CHAPTER - IV

SURJECTIVITY THEOREMS BY USING WEAK DIRECTIONAL CONTRACTIONS AND ITS APPLICATIONS

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4.1 In the present chapter we have established a surjectivity theorem for some non-linear operators by using the concept of weak direction contractor.

M. Altman [2,3,4] obtained some surjectivity theorems or existence theorems for a non-linear map which has a directional contractor. A transfinite induction argument used by Gavurin [36] applied in his work to prove surjectivity theorems for nonlinear maps. The following theorem was applied to the surjectivity theorem of Altman.

**THEOREM A:**

Let \((X,d)\) be a complete metric space of a selfmap of \(X\). If there exists a l.s.c function \(\phi\) from \(X\) into the non-negative real numbers such that

\[
d(x,fx) \leq \phi(x) - \phi(fx), \quad x \in X,
\]

then \(f\) has a fixed point.

By using Theorem A and the concepts of directional contractor, bounded directional contractor, the surjectivity theorem of Altman [2] was proved.
Let \( P: D(P) \subseteq X \rightarrow Y \) be a nonlinear operator from a linear subspace \( D(P) \) of \( X \) to a Banach space \( Y \), and \( \overline{(x)}: Y \rightarrow D(P) \) a bounded linear operator associated with \( x \in D(P) \). There exists a positive number \( q = q(P) < 1 \) such that for any \( x \in D(P) \) and \( y \in Y \), there exists \( 0 < \epsilon(x,y) \leq 1 \) satisfying

\[
\| P(x + \epsilon(x)y) - Px - \epsilon y \| \leq q \| y \|. \tag{4.1.1}
\]

Then \( \overline{(x)} \) is a directional contractor for \( P \) at \( x \in D(P) \) and \( \overline{(x)}: D(P) \subseteq X \rightarrow L(Y,X) \) is called a directional contractor for \( P \), where \( L(Y,X) \) denotes the set of all linear continuous maps of \( Y \) into \( X \). If there exists a constant \( B > 0 \) such that \( \| \overline{(x)} \| \leq B \) for all \( x \in D(P) \), then \( \overline{(x)} \) is called a bounded directional contractor for \( P \).

4.2 Now we have established the concept of weak-directional contractor as following:

Let \( P: D(P) \subseteq X \rightarrow Y \) be a nonlinear operator from a linear subspace \( D(P) \) of \( X \) to a Banach space \( Y \), and \( \overline{(x)}: Y \rightarrow D(P) \) a bounded linear operator associated with \( x \in D(P) \). Suppose there exists a positive number \( q = q(P) < 1 \) such that for any \( x \in D(P) \) and \( y \in Y \),

\[
0 \neq \omega \in X \text{ and there exists } 0 < \epsilon(x,y) \leq 1
\]

satisfying
(4.2.1) \[ \| p(x + \epsilon f(x) - px - \epsilon y) \| \leq M \epsilon \| y \| \]

where \( M = M(q) \) such that \( \| M \| \leq 1 \).

Then \( \Gamma x \) is a weak-directional contractor for \( P \) at \( x \in D(P) \) and \( \Gamma : D(P) \subseteq X \rightarrow L(Y, X) \) is called a weak directional contractor for \( P \), where \( L(Y, X) \) denotes the set of all linear continuous maps of \( Y \) into \( X \). If there exists a constant \( B(>0) \) such that \( \| \Gamma(x) \| \leq B \) for all \( x \in D(P) \), then \( \Gamma \) is called bounded weak directional contractor for \( P \). Then it follows from this definition that \( \Gamma(x)y = 0 \) implies \( y = 0 \) i.e. \( \Gamma(x) \) is injective, and an inverse Gâteaux derivative is a directional contractor. Moreover \( P: D(P) \subseteq X \rightarrow Y \) is said to have closed graph if \( x_n \rightarrow x, \ x_n \in D(P) \) and \( px_n \rightarrow y \) imply \( x \in D(P) \) and \( y = Px \).

**Remark:**

It may be observe that every weak-directional contractor is a directional contractor.

By applying the ideas of Ray and Walker [62] and Bae and Yie [5] we have the following surjectivity theorem.

