Chapter 5
Two-Phase Start-up Demonstration Tests

5.1 Introduction

Business people demonstrates or convinces customer about the reliability of heavy machine by start-ups and such a procedure is normally termed as start-up demonstration test. Most of the earlier studies in the start-up demonstration tests were assuming that the consecutive attempts are independent. But this may not be situation as success yields success with higher probability. For example a team winning a game has higher chance for winning the next match as the win in a match will increase the morale of the team as a whole. Also vendor is always motivated to sell his product. Hence he will make attempts to convince the customer about the quality of the product with necessary steps like looking for new or alternative product if the initial attempts make to have positive impact on the customer. But at the same time as the customer is vigilant he might be having certain conditions for the rejection of the product. Hence in this chapter we will introduce two test procedures which a vendor can be applied for selling his product.

5.2 Models

The proposed models consist of two phases. We will incorporate repair action in phase one. Let us consider the first model. The product is accepted in the first phase if $k_1$ consecutive successes occur prior to $c_1$ consecutive failure and $d_1$ failed attempts and we reject it in the phase one if we observe $d_1$ failures ahead of $k_1$ consecutive successes and $c_1$ consecutive failure. If $c_1$ consecutive failures occur prior to $k_1$ consecutive successes and $d_1$ failures, we will take product for repair and the repaired product, in second phase, is accepted if $k_2$ consecutive

---

2 A part of this chapter is communicated in Nair and Thomas(2011)
successes occur before we observe \( d_2 \) failures and we reject the product otherwise. If we take \( c_i = d_i + 1 \) our model reduces to the case with rejection of units upon observing \( d \) failures proposed by Balakrishnan and Chan (2000) and later studied by Martin (2004) and Smith and Griffith (2005). When \( c_i = 1 \) our model reduces to single corrective model discussed by Balakrishnan et. al (1995,1997).

In the second model we will sent the product for repair if accidental causes exceeds specified limits and once a permanent failure observed for the product, leading to consecutive failure, we will reject the product. In phase one, we will accept if \( k_1 \) consecutive successes prior to \( c_i \) consecutive failures and \( d_i \) random failures and if \( c_i \) consecutive failures occur before \( k_1 \) consecutive successes and \( d_i \) random failures, we reject the product. The product is sent for repair if \( d_i \) random failures come ahead of the other two events. Now in order to make sure that the test terminates with probability one, in the second phase we reject the product if \( d_2 \) random failures occur ahead of the \( k_2 \) consecutive successes. On the contrary we accept the product.

It is natural to think that success (failure) leads to success (failure) with high probability. Hence throughout this paper we will assume that the probability of success (failure) depends on the number of just preceding consecutive success (failure), discussed as \( l \) dependent sequences by Aki and Hirano (2000). Independent and identical and Markov dependent trials comes as special cases to our proposed model. So let \( p_{sa}(x) \) denotes the probability for a success following a success run of length \( x \) and \( p_{fs}(x) \) denotes the probability for a success following a failure run of length \( x \). Similarly \( p_{sf}(x) \) \( (p_{fs}(x)) \) denote probability for a failure following a failure (success) run of length \( x \).

For both the models of start-up demonstration tests, we proceed by embedding a Markov chain with each phase of the test. The states of the embedded
Markov chain, in both phases, are given by the triplet \((x, y, z)\) where \(x\) denotes the length of the current run, \(y\) denotes the type of the current run i.e. \(s\) if a success run is going on and \(f\) otherwise. \(z\) corresponds to the number of failures occurred till now. Also throughout this paper we will assume that \(1^t\) unit vector of appropriate dimension so that the matrix multiplication is conformable.

