CHAPTER VIII

FIXED POINT THEOREMS

IN NORMED SPACES

130 - 137
CHAPTER - VIII

FIXED POINT THEOREMS IN NORMED SPACES

8.1 In this chapter we have extended the contractive conditions for a pair of map by using Mann iteration scheme in lieu to Ishikawa iterates by weakening the norm conditions; i.e, by taking some index over the norm, which does not necessarily satisfy the triangle inequality. The technique of the proof is based on the binominal expansion method.

In 1983, Naimpally and Singh [51] extended the corresponding results of Rhoades [64], and Hicks and Kubicek [72] and obtained that, for mapping T which satisfy following conditions, if the sequence of Ishikawa iterates converges, it converges to the fixed point of T.

Let X be a Banach space and C be a nonempty subset of X. Let T: C → C be a mapping. The iteration scheme called I-scheme, was defined as follows:

(8.1.1) \( x_0 \in C \);

(8.1.2) \( y_n = \beta_n T x_n + (1 - \beta_n) x_n \), \( n \geq 0 \);

(8.1.3) \( x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T y_n \), \( n \geq 0 \);

In the Ishikawa scheme, \( \{ a_n \}, \{ b_n \} \) satisfy \( 0 \leq a_n \leq b_n \leq 1 \) for all \( n \),

\[
\lim_{n \to \infty} b_n = 0, \quad \text{and} \quad \sum a_n b_n = \infty.
\]

The following assumptions were used:

(8.1.4) \( 0 \leq a_n, \ b_n \leq 1 \) for all \( n \).

(8.1.5) \( \lim \ a_n = a > 0 \),

(8.1.6) \( \lim \ b_n = \beta < 1 \).

The two contractive conditions were used as following:

There exists a constant \( k, \ 0 \leq k < 1 \) such that for all \( x, y \in X \),

(8.1.7) \[
||Tx - Ty|| \leq k \max \{ ||x - y||, ||x - Tx||, \}
\]

\[
\min \{ ||y - Ty||, ||x - Ty|| + ||y - Tx|| \}.
\]

(8.1.8) At least one of the following conditions holds:

(i) For each \( x, y \in X \),

\[
||x - Tx|| + ||y - Ty|| \leq a \ ||x - y||, \ 1 \leq a < 2;
\]

(ii) For each \( x, y \in X \),

\[
||x - Tx|| + ||y - Ty|| \leq b \ ||x - Ty|| + ||y - Tx|| + \]

\[
||x - y||, \ 1/2 \leq b < 2/3;
\]

(iii) For each \( x, y \in X \),
For each \( x, y \in X \),
\[
||Tx - Ty|| \leq k \max \{ ||x - y||, ||x - Tx||, ||y - Ty||, \\
\max \{ ||x - Ty|| + ||y - Tx|| / 2 \}, 0 \leq k < 1.\]

**8.2 MAIN RESULTS:**

**THEOREM 8.1:**

Let \( X \) be a closed, convex, bounded subset of a normed space \( N \) and let \( T_1 \) and \( T_2 \) be self mappings of \( X \) satisfying any one of the following:

For all \( x, y \) in \( X \) and \( p \) is any integer.

\[ (8.2.1) \quad \begin{align*}
||T_1 x - T_2 x||^p & \leq q \max \{ c ||x - y||^p, ||x - T_1 x||^p, \\
& \quad ||y - T_2 y||^p, ||x - T_2 y||^p + ||y - T_1 x||^p \}, 0 < q < 1.
\end{align*} \]

\[ (8.2.2) \quad \begin{align*}
||x - T_1 x||^p + ||y - T_2 y||^p & \leq a ||x - y||^p, 1 \leq a < 2.
\end{align*} \]

\[ (8.2.3) \quad \begin{align*}
||x - T_1 x||^p + ||y - T_2 y||^p & \leq b ||x - T_2 y||^p + \\
& \quad ||y - T_1 x||^p + ||x - y||^p, 1/2 \leq b < 2/3.
\end{align*} \]