**Theorem 4.1:**

A nonlinear map \( P: D(P) \subseteq X \rightarrow Y \) which has closed
graph and a bounded weak-directional contractor \( \Gamma \) is surjective.

**PROOF:**

We define a new metric \( f \) on \( D(P) \) by

\[
f(x, y) = \max \{ ||x-y||, (1+q)^{-1} ||P(x+y)|| \}.
\]

Since \( P \) has closed graph, \( (D(P), f) \) is a complete metric space. Suppose \( w \notin R(P) \) (the range of \( P \)), then we get a contradiction by applying Theorem A. For any \( x \in D(P) \) we set \( y = P(x) - w \). Since \( P \) has a bounded weak-directional contractor \( \Gamma \) we have, for some \( 0 < \epsilon \leq 1 \)

\[
(4.2.2) \quad \frac{1}{||P(x + \epsilon [x,y]) - P(x)||} \leq q \epsilon ||y||.
\]

We set \( \epsilon [x,y] = h \), so,

\[
||h|| = ||\epsilon [x,y]|| \leq \epsilon B ||y|| = \epsilon B ||w-Px||.
\]

From (4.2.1),

\[
||P(x+h) - w + (1-\epsilon)(w-Px)|| \leq q \epsilon ||w-Px||
\]

and hence

\[
||P(x+h) - w|| - (1-\epsilon) ||w-Px|| \leq q \epsilon ||w-Px|| ||M||.
\]

Therefore we have

\[
\epsilon ||w-Px|| - q \epsilon ||w-Px|| ||M|| \leq ||w-Px|| - ||w-P(x+h)||,
\]

that is,
(4.2.3) \[ \epsilon (1-q ||M||) ||w-Px|| \leq ||w-Px|| - \\
||w - P(x+h)||. \]

But from (4.2.2)
\[ ||P(x+h) - Px|| ||M||^{-1} - \epsilon ||y|| ||M||^{-1} \leq \epsilon \in ||y||. \]

Hence,
\[ ||P(x+h) - Px|| ||M||^{-1} \leq (q+||M||^{-1}) \epsilon ||y||. \]

From (4.2.2),
\[ ||P(x+h) - Px|| ||M||^{-1} \leq (q+||M||^{-1})(1+q||M||^{-1}). \]

or
\[ ||P(x+h) - Px|| ||M||^{-1} \leq (q+||M||^{-1}) \]

\[ \leq (||w-Px|| - ||w-P(x+h)||) \]

Since \( M = M(q) \) such that \( ||M|| \leq 1 \), we have
\[ ||P(x+h) - Px|| \leq (1+q) (1-q)^{-1} (||w-Px|| - ||w-P(x+h)||). \]

And
\[ ||h|| \leq \epsilon B ||w-Px|| \]
\[ \leq B (1-q ||M||)^{-1} (||w-Px|| - ||w-P(x+h)||) \]

or
\[ ||h|| \leq B ||M||^{-1} (||M||^{-1} - q)^{-1} (||w-Px|| - ||w-P(x+h)||) \]

or
\[ ||h|| \leq B (l-q)^{-1} (||w-Px|| - ||w-P(x+h)||) \]

Let \( a = \max (B,l) \) and \( \phi (x) = a(l-q)^{-1} ||Px-w|| \).

Then \( \phi \) is continuous with respect to the metric \( \phi \).

Therefore if we set \( fx = x+h \), then \( fx \neq x \). Indeed if
\[ h = 0 \], then from (4.1.1)
\[ \epsilon ||y|| ||M||^{-1} \leq \epsilon \in ||y||. \]
or \( \epsilon \| y \| \leq \frac{q}{\| M \|} \epsilon \| y \| \)

or \( \epsilon \| y \| \leq q \epsilon \| y \| \) and \( q < 1 \).

But since \( w \notin R(P) \), \( y = Px - w \neq 0 \).

Therefore \( fx \neq x \) and \( \phi(x, fx) \leq \phi(x) - \phi(fx) \).

It is a contradiction to Theorem A. So we conclude that \( w \in R(P) \).

