Chapter 4
Optimal Protection Policy for Repair Facility with Interruption

4.1 Introduction

Vacation is a concept that is used widely in queuing literature. This chapter is motivated by the concept of vacation. Consider the case of software development. Assume that the development is done by an individual. One of the major threats of employing a single individual for the execution of the project is that the project may need to be dealt from scrap if the concerned person leaves the company without notice. This is because the approach and logic applied varies from person to person. Hence the company should keep alternatives open to tackle such situations. But keeping a standby from the beginning is not admissible from the cost perspective. It may be economical to restart if the concerned person leaves abruptly than keeping a standby from the beginning. Furthermore when the project enters some subsequent stages it might be desirable to have a standby than starting from scrap. Hence the most admissible choice will be to go for maintaining a standby resource once the development enters some critical stage. The time at which the standby person should be hired is a serious decision making problem.

In this chapter we deal with these types of problems. We assume that the spans of various stages are exponentially distributed with parameters depending on the stage.

4.2 Model

Consider a multi-state repair facility with \( n \) states. The repair facility can be interrupted because of many reasons like shut down at the repair facility, attrition of the service personal etc. Here we assume that the interruption will result in restarting the repair facility when the repair process is in certain class of states. But since restarting the repair facility from the scrap cannot be financially
viable always, we introduce the concept of protecting some of the repair states from interruption. When the repairs are in these states, the shocks or the interruption will not have any impact of the repairs, that is, the repairs will continue uninterrupted. Hence it will be desirable to protect every step of the repair. This incurs a heavy financial commitment as the cost of protecting a state be too high. Hence it will be interesting to find the number of states to be protected.

Suppose there are \(n\) states in the repair procedure. Assume that the initial \(k\) are unprotected. Hence the number of protected states will be \(n-k\). Let \(U\) and \(P\) respectively denote the set of unprotected and protected states. Once the process enters a protected state, it is protected till the completion of the repair facility. Hence transitions are assumed to be from unprotected state to same class or unprotected to protected and within the protected class. Transitions are not allowed from a protected state to unprotected state.

Let us assume that the repair times are distributed as an Erlang random variable with parameters \((n, \lambda)\). Hence the state at which the process is in at time \(t\) is, one plus the number of renewals by the time \(t\). Hence the probability that the process is in state \(m+1\) at time \(t\) is given by \(\exp(-\lambda t)(\lambda t)^m / m!\), \(m = 0, 1, 2, \ldots, n-1\).

Then
\[
P(X(t)\in U) = \sum_{m=0}^{k-1} \frac{\exp(-\lambda t)(\lambda t)^m}{m!}
\]
(4.1)

\[
P(X(t)\in P) = \sum_{m=k}^{n-1} \frac{\exp(-\lambda t)(\lambda t)^m}{m!}
\]
(4.2)

\[
P(\text{Repair is completed by time } t) = \sum_{m=n}^{\infty} \frac{\exp(-\lambda t)(\lambda t)^m}{m!}
\]
(4.3)

Let \(X\), \(X_u\) and \(X_p\) respectively be the random variable denoting repair time, the time spent in the unprotected states and the time spent in protected state till the completion of the repair. Let \(X(t)\) denote the state occupied by the repair facility.
at time $t$. Let random variable $Y$ denote the time for the occurrence of the shock or the interruption. We will assume that the shocks occur according to an exponential distribution with parameter $\delta$. Let $c$ denote the cost incurred per unit time when the system is under repair. Let $c_p$ denote the protection cost per unit time. Our objective is to find $k$ so that the expected cost will be minimized during the procedure. Let $L(k)$ denote the expected cost with $k$ unprotected states.

We can broadly classify the whole time interval into three distinct cases, assuming that the shock has occurred at time $t$:

(i) repair process is still in unprotected states at time $t$ and the cost will be $ct$,

(ii) repair process is in protected state at time $t$ and

(iii) repair process has completed by the time $t$. Let $L(k|t)$ denote the expected cost with $k$ unprotected states when the occurrence of the shock is given to be at $t$. Then under situation (i), the cost will be

$$
(ct + L(k))P\left(X(t) \in U\right)
$$

(4.4)

Now under (ii), assume that the repair process is in state $i$, $i \in P$ at time $t$, then expected time for the completion of repair after the $i^{th}$ state is given by $\frac{(n-i)}{\lambda}$. Also since the sojourn time in state $i$ is assumed to be exponentially distributed, the expected time in the state is independent the time for which it had been working in state $i$ which implies that the expected time of stay in state $i$ after time $t$ is given by $\frac{1}{\lambda}$. Hence the cost incurred for repair, if $X(t) \in i$, is given by

$$
c\left(t + \frac{n-i+1}{\lambda}\right)
$$

(4.5)

Now in (iii), the repair process has completed by the time $t$. Hence if we consider the transitions from each state as a renewal in a Poisson Process, repair has completed by the time $t$ imply that the number of renewals for the corresponding
renewal process is \( n, n+1, \ldots \). Let the process is in at state \( i \) at time \( t \) that is \((i-1)\) renewals had occurred by the time \( t \).