Let \(\{A_n^{(i)}, n \geq 1\}\) be the Markov chain associated with the first phase of the test. Then the transition probabilities of the Markov chain \(\{A_n^{(i)}, n \geq 1\}\) are given by

\[
P\left(A_{n+1}^{(i)} = (x+1, s, z) \mid A_n^{(i)} = (x, s, z)\right) = p_{ss}(x) \quad \text{if} \quad 0 \leq z \leq d_i - 1, \quad 0 \leq x \leq k_i - 1
\]

\[
P\left(A_{n+1}^{(i)} = (x+1, f, z+1) \mid A_n^{(i)} = (x, f, z)\right) = p_{ff}(x) \quad \text{if} \quad 0 \leq z \leq d_i - 1, \quad 0 \leq x \leq c - 1
\]

\[
P\left(A_{n+1}^{(i)} = (1, f, z+1) \mid A_n^{(i)} = (x, s, z)\right) = p_{sf}(x) \quad \text{if} \quad 0 \leq z \leq d_i - 1, \quad 0 \leq x \leq k_i - 1
\]

\[
P\left(A_{n+1}^{(i)} = (1, s, z) \mid A_n^{(i)} = (x, f, z)\right) = p_{fs}(x) \quad \text{if} \quad 0 \leq z \leq d_i - 1, \quad 0 \leq x \leq c - 1
\]

and

\[
P\left(A_{n+1}^{(i)} = (x, f, d_i) \mid A_n^{(i)} = (x, f, d_i)\right) = 1 \quad 0 \leq x \leq c - 1
\]

\[
P\left(A_{n+1}^{(i)} = (k_i, s, z) \mid A_n^{(i)} = (k_i, s, z)\right) = 1 \quad 0 \leq z \leq d_i - 1
\]

\[
P\left(A_{n+1}^{(i)} = (c, f, z) \mid A_n^{(i)} = (c, f, z)\right) = 1 \quad 0 \leq z \leq d_i
\]

Based on the decision whether we accept, reject or repair we can divide the states of the Markov chain into four classes viz., Non-decisive, accepting, rejecting and repair classes. We will continue with making attempts as long as the Markov chain is in non-decisive class. Once the Markov chain enters the accepting, rejecting or repair class, we will accept, reject or repair the product. Hence the states in the accepting, rejecting and repair classes are all absorbing. Clearly the last three equations correspond to the absorbing states. Now we can partition the transition probability matrix corresponding to the first phase of the start-up demonstration test as shown
\[
P_i = \begin{bmatrix}
    R^{(i)} & Q_a^{(i)} & Q_r^{(i)} & Q_t^{(i)} \\
    0 & I & 0 & 0 \\
    0 & 0 & I & 0 \\
    0 & 0 & 0 & I
\end{bmatrix}
\]

where \( R^{(i)} \) denote the matrix of transition probabilities between the non-decisive states or non-absorbing states. \( Q_a^{(i)}, Q_r^{(i)} \) and \( Q_t^{(i)} \) respectively denote the transition probabilities from non-decisive states to the accepting, repair/corrective and rejecting states.

Let \( \tau^{(i)}_{a}, \tau^{(i)}_{r} \) and \( \tau^{(i)}_{t} \) denote the number of transition required either to accept or to repair or to reject the product in the first phase. Then by using the Chappman-Kolmogrov equation and the first phase t.p.m we have

\[
\begin{align*}
    P\left(\tau^{(i)}_{a} = n\right) &= \alpha^{(i)} \left(R^{(i)}\right)^{n-1} Q_a^{(i)} 1', \\
    P\left(\tau^{(i)}_{r} = n\right) &= \alpha^{(i)} \left(R^{(i)}\right)^{n-1} Q_r^{(i)} 1', \\
    P\left(\tau^{(i)}_{t} = n\right) &= \alpha^{(i)} \left(R^{(i)}\right)^{n-1} Q_t^{(i)} 1'.
\end{align*}
\]

(5.1)

where \( 1' \) denote the unit vector of appropriate dimension so that the matrix multiplication is conformable and \( \alpha^{(i)} \) denote the initial probability distribution.