Let \( X \) be a complete metric space, \( Y \) a Banach space, \( P : D(P) \subseteq X \rightarrow Y \), and \( x \in X \). Then \( X(P) \) is a set of special contractor directions for \( P \) at \( x \in D(P) \) if there exist a positive \( q(P) < 1 \) and \( B \) which has the following property. For each \( y \in X(P) \), there exist a positive \( \epsilon = \epsilon(x, y) \leq 1 \) and an element \( \bar{x} \in D(P) \) such that

\[
(4.2.4) \quad \| P\bar{x} - Px - \epsilon y \| \leq q \epsilon \| y \| \quad \text{and}
\]

\[
(4.2.5) \quad \| d(\bar{x}, x) \| \leq \epsilon B \| y \| ,
\]

where \( d \) is the metric on \( X \).

APPLICATION:

Further, we apply the above results to obtain the solution of certain functional equations.

THEOREM 4.2:

Let \( P : D(P) \subseteq X \rightarrow Y \) have closed graph, \( Y \) a
Banach space. Suppose that $y_0$ is such that for each $x \in D(P)$, the element $y_0 - Px$, belongs to the closure of a set $\overline{x(P)}$ defined by means of (4.2.4) and (4.2.5). Then the equation $Px - y_0 = 0$, $x \in D(P)$ has a solution.

**Proof:**

For any $x \in D(P)$, we suppose that $Px - y_0 \neq 0$. Then we find a contradiction by applying Caristi's fixed point theorem. Further we set $y = y_0 - Px \neq 0$, and then by hypothesis $y \in \overline{x(P)}$. So we choose $y'$ in $\overline{x(P)}$ and $c > 0$ such that $||y - y'|| < c ||y||$.

Of course $c < 1$ and does not depend on $x$. Since $y' \in \overline{x(P)}$, there exists $\lambda$ such that

$$||P\lambda - P\lambda - \epsilon y'|| ||M|| < \epsilon q ||y'||.$$  

From this inequality,

$$||P\lambda - y_0 + y_0 - Px - \epsilon y'|| ||M|| < \epsilon q ||y'||.$$  

Since $y = y_0 - Px$,

$$||P\lambda - y_0 + y - \epsilon y'|| ||M|| < \epsilon q ||y'||.$$  

If we choose $q > 0$ such that $q < q' < 1$, then we may choose $c > 0$ sufficiently small so that $(c+1) < q' q^{-1}$. Since $||y - y'|| < c ||y||$, we have $||y'|| < (1 + c) ||y||$.

From this and (4.2.7) we have
\[ ||P\bar{x} - y_0 + y - \varepsilon y'|| ||M||^{-1} \leq \varepsilon q'||y'||. \]

On the other hand, we have
\[ \zeta \left( ||P\bar{x} - y_0 + (1-\varepsilon)y|| - ||P\bar{x} - y_0 + y - \varepsilon y'|| \right) ||M||^{-1} \]
\[ \leq \varepsilon ||y - y'|| \leq \varepsilon c ||y||. \]

Hence from (4.2.7) and this inequality,
\[ ||P\bar{x} - y_0 + (1-\varepsilon)y|| ||M||^{-1} - \varepsilon c ||y|| \leq \varepsilon q'||y'||. \]

Therefore, we get
\[ ||P\bar{x} - y_0|| ||M||^{-1} - (1-\varepsilon)||y|| ||M||^{-1} - \varepsilon c ||y|| \leq \varepsilon q'||y'||. \]

So from this,
\[ \varepsilon \left( ||M||^{-1} - q' - c \right) ||y|| \leq ||y|| ||M||^{-1} - ||P\bar{x} - y_0|| ||M||^{-1} \]

or
\[ \left( 1 - \frac{q}{||M||^{-1}} - \frac{c}{||M||^{-1}} \right) ||y|| \leq ||y|| ||P\bar{x} - y_0|| \]

which implies
\[ \varepsilon (1 - q' - c) ||y|| \leq ||y|| ||P\bar{x} - y_0||. \]

If we choose \( c > 0 \) so that \( \alpha = 1 - q' - c > 0 \), then we get
\[ (4.2.8) \quad \varepsilon \alpha ||y_0 - Px|| \leq ||Px - y_0|| - ||P\bar{x} - y_0||. \]