\[ (8.2.4) \quad \begin{align*}
||x - T_1 x||^p + ||y - T_2 y||^p & \leq c \max \{ ||x - T_2 y||^p, ||y - T_1 x||^p \}, 1 \leq c < 3/2.
\end{align*} \]

\[ (8.2.5) \quad \begin{align*}
||T_1 x - T_2 y||^p & \leq k \max \{ c ||x - y||^p, \\
& \quad ||x - T_1 x||^p, \quad ||y - T_2 y||^p, ||x - T_2 y||^p + \\
& \quad ||y - T_1 x||^p / 2 \}, 0 \leq k < 1.
\end{align*} \]
Let the sequence \( \langle x_n \rangle \) be defined in accordance with Mann iteration process associated with two mappings \( T_1 \) and \( T_2 \) as follows:

\[
(8.2.6) \quad x_{2n+1} = (1 - c_{2n}) x_{2n} + c_{2n} \; T_1 x_{2n}
\]

\[
(8.2.7) \quad x_{2n+2} = (1 - c_{2n+1}) \; x_{2n+1} + c_{2n+1} \; T_2 x_{2n+1}
\]

For \( n \geq 0 \) where \( c_0 = l, 0 < c_n < 1 \) for \( n > 0 \), and \( \lim c_n = h > 0 \). If \( \langle x_n \rangle \) converges to \( z \) in \( X \) then 

\( z \) is a common fixed point of \( T_1 \) and \( T_2 \).

**PROOF:**

\[
\| z - T_2 z \| \leq \left[ \| z - x_{2n+1} \| + \| x_{2n+1} - T_2 z \| \right]^P
\]

\[
\leq \left[ \| x_{2n+1} - T_2 z \| \right]^P \left[ 1 + \frac{\| z - x_{2n+1} \|}{\| x_{2n+1} - T_2 z \|} \right]^P
\]

\[
= (1 - c_{2n}) x_{2n} + c_{2n} \; T_1 x_{2n} - T_2 z
\]

\[
= \left[ 1 + \frac{\| z - x_{2n+1} \|}{\| x_{2n+1} - T_2 z \|} \right]^P
\]

\[
= \left[ 1 - (1 - c_{2n}) x_{2n} - (1 - c_{2n}) \; T_2 z + c_{2n} \; T_1 x_{2n}
\right.\]

\[
- c_{2n} \; T_2 z \left. \right\| \left[ 1 + \frac{\| z - x_{2n+1} \|}{\| x_{2n+1} - T_2 z \|} \right]^P
\]


\[
\leq \left[ (1- c_{2n}) ||x_{2n} - T_2 z || + c_{2n} ||T_1 x_{2n} - T_1^2 z || \right]^p
\]

\[
\leq (1-c_n)^p ||x_{2n} - T_2 z ||^p \left[ 1 + \frac{||x_{2n} - x_{2n+1}||}{||(x_{2n} - T_2 z)||} \right]^p
\]

\[
\leq (1-c_n)^p ||x_{2n} - T_2 z ||^p \left[ 1 + \frac{c_n (||T_1 x_{2n} - T_2 z||)}{(1-c_n)||x_{2n} - T_2 z||} \right]^p
\]

If \( x_{2n}, z \) satisfy (8.2.1), then

\[
||T_1 x_{2n} - T_2 z||^p \leq q \max \left[ c, ||x_{2n} - z||^p, \right. \left. ||x_{2n} - T_1 x_{2n}||^p, \right. \left. ||z - T_2 z||^p, ||x_{2n} - T_2 z||^p + ||z - T_1 x_{2n}||^p \right]
\]
from (8.2.6) we see that as $n \to \infty$

$$||x_{2n} - T_1 x_{2n}||^p \to 0 \quad \text{and} \quad ||z - T_1 x_{2n}||^p \to 0.$$  

Therefore, $||T_1 x_{2n} - T_2 z||^p \leq q ||z - T_2 z||^p$ as $n \to \infty$.

