( \{ x_{(2p+1)n} \} \), where \( p \in \mathbb{N} \) is fixed and \( n \in \mathbb{N} \), then \( T_1 \) and \( T_2 \) have a unique fixed point.

Our next theorem deals with a sequence of continuous mappings of a Hausdorff space into itself and let \( f \) be a continuous mapping of \( X \times X \) into the non-negative reals such that:

\[
\begin{align*}
(1.10.8) \quad & f(x,y) = f(y,x) \ ; \ \forall x,y \in X, \\
(1.10.9) \quad & f(x,y) \not< 0 \ ; \ \forall x \not< y \in X, \\
(1.10.10) \quad & \{ f(T_i x, T_{i+1} y) \}^2 \leq \alpha \{ f(x, T_i x) \}^2 \\
& \quad f(y, T_{i+1} y) + f(x, y) + \\
& \quad \beta \{ f(x, y) \}^2.
\end{align*}
\]
for all $x, y \in X; \alpha, \beta \in \mathbb{R}^+: \alpha + \beta < 1$ and $T_{k+1} = T_k$.

(1.10.11) $\{ f(x, y) \}^3 \geq (f(x, x))^2 f(y, y); \quad \forall x, y \in X.$

If for some $x_0 \in X$ the sequence $\{x_n\}$, where $x_1 = T_1 x_0$,

\[ x_2 = T_2 x_1, \quad \ldots, \]

\[ x_k = T_k x_{k-1}, \quad x_{k+1} = T_1 x_k, \quad x_{k+2} = T_2 x_{k+1}, \ldots \]

\[ x_{2k} = T_k x_{2k-1}. \]

\[ \begin{align*}
    x_{n+1} &= T_1 x_n, \quad x_{n+2} = T_2 x_{n+1} \quad \ldots, \\
    x_{(n+1)k} &= T_k x_{(n+1)k-1}
\end{align*} \]

for $n = 0, 1, 2, \ldots$, has a convergent subsequence of these types $\{x_{(m_k+1)n}\}$, where $m \in \mathbb{N}$ is fixed and $n \in \mathbb{N}$, then $T_1, T_2, \ldots, T_k$ has a unique common fixed point.

\[
\star \star \star 
\]