Then the

\[
\begin{align*}
    P(\text{Acceptance of the product in phase 1}) &= \sum_{n=1}^{\infty} \alpha^{(i)} \left(R^{(i)}\right)^{n-1} Q_a^{(i)} 1' = \alpha^{(i)} [I - R^{(i)}]^{-1} Q_a^{(i)} 1', \\
    P(\text{Rejection of the product in phase 1}) &= \sum_{n=1}^{\infty} \alpha^{(i)} \left(R^{(i)}\right)^{n-1} Q_r^{(i)} 1' = \alpha^{(i)} [I - R^{(i)}]^{-1} Q_r^{(i)} 1', \\
    P(\text{Repair product}) &= \sum_{n=1}^{\infty} \alpha^{(i)} \left(R^{(i)}\right)^{n-1} Q_t^{(i)} 1' = \alpha^{(i)} [I - R^{(i)}]^{-1} Q_t^{(i)} 1'.
\end{align*}
\]

(5.2)

As in phase one we can associate a Markov chain in the second phase also. Let \( \left\{ A^{(2)}_n, n \geq 1 \right\} \) be the Markov chain associated with the second phase of the test.

Here again we will denote the states of the Markov chain by triplet as denoted in the phase one. Transition probabilities in the second phase is given by
Since there is no scope for repair in the second phase, we can partition the
states of the Markov chain into subclasses (i) Non-decisive class (ii) accepting
class and (iii) rejecting class. On achieving any states in the class leads to the
acceptance or rejection of the product then such a class is respectively called as
accepting and rejecting class. As in phase one the transition probability matrix
associated with the embedded Markov chain is given by

\[
P_2 = \begin{bmatrix}
R^{(2)} & Q_a^{(2)} & Q_r^{(2)} \\
0 & I & 0 \\
0 & 0 & I
\end{bmatrix}
\]

with \( R^{(2)} \) denoting the matrix of transition probabilities between the non-
decisive states, \( Q_a^{(2)} \) and \( Q_r^{(2)} \) respectively denote the matrix of transition
probability from a non-decisive state to an accepting, rejecting state. As the test is
stopped once we reach any of the accepting or rejecting states, those states are all
absorbing states. Also let the number of trials required either to accept or to reject
the product in the second phase be denoted respectively by \( \tau_a^{(2)} \) and \( \tau_r^{(2)} \). Then by
exploiting Chappman-Kolmogrov (Bhat 2002, Feller 1968) equation and the
second phase t.p.m we have

\[
P\left(\tau_a^{(2)} = n\right) = \alpha^{(2)} \left(R^{(2)}\right)^{n-1} Q_a^{(2)} 1', \quad P\left(\tau_r^{(2)} = n\right) = \alpha^{(2)} \left(R^{(2)}\right)^{n-1} Q_r^{(2)} 1',
\]  

(5.3)
\[ P(\text{Acceptance of the product in phase 2}) = \sum_{n=1}^{\infty} \alpha^{(2)}(R^{(2)})^{n-1} Q^{(2)}_1 = \alpha^{(2)}(I - R^{(2)})^{-1} Q^{(2)}_1, \]
\[ P(\text{Rejection of the product in phase 2}) = \sum_{n=1}^{\infty} \alpha^{(2)}(R^{(2)})^{n-1} Q^{(2)}_i = \alpha^{(2)}(I - R^{(2)})^{-1} Q^{(2)}_1, \]

Hence from equations (5.2) and (5.4) we have

\[ P(\text{Acceptance of the product}) = \alpha^{(i)}(I - R^{(i)})^{-1} \left[ Q^{(i)}_1 + Q^{(i)}_1 \alpha^{(2)}(I - R^{(2)})^{-1} Q^{(2)}_1 \right], \]
\[ P(\text{Rejection of the product}) = \alpha^{(i)}(I - R^{(i)})^{-1} \left[ Q^{(i)}_1 + Q^{(i)}_1 \alpha^{(2)}(I - R^{(2)})^{-1} Q^{(2)}_1 \right], \]

(5.5)

Now we will consider the states in the different classes in both phases of the test. Since both tests vary only in the first phase and Markov chain \( \{A^{(2)}_n, n \geq 1\} \) and their partitions are the same for both phases of the test. They differ only in their partition of the states of the Markov chain \( \{A^{(i)}_n, n \geq 1\} \). First we shall consider the case for the first model. States in the non-decisive class, which forms a \( 2 + (k_i - 1)d_i + (d_i - c_i + 1)(c_i - 1) + (c_i - 1)(c_i - 2)/2 \) dimensional vector, can be further subdivided into three sub classes, namely,

