CHAPTER 2

Mathematical Background

2.1 Overview of Perturbation Method

Before turning to the details, we shall outline the standard method of perturbation theory used here but in a more generalized form that can be adapted to multidimensional systems defined for example by partial differential and difference equations. Let $f$ be the input $g$ the output and $L, N$ linear and nonlinear operators acting on the output signals so that the system dynamics is described by

$$L(g) + \epsilon N(g) = f$$

where $\epsilon$ is a small parameter. Suppose we expand the solution in powers of $\epsilon$:

$$g = g(\epsilon) = \sum_{m=0}^{\infty} \epsilon^m g_m$$

Substituting this into the system dynamics and equating coefficients of various powers of $\epsilon$ gives us

$$L(g_0) = f,$$

$$L(g_1) = -N(g_0),$$

$$L(g_2) + N'(g_0)g_1 = 0,$$

etc. where $N'(g_0)$ is the generalized derivative of $N$ with respect to its argument $g_0$. For example suppose

$$N(g) = \sum_{r=1}^{\infty} N_r(g)$$
where $N_r$ is a homogeneous functional of $g$ of degree $r$. For example, if the signal space is the space of continuous $n$ dimensional signals, then $N_r$ may have the form

$$N_r(g)(x) = \int_{\mathbb{R}^r} K_r(x, y_1, \ldots, y_r)g(y_1)\ldots g(y_r)dy_1\ldots dy_r$$

Then, the coefficient of $\epsilon^n$ gives

$$L(g_n) + N(g_0, g_1, \ldots, g_{n-1}, n) = 0$$

where $N(g_0, \ldots, g_{n-1}, n)$ is the coefficient of $\epsilon^{n-1}$ in $N(g)$. It is seen that the coefficient of $\epsilon^{n-1}$ in $N_r(g)$ is given by a sum of terms of the form

$$\int K_r(x, y_1, \ldots, y_r)g_{i_1}(y_1)\ldots g_{i_r}(y_r)dy_1\ldots dy_r$$

over all $i_1, \ldots, i_r$ such that

$$i_1 + \ldots i_r = n - 1$$

A quick way to understand the role of harmonic analysis of nonlinear systems is to consider the following single input single output differential equation:

$$y'(t) = f(y(t), \theta) + x(t)$$

where $f$ is a nonlinear function and $\theta$ is an unknown parameter vector to be estimated. We assume that $f$ can be decomposed into a strong linear part and a weak nonlinear part as follows:

$$f(y, \theta) = a(\theta)y + \epsilon.g(y, \theta)$$

where $\epsilon$ is a small parameter to be set equal to unity at the end. The differential equation thus assumes the form

$$y'(t) = a(\theta)g(t) + \epsilon.g(y(t), \theta) + x(t)$$

We expand $y(t)$ in powers of $\epsilon$:

$$y(t) = y(t, \epsilon) = \sum_{m=0}^{\infty} y_m(t)\epsilon^m$$
Plugging this into the above differential equation and equating coefficients of $\epsilon^m, m = 0, 1, 2$, gives us

\[
y_0'(t) = a(\theta)y_0(t) + x(t),
\]
\[
y_1'(t) = a(\theta)y_1(t) + g(y_0(t), \theta),
\]
\[
y_2'(t) = a(\theta)y_2(t) + g'(y_0(t), \theta)y_1(t)
\]

and in general,

\[
y_m'(t) = a(\theta)y_m(t) + C_m(y_0(t), ..., y_{m-1}(t), \theta), m = 1, 2, ...
\]

where $C_m(y_0, y_1, ..., y_{m-1}, \theta)$ is the coefficient of $\epsilon^{m-1}$ in the Taylor series expansion of

\[
g(y_0 + \epsilon y_1 + \epsilon^2 y_2 + ..., \theta)
\]

about $y_0$. We note that $C_m$ is expressible as a function of $y_0, ..., y_{m-1}$ and $\theta$ and does not depend on $y_m, y_{m+1}, ...$. This fact permits the various approximants to the solution $y_0, y_1, y_2, ...$ to be determined recursively. Thus,

\[
y_0(t) = h(t, \theta) * x(t)
\]

where

\[
h(t, \theta) = \exp(ta(\theta))u(t)
\]

and

\[
y_1(t) = h(t, \theta) * g(y_0(t), \theta), ...
\]

\[
y_m(t) = h(t, \theta) * C_m(y_0(t), ..., y_{m-1}(t), \theta), m = 1, 2, ...
\]

Now suppose we input a harmonic process:

\[
x(t) = \sum_{k=1}^{p} A(k)\exp(j\omega_k t)
\]

where the $A(k)$'s are complex numbers and the $\omega_k$'s are distinct real numbers. Then, in steady state, assuming $a(\theta) < 0$ (this guarantees stability of the linearized system), we have

\[
y_0(t) = \sum_{k=1}^{p} H(\omega_k, \theta)A(k)\exp(j\omega_k t)
\]

\[
y_1(t) = h(t, \theta) * g(y_0(t), \theta)
\]
But assuming
\[ g(y, \theta) = \sum_{k=2}^{\infty} a_k(\theta) y^k \]
we get
\[ g(y_0(t), \theta) = \sum_{n, r_1 + \ldots + r_p = n} a_n(\theta) \frac{n!}{r_1! \ldots r_p!} A(1)^{r_1} \ldots A(p)^{r_p} H(\omega_1, \theta)^{r_1} \ldots H(\omega_p)^{r_p} \exp(j(r_1 \omega_1 + \ldots + r_p \omega_p)t) \]
so that
\[ y_1(t) = \sum_{n, r_1 + \ldots + r_p = n} a_n(\theta) \frac{n!}{r_1! \ldots r_p!} A(1)^{r_1} \ldots A(p)^{r_p} H(\omega_1, \theta)^{r_1} \ldots H(\omega_p)^{r_p} \exp(j(r_1 \omega_1 + \ldots + r_p \omega_p)t) \exp(j(r_1 \omega_1 + \ldots + r_p \omega_p)t) \]
It is clear that a harmonic analysis of \( y_0 + y_1 \) will give plenty of information about the parameter vector \( \theta \).

### 2.2 Overview of Stochastic Differential Equations for Modeling

Formally, the effect of noise on a circuit can be described by adding white noise processes on the RHS of ordinary differential equations (ODEs). However, white noise has delta correlation functions and hence it is meaningless to speak of non-linear functions of the noise which result from solving the system. In order to circumvent the problem, Ito proposed the replacement of ODE with white noise by stochastic differential equations (SDEs) in which if \( w(t) \) is white noise, then \( w(t)dt \) is replaced by the differential \( dB(t) \) where \( B(.) \) is a process of unbounded variation but finite quantum variation and Ito developed method for adapted process with respect to Brownian motion (B.M.). Many nice properties of this stochastic integral follow from this. The stochastic differential equations are then interpreted in the Ito theory as stochastic integral equations. From the Ito formalism, it is possible to derive partial differential equations that describe the evolution in time of the pdf or the states. These are called Fokker-Plank or Kolmogorov equations. Now, thanks to the pioneering work of many probabilists like P.A. Meyer and M.Yor, there exists a
far reaching generalization of the Ito theory for arbitrary martingales and this theory can be used to analyze the effects of discontinuous noise like Poisson processes on dynamical systems.