But from (4.2.6) we have
\[ ||P\bar{x} - Px|| ||M||^{-1} \leq \varepsilon (q + ||M||^{-1}) ||y'|| \]
\[ \leq \varepsilon (q + ||M||^{-1}) (c+1) ||y|| \]

or
\[ ||P\bar{x} - Px|| \leq \varepsilon (q + 1) (c+1) ||y||. \]
Therefore from (4.2.8),
\[ |P\vec{x} - Px| |M|^{-1} \leq (q+|M|^{-1})(c+1) \alpha^{-1} (|Px-y_o| - |P\vec{x}-y_o|).\]

Since \(d(x,x) \leq d(x,y) \leq B(c+1) \|y\|\),
\[d(\vec{x},x) \leq B(c+1) \alpha^{-1} (|Px-y_o| - |P\vec{x}-y_o|).\]

Further, we define a new metric \(\bar{f}\) on \(D(P)\) by
\[\bar{f}(x,y) = \max \{d(x,y), (1+q)^{-1} |Px-Py|\}.\]

And we set \(fx = \vec{x}\). Since \(y' \neq 0 (c<1)\),
we have \(x \neq \vec{x}\). So we set \(\phi(x) = a(c+1) \alpha^{-1} |Px-y_o|\)
where \(a = \max \{B,1\}\). Then
\[\bar{f}(x,fx) \leq \phi(x) - \phi(fx).\]

So we have a contradiction to Theorem A. If there exists
continuous increasing function
\[B : (0,\infty) \longrightarrow (0,\infty)\] such that
\[\|\sqrt{(x)}\| \leq B (\|x\|) \text{ for all } x \in D(P),\]
then \(P\) is said to have a pointwise bounded directional
contractor \(\square\).

The following theorem of Bae and Yie [5] is a
localized version of Caristi's theorem.

**THEOREM B:**

Let \((R,d)\) be a complete metric space and \(\phi\) a l.s.c.
function from $\mathbb{R}$ into $(0, \infty)$.

Let $c$ be a continuous non-increasing function from $[0, \infty)$ into $(0, \infty)$ and $x_0 \in \mathbb{R}$ fixed. Furthermore, suppose that there exist a $z \in \mathbb{R}$ and a $K > 0$ (possibly $K = \infty$) satisfying

$$\int_0^K c(s) ds - \phi(z) \quad (\text{When } K = \infty, \quad \int_0^\infty c(s) ds > \phi(z)).$$

If $f$ is a self map of $M$ satisfying

$$(*) \quad c(d(x_0, z)) d(x, fx) \leq \phi(x) - \phi(f(x))$$

whenever $x \in \mathbb{R}$ with $d(x_0, x) < \int_0^K c(s) ds$, then $f$ has a fixed point in $\mathbb{R}$.

Further, we apply our main result to derive some results concerning the algebraic property of nonlinear operators in the following theorems;

**THEOREM 4.3:**

If a nonlinear operator $P: X \rightarrow Y$ has closed graph and a pointwise bounded weak directional contractor

$$\Gamma$$

with $B(s)$, then $P(B(o, K))$ contains

$$B(P(o), (1-q) \int_0^K B(s)^{-1} ds)$$

for any $K > 0$.

**PROOF:**

Let $w \in B(P(o), (1-q) \int_0^K B(s)^{-1} ds)$.

We set $c(s) = B(s)^{-1}$. Then $||P(o) - w|| < (1-q) \int_0^K B(s) ds = (1-q) \int_0^K c(s) ds$. 
We take a sufficiently small \( \varepsilon > 0 \) such that
\[
||P(0) - w|| \leq (1 - q) \int_0^K c(s) \, ds.
\]

Further, we define a new metric \( \rho \) on the set \( R = B(0, K - \varepsilon) \) by
\[
\rho(x, y) = \max \{ ||x - y||, B(0)(1+q)^{-1} ||Px - Py|| \}.
\]

Since \( P \) has closed graph, \( (R, \rho) \) is a complete metric space. Again we set
\[
\phi(x) = \frac{1}{||M||} \left( ||M||^{-1} (||M||^{-1} - q)^{-1} ||Px - w|| \right), \text{ so that}
\]
\[
\phi: (R, \rho) \to [0, \infty) \text{ is continuous and}
\]
\[
\phi(0) \leq \int_0^K c(s) \, ds.
\]

Further, we suppose that \( w \notin \overline{B}(0, K - 2\varepsilon) \). Then we find a contradiction, from the above fixed point theorem of Bae and Yie [5]. For any \( x \in R - \overline{B}(0, K - 2\varepsilon) \), we have
\[
\int_0^{K - 2\varepsilon} c(s) \, ds < o - \phi(x) \text{ since } \rho(o, x) = ||x|| > k - 2\varepsilon.
\]

Now we set \( f(x) = 0(\neq x) \). For \( x \in \overline{B}(0, K - 2\varepsilon) \)

we choose \( k > 0 \) so small that

\( k ||x|| (w-Px) || < \varepsilon \) and \( k < 1 \). Then \( k \) depends on \( x \).