(i) \( \{(0, s, 0), (x, s, z) : 1 \leq x \leq k_i - 1, \quad 0 \leq z \leq d_i - 1\} \) with \( 1 + (k_i - 1)d_i \) states

(ii) class with \( 1 + (d_i - c_i + 1)(c_i - 1) \) states given by
\[
\{(0, f, 0), (1, f, 1), \ldots, (c_i - 1, f, c_i - 1), (1, f, 2), \ldots, (c_i - 1, f, c_i), \ldots, \\
(1, f, d_i - c_i), \ldots, (c_i - 1, f, d - 1)\}.
\]

(iii) Class given by \( \{(1, f, d_i - c_i + 2), \ldots, (c_i - 2, f, d_i - 1), (1, f, d_i - c_i + 3), \ldots, (c_i - 3, f, d_i - 1), \ldots, (1, f, d_i - 1)\} \), with dimension \( (c_i - 1)(c_i - 2)/2 \).

Accepting class, a row vector with dimension \( d_i \) is given by \( \{(k_i, s, 0), (k_i, s, 1), \ldots, (k_i, s, d_i - 1)\} \). Rejecting class given by \( \{(1, f, d_i), (2, f, d_i), \ldots\} \).
(3, f, d_i), ..., (c, f, d_i}) has \( c_1 \) states. A \((d_i - c_1)\) states class, with elements given by \( \{(c_1, f, c_1), (c_1, f, c_1 + 1), ..., (c_1, f, d_i - 1)\} \) forms the repair class.

For the second model, the rejecting class and the repair class got replaced each other. Hence the class of states \( \{(1, f, d_i), (2, f, d_i), (3, f, d_i), ..., (c, f, d_i)\} \) forms the rejection class, corresponds to the rejection of the products. Clearly there are \( c_1 \) states in the class. Once the Markov chain enters any of the states in the repair class, with \((d_i - c_1)\) states, given by \( \{(c_1, f, c_1), (c_1, f, c_1 + 1), ..., (c_1, f, d_i - 1)\} \) then the product is sent for repair/corrective action. The non-decisive class and the accepting class remains to be the same.

Now let us consider the partition of states in the second phase. Since the conditions of acceptance and rejection remain same in both the models, the partition is same for both the models. The class \((k_2, s, z), 0 \leq z \leq d_2 - 1\) with \( d_2 \) states forms the accepting class, where as the class of \( d_2 \) states given by \((x, f, d_2), 1 \leq x \leq d_2\) . Non-decisive class consists of \( r \) states , where

\[
    r = 2 + (k_2 - 1)d_2 + d_2(d_2 - 1)/2 ,
\]

states and is given by

\[
    \{(0, s, 0), (1, s, 0), ..., (k_2 - 1, s, 0), (1, s, 1), ..., (k_2 - 1, s, d_2 - 1), (0, f, 0), (1, f, 1), ..., (d_2 - 1, f, d_2 - 1), (1, f, 2), ..., (d_2 - 2, f, d_2 - 1), ..., (1, f, d_2 - 1)\} .
\]

5.3 Results

In this section we will derive the expressions for the probability generating functions (pgfs) of the random variable the number of trials under different scenario. Let \( N \) and \( N^{(i)} \), \( i = 1, 2 \) denote the number of trials in the test and in the \( i^{th} \) state respectively. Then we have
Theorem 5.1: If \( \phi_{N}(z), \phi_{N^{(i)}}(z), \ i = 1, 2 \) denote the pgf of \( N, N^{(i)}, i = 1, 2 \) Then

i. \( \phi_{N}(z) = [z-1] \alpha^{(1)} \left[ I - zR^{(1)} \right]^{-1} \left( 1 + zQ_{1}^{(1)}1^{'} \right) \) \( \alpha^{(2)} \left[ I - zR^{(2)} \right]^{-1} \) \( 1^{'} \) \( (5.6) \)

ii. \( \phi_{N^{(i)}}(z) = [z-1] \alpha^{(i)} \left[ I - zR^{(i)} \right]^{-1} 1^{'} + 1, \ i = 1, 2 \)

Proof:

We have \( \phi_{N}(z) = E\left( z^{n} \right) \)

\[ = \sum_{n=1}^{\infty} \left\{ z^{n} \alpha^{(1)} \left( R^{(1)} \right)^{n} \left( Q_{1}^{(1)}1^{'} + Q_{1}^{(1)}1^{'} \right) + \sum_{\mu=1}^{n-1} \alpha^{(1)} \left( R^{(1)} \right)^{\mu} Q_{1}^{(1)}1^{'} \alpha^{(2)} \left( R^{(2)} \right)^{\mu-1} \right\} \]

\[ = \sum_{n=1}^{\infty} z^{n} \alpha^{(1)} \left( R^{(1)} \right)^{n} \left( Q_{1}^{(1)}1^{'} + Q_{1}^{(1)}1^{'} \right) = \sum_{n=1}^{\infty} z^{n} \alpha^{(1)} \left( R^{(1)} \right)^{n} \left( 1^{'} - R^{(1)}1^{'} - Q_{1}^{(1)}1^{'} \right) \]

\[ = z \sum_{n=1}^{\infty} \alpha^{(1)} \left( zR^{(1)} \right)^{n} 1^{'} - \sum_{n=0}^{\infty} \alpha^{(1)} \left( zR^{(1)} \right)^{n} 1^{'} + \alpha^{(1)} \left( zR^{(1)} \right)^{n} 1^{'} - \sum_{n=1}^{\infty} z^{n} \alpha^{(1)} \left( R^{(1)} \right)^{n} Q_{1}^{(1)}1^{'} \]

\[ = [z-1] \alpha^{(1)} \left[ I - zR^{(1)} \right]^{-1} 1^{'} + 1 - \sum_{n=1}^{\infty} z^{n} \alpha^{(1)} \left( R^{(1)} \right)^{n} Q_{1}^{(1)}1^{'} \] \( (5.7) \)

\[ \sum_{n=1}^{\infty} \sum_{\mu=1}^{n-1} \alpha^{(1)} \left( R^{(1)} \right)^{\mu} Q_{1}^{(1)}1^{'} \alpha^{(2)} \left( R^{(2)} \right)^{\mu-1} \left\{ Q_{1}^{(2)}1^{'} + Q_{1}^{(2)}1^{'} \right\} = \sum_{n=1}^{\infty} \sum_{\mu=1}^{n-1} \alpha^{(1)} \left( R^{(1)} \right)^{\mu} Q_{1}^{(1)}1^{'} \alpha^{(2)} \left( R^{(2)} \right)^{\mu-1} \left\{ 1^{'} - R^{(2)}1^{'} \right\} \]

Interchanging the order of summation, we have

\[ = \sum_{\mu=1}^{\infty} \alpha^{(1)} \left( R^{(1)} \right)^{\mu} Q_{1}^{(1)}1^{'} \alpha^{(2)} \left( R^{(2)} \right)^{\mu-1} \left\{ 1^{'} - R^{(2)}1^{'} \right\} \sum_{n=\mu+1}^{\infty} z^{n} \left( R^{(2)} \right)^{n-\mu-1} \left[ 1^{'} - R^{(2)}1^{'} \right] \]

\[ = \sum_{n=\mu+1}^{\infty} z^{n} \left( R^{(2)} \right)^{n-\mu-1} \left[ 1^{'} - R^{(2)}1^{'} \right] = z^{\mu} \sum_{n=\mu+1}^{\infty} z^{n-\mu} \left( R^{(2)} \right)^{n-\mu-1} \left[ 1^{'} - R^{(2)}1^{'} \right] \]
\[ = z^\mu \left\{ \sum_{n=1}^{\infty} \left( zR^{(2)} \right)^{(n-1)} \mathbf{1}' - \sum_{n=1}^{\infty} \left( zR^{(2)} \right)^{(n)} \mathbf{1}' + \mathbf{1}' \right\} \]
\[ = z^\mu \left[ z \sum_{n=1}^{\infty} \left( R^{(2)} \right)^{(n-1)} \mathbf{1}' - \sum_{n=0}^{\infty} \left( R^{(2)} \right)^{(n)} \mathbf{1}' + \mathbf{1}' \right] \]
\[ = z^\mu \left\{ (z-1) \left[ I - zR^{(2)} \right]^{-1} \mathbf{1}' + \mathbf{1}' \right\} \]  
(5.9)