Let \( Y_1, Y_2, \ldots \) denote the interoccurrence times and \( N(t) \) denote the number of renewals by the time \( t \) for the renewal process. Let \( S_k = \sum_{i=1}^{k} Y_i \). Then it is known that the distribution of \( S_k \) given that \( N(t) = i \) is the distribution of the \( k^{th} \) order statistic of \( i \) uniform random variables over the range \((0,t)\).

Hence

\[
f_x(x|N(t) = i-1) = \frac{(i-1)!}{(n-1)!i(i-n-1)!} \left( \frac{x}{t} \right)^{i-1} \left( 1 - \frac{x}{t} \right)^{i-n-1} \frac{1}{i}.\]

Hence

\[
E(X|N(t) = i-1) = \int_0^t x \frac{(i-1)!}{(k-1)!(i-k-1)!} \left( \frac{x}{t} \right)^{i-1} \left( 1 - \frac{x}{t} \right)^{i-n-1} \frac{1}{t} dx.
\]

Putting \( y = \frac{x}{t} \) the integral reduces to a beta integral and on simplification we get

\[
E(X|N(t) = i-1) = \frac{(i-1)!}{(n-1)!i(i-n-1)!} \frac{n!(i-k-1)!}{i!} = \frac{nt}{i}.
\]

Hence

\[
E(X|T = t) = \sum_{i=1}^{\infty} \frac{nt}{i} \frac{\exp(-\lambda t)(\lambda t)^{i-1}}{(i-1)!}
\]

\[
= \frac{\exp(-\lambda t)n}{\lambda} \sum_{i=1}^{\infty} \frac{(\lambda t)^i}{(i)!}
\]

Now it is remaining to find out the protection cost under the condition (ii). Let us assume that \( X(t) \in i, i \in P \). Then the time for which protection is given by time \( t \) is \( t - X_u \). Then the distribution of \( X_u \) is the \( k^{th} \) order statistic of a random sample of size \( i-1 \) taken from a uniform distribution over the interval \((0,t)\). Hence
\[
f_{x_i}(x|X(t)=i) = \frac{(i-1)!}{(k-1)!(i-k-1)!} \left(\frac{x}{t}\right)^{k-1} \left(1 - \frac{x}{t}\right)^{i-k-1} \frac{1}{t}
\]

Hence the expected time in protected states till time \( t \) when the process is in state \( i \in P \) at time \( t \) is

\[
I = \int_0^t (t-x) \frac{(i-1)!}{(k-1)!(i-k-1)!} \left(\frac{x}{t}\right)^{k-1} \left(1 - \frac{x}{t}\right)^{i-k-1} \frac{1}{t} \, dx
\]

Putting \( y = \frac{x}{t} \) we have

\[
I = \frac{(i-1)!}{(k-1)!(i-k-1)!} \int_0^1 t(1-y) y^{k-1} (1-y)^{i-k-1} \, dy
\]

(4.6)

Following similar argument in condition (ii), expected time to the completion of the repair after time \( t \) and \( X(t) \in i \) is \( \frac{n-i+1}{\lambda} \).

Hence

\[
L(k|t) = (ct+L(k) \sum_{i=0}^{k-1} P(X(t)=i) + \sum_{i=k}^{n-1} c \left( t + \frac{n-i+1}{\lambda} \right) P(X(t)=i)
\]

\[
+ \frac{c_p}{\lambda} \sum_{i=k}^{n-1} \frac{t(i-k)}{i} P(X(t)=i) + \frac{c_p}{\lambda} \sum_{i=k}^{\infty} (n-i+1) P(X(t)=i)
\]

\[
+ c_p \sum_{i=k}^{n-1} \frac{t(i-k)}{i} P(X(t)=i) + \left( \frac{(c+c_p)n-t}{\lambda} - \frac{c_p{k}t}{\lambda} \right) \sum_{i=n+1}^{\infty} P(X(t)=i)
\]

Integrating over the range of \( t \), we have

\[
L(k) = \int_0^\infty L(k|t) \delta \exp(-\delta t) \, dt
\]

(4.7)
where

\[ I_1 = \sum_{i=0}^{n-1} \frac{t^i}{i!} \exp(-\lambda t) \exp(-\delta t) dt = \sum_{i=0}^{n-1} \frac{\lambda^i \delta}{i!} \int_0^\infty \exp(-(\lambda + \delta) t) t^i dt = \sum_{i=0}^{n-1} (i+1) \delta \lambda^i \]