Now we can choose \( \varepsilon > 0 \) such that

(4.1.1) holds for \( y = k(w-Px) \). Then we set
\[
h = \varepsilon k \sqrt{x} (w-Px) \text{ and } f(x) = x + h. \text{ So } x \neq x \text{ and}
\]
\[
f(x) \in \overline{B}(0, K - \delta). \text{ From the choice of } \varepsilon \text{ we have}
\[ (4.2.9) \quad \| P(x+h) - Px - \epsilon k(w-Px) \| \| M \|^{-1} \leq q \in k \| w-Px \| , \]

and

\[ \| h \| = \| \epsilon k \left(\frac{x}{\epsilon} (w-Px) \right) \| \]

\[ - \epsilon B(|x| \| M \|^{-1}) k \| w-Px \|. \]

So,

\[ (4.2.10) \quad c(|x| \| M \|^{-1}) \| fx-x \| \leq k \| w-Px \|. \]

From (4.2.9), we have

\[ (4.2.11) \quad \| P(x+h) - Px \| \| M \|^{-1} \leq (\| M \| + q) \epsilon k \| w-Px \| \]

and

\[ \| P(x+h) - w \| \| M \|^{-1} - (1 - \epsilon k) \| w-Px \| \| M \|^{-1} \]

\[ \leq q \in k \| w-Px \| , \]

which implies

\[ \epsilon k \| w-Px \| \| M \|^{-1} - q \in k \| w-Px \| \]

\[ \leq \| w-Px \| \| M \|^{-1} - \| w-P(fx) \| \| M \|^{-1} \]

or \( (\| M \|^{-1} - q) \epsilon k \| w-Px \| \leq (\| w-Px \| - \| w-P(fx) \|) \| M \|^{-1} \)

or \( (\| M \|^{-1} - q) \epsilon k \| w-Px \| \leq (\| w-Px \| - \| w-P(fx) \|) \| M \|^{-1} \).

From (4.2.11), we have

\[ \| P(fx) - Px \| \| M \|^{-1} \leq \| M \|^{-1} + q \) \| M \|^{-1} - q^{-1} \]

\[ \| w - P(fx) \| \| M \|^{-1} \]

or \( \| P(fx) - Px \| \leq (\| M \|^{-1} + q) \| M \|^{-1} - q^{-1} \)

\[ \| w - P(fx) \| \| M \|^{-1} \]
or \[ ||P(fx) - Px|| \leq (1+q)(1-q)^{-1} (||w-Px|| - ||w-P(fx)||), \]

that is,

\[
(1+q)^{-1} ||P(fx) - Px|| \leq \phi(x) - \phi(fx), \]

and from (4.2.10) (4.2.12),

\[
c(||x|| ||M||^{-1}) ||f(x-x)|| \leq \phi(x) - \phi(fx). \]

Hence, if \( f(x,fx) = ||x-fx|| \), then (4.2.12) and

\[
c(\phi(0,x)) \leq c(||x||) \text{ give (*):} \text{ in the above fixed point theorem.} \]

If \( f(x,fx) = B(o)(1+q)^{-1} \text{||P - P(fx)||,} \) then we have

\[
c(||x|| ||M||^{-1}) f(x,fx) = c(||x|| ||M||^{-1}) (1+q)^{-1} B(o) \text{||P - P(fx)||} \\
\leq (1+q)^{-1} \text{||P - P(fx)||} \\
\leq \phi(x) - \phi(fx). \]

Thus by the above fixed point theorem, \( f \) has a fixed point in \( R \), a contradiction, and consequently

\[ w \in P(\overline{B(o,2\delta)}) < P(B(o,K)). \]

This completes the proof.