If $x_2, z$ satisfy (8.2.2), then

$$||T_1 x_{2n} - T_2 z||^p \leq 1/2 \left[ ||x_{2n} - z||^p + \rho_{c_1} ||T_1 x_{2n} - x_{2n}||^{p-1} ||x_{2n} - T_2 z|| + \ldots \right].$$

from (8.2.6) we see that as $n \to \infty$

$$||x_{2n} - T_1 x_{2n}||^p \to 0 \quad \text{and} \quad ||z - T_1 x_{2n}||^p \to 0.$$  

Therefore, $||T_1 x_{2n} - T_2 z||^p \leq 0$ as $n \to \infty$.

If $x_2, z$ satisfy (8.2.4), then

$$||T_1 x_{2n} - T_2 z||^p \leq 1/2 \left[ b ||x_{2n} - T_2 z||^p + ||z - T_1 x_{2n}||^p + ||x_{2n} - z||^p + \rho_{c_1} ||T_1 x_{2n} - x_{2n}||^{p-1} \right].$$

$$||x_{2n} - T_2 z|| + \ldots \ldots +$$

$$\rho_{c_1} ||T_1 x_{2n} - z||^{p-1} ||z - T_2 z|| + \ldots \ldots \right].$$

From (8.2.6) we see that as $n \to \infty$

$$||x_{2n} - T_1 x_{2n}||^p \to 0 \quad \text{and} \quad ||z - T_1 x_{2n}||^p \to 0.$$  

Therefore, $||T_1 x_{2n} - T_2 z||^p \leq (b/2) ||z - T_2 z||^p$ as $n \to \infty$.  

If \( x_{2n}, z \) satisfy (8.2.4), then
\[
\left\| T_1 x_{2n} - T_2 z \right\|^p \leq 1/3 \left\{ \left\| x_{2n} - T_2 z \right\|^p + \left\| z - T_1 x_{2n} \right\|^p \right\} + \left( \rho_1 \right) \left\| x_{2n} - T_2 z \right\|^p + \ldots + \left( \rho_1 \right) \left\| z - T_2 z \right\|^p.
\]
From (8.2.6) we see that as \( n \to \infty \)
\[
\left\| x_{2n} - T_1 x_{2n} \right\|^p \to 0 \quad \text{and} \quad \left\| z - T_1 x_{2n} \right\|^p \to 0.
\]
Therefore, \( \left\| T_1 x_{2n} - T_2 z \right\|^p \leq (c|3|) \left\| z - T_2 z \right\|^p \text{ as } n \to \infty. \)

If \( x_{2n}, z \) satisfy (8.2.5), then
\[
\left\| T_1 x_{2n} - T_2 z \right\|^p \leq k \max \left\{ \left\| x_{2n} - z \right\|^p , \left\| x_{2n} - T_1 x_{2n} \right\|^p , \left\| z - T_2 z \right\|^p , \left\| x_{2n} - T_2 z \right\|^p + \left\| z - T_2 z \right\|^p \right\}.
\]
From (8.2.6) we see that as \( n \to \infty \)
\[
\left\| x_{2n} - T_1 x_{2n} \right\|^p \to 0 \quad \text{and} \quad \left\| z - T_1 x_{2n} \right\|^p \to 0.
\]
Therefore, \( \left\| T_1 x_{2n} - T_2 z \right\|^p \leq k \left\| z - T_2 z \right\|^p \text{ as } n \to \infty. \)

Hence substituting the value of \( \left\| T_1 x_{2n} - T_2 z \right\|^p \)
we see that as \( n \to \infty \) , (8.2.6) reduces to
\[
\left\| z - T_2 z \right\|^p \leq \left\| z - T_2 z \right\|^p (1-h) + h \max \left\{ q^1 \left| p \right| , \left( b \right) \left| 2 \right| ^p , \left( c \right) \left| 3 \right| ^p , \left( k \right) \left| 1 \right| ^p \right\}, \text{ a contradiction. Hence } z = T_2 z, \text{ i.e } z \text{ is a fixed point of } T_2.\]
Similarly, one can prove that \( z \) is a fixed point of \( T_1 \). Hence, \( z \) is a common fixed point of \( T_1 \) and \( T_2 \).

**REMARK:**

It is not known whether the result of Theorem 8.1 follows without boundedness of the space.