Hence

\[ I_1 = \frac{1}{\delta} \frac{(n+1) \delta + \lambda}{\lambda + \delta} \left( \frac{\lambda}{\lambda + \delta} \right)^n \]  \hspace{1cm} (4.8)

\[ I_2 = \sum_{i=0}^{k-1} \frac{t^i}{i!} \exp(-\lambda t) \exp(-\delta t) dt = \sum_{i=0}^{k-1} \frac{\lambda^i \delta}{i!} \int_0^\infty \exp(-(\lambda + \delta) t) t^i dt = \sum_{i=0}^{k-1} \delta \lambda^i \]

Hence

\[ I_2 = 1 - \left( \frac{\lambda}{\delta + \lambda} \right)^k \]  \hspace{1cm} (4.9)

\[ I_3 = \sum_{i=1}^{n} \frac{n-i+1}{\lambda} \int_0^\infty \frac{t^{i-1}}{(i-1)!} \exp(-\lambda t) \exp(-\delta t) dt = \sum_{i=0}^{n-1} \frac{(n-i+1) \delta}{\lambda} \int_0^\infty \frac{t^{i-1}}{(i-1)!} \exp(-(\lambda + \delta) t) dt = \sum_{i=0}^{n-1} \frac{(n-i+1) \delta \lambda^{i-2}}{(\lambda + \delta)^i} \]

Hence

\[ I_3 = \frac{\lambda^{k-1} (n-k) \delta - \lambda + \lambda \left( \frac{\lambda}{\lambda + \delta} \right)^{k-n}}{\delta (\lambda + \delta)^k} \]  \hspace{1cm} (4.10)
Now

\[
I_4 = \sum_{i=k+1}^{n} \frac{i-k}{i} \int_0^\infty \frac{t \exp(-\lambda t)(\lambda t)^{-1}}{(i-1)!} \delta \exp(-\delta t) dt
\]

\[= \sum_{i=0}^{n-1} \frac{i-k}{i} \frac{\lambda^{-1} \delta}{(\lambda+\delta)^i} \int_0^\infty \exp(-(\lambda+\delta)t) t^i dt
\]

\[= \sum_{i=0}^{n-1} \frac{(i-k)}{i} \frac{i \delta \lambda^{-1}}{(\lambda+\delta)^i} = \sum_{i=0}^{n-1} \frac{(i-k) \delta \lambda^{-1}}{(\lambda+\delta)^i}
\]

\[
I_4 = \frac{\lambda^k (\delta + \lambda)^{n+1} - \lambda^n (\delta + \lambda)^k \left[ (1-k+n) \delta + \lambda \right]}{\delta (\delta + \lambda)^{n+k}} \tag{4.11}
\]

\[
I_5 = \int_0^\infty \frac{\exp(-\lambda t) n \lambda}{t} \sum_{i=n+1}^{\infty} \frac{(\lambda t)^i}{i!} \delta \exp(-\delta t) dt
\]

\[= \frac{n \delta}{\lambda} \sum_{i=n+1}^{\infty} \frac{\lambda^i}{i!} \int_0^\infty \exp(-(\lambda+\delta)t) t^i dt
\]

\[= \frac{n \delta}{\lambda} \sum_{i=n+1}^{\infty} \frac{\lambda^i}{i!} \frac{i!}{(\lambda+\delta)^i}
\]

\[= \frac{n \delta}{\lambda} \sum_{i=n+1}^{\infty} \frac{\lambda^i}{(\lambda+\delta)^i} \tag{4.12}
\]

On Simplification we get

\[
I_5 = \frac{\lambda^{n+1}}{(\delta + \lambda)^{n+1} \delta} \tag{4.12}
\]

Now making use of equations (4.8) to (4.12) in equation (4.7), we will have the expression for \(L(k), k = 1, 2, ..., n\).

For \(k = 0\), that is the case when all the states are protected, the expected cost will be independent of the time at which shock occurs and the expected time to complete the repair process is given by \(\frac{n}{\lambda}\). Hence the expected cost in this will be \((c_{+} + c_{p}) \frac{n}{\lambda}\).
4.3 Numerical Illustration

The results obtained in this chapter are illustrated with the help of a numerical example. The illustrations are performed with the help of MATHEMATICA®. The results are validated with various values of the parameters $\lambda, \delta, c$ and $c_p$, the values of the number of states are assumed to be constant at $n = 20$. The results are summarized in table 4.1. The figures in the boldface letter indicated the optimal values.

*Table 4.1* Long run cost for various values of $(\lambda, \delta, c, c_p)$ and constant at $n = 20$

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4.4 Conclusion

We introduced the concept of protection during the repair process in this chapter. An optimal modelling assuming that the sojourn times in each state as exponential distribution is also done. Since the most of the lifetime distribution can be approximated by Coxian distribution, the results can be applied to the single state systems, splitting the total stay as to be combination of exponential random variables.