\[ \sum_{n=1}^{\infty} z^n \sum_{\mu=1}^{n-1} \alpha^{(i)} \left( R^{(i)} \right)^{\mu-1} Q_i^{(i)} \mathbf{1}' \alpha^{(2)} \left( R^{(2)} \right)^{n-\mu-1} \left\{ Q_a^{(i)} \mathbf{1}' + Q_r^{(i)} \right\} \mathbf{1}' \]
\[ = \sum_{\mu=1}^{\infty} \alpha^{(i)} \left( R^{(i)} \right)^{\mu-1} Q_i^{(i)} \mathbf{1}' \alpha^{(2)} \left( z - 1 \right) \left[ I - zR^{(2)} \right]^{-1} \mathbf{1}' + \sum_{\mu=1}^{\infty} \alpha^{(i)} z^\mu \left( R^{(i)} \right)^{(\mu-1)} Q_i^{(i)} \mathbf{1}' \]

Using (5.9)
\[ = \sum_{\mu=1}^{\infty} \alpha^{(i)} z \left( zR^{(i)} \right)^{\mu-1} Q_i^{(i)} \mathbf{1}' \alpha^{(2)} \left( z - 1 \right) \left[ I - zR^{(2)} \right]^{-1} \mathbf{1}' + \sum_{\mu=1}^{\infty} \alpha^{(i)} z^\mu \left( R^{(i)} \right)^{(\mu-1)} Q_i^{(i)} \mathbf{1}' \]
(5.10)

Now using (5.8) and (5.10) in (5.7), we have
\[ \phi_n(z) = [z-1] \alpha^{(i)} \left[ I - zR^{(i)} \right]^{-1} \mathbf{1}' + 1 - \sum_{n=1}^{\infty} z^n \alpha^{(i)} \left( R^{(i)} \right)^{n-1} Q_i^{(i)} \mathbf{1}' + \]
\[ \sum_{\mu=1}^{\infty} \alpha^{(i)} z \left( zR^{(i)} \right)^{\mu-1} Q_i^{(i)} \mathbf{1}' \alpha^{(2)} \left( z - 1 \right) \left[ I - zR^{(2)} \right]^{-1} \mathbf{1}' + \sum_{\mu=1}^{\infty} \alpha^{(i)} z^\mu \left( R^{(i)} \right)^{(\mu-1)} Q_i^{(i)} \mathbf{1}' \]
\[ \phi_n(z) = [z-1] \alpha^{(i)} \left[ I - zR^{(i)} \right]^{-1} \left\{ \mathbf{1}' + zQ_i^{(i)} \mathbf{1}' \left[ I - zR^{(2)} \right]^{-1} \mathbf{1}' \right\} \]

(ii) Also we have,
\[ \phi_n \left( N^{(i)} = n \right) = \phi_n \left( \tau_{a}^{(i)} = n \right) + \phi_n \left( \tau_{r}^{(i)} = n \right) + \phi_n \left( \tau_{t}^{(i)} = n \right) \]
\[ = \alpha^{(i)} \left( R^{(i)} \right)^{n-1} \left( Q_a^{(i)} \mathbf{1}' + Q_r^{(i)} \mathbf{1}' + Q_t^{(i)} \mathbf{1}' \right) \mathbf{1}' = \alpha^{(i)} \left( R^{(i)} \right)^{n-1} \left( \mathbf{1}' - R^{(i)} \mathbf{1}' \right) \]
\[
\phi_{N(i)}(z) = \sum_{n=1}^{\infty} z^n P(N^{(i)} = n) = \sum_{n=1}^{\infty} z^n \alpha^{(i)}(R^{(i)})^{(n-1)} \left(1' - R^{(i)} 1' \right)
\]

\[
= z \sum_{n=1}^{\infty} z^{(n-1)} \alpha^{(i)}(R^{(i)})^{(n-1)} 1' - \sum_{n=1}^{\infty} z^n \alpha^{(i)}(R^{(i)})^n 1'
\]

\[
= z \sum_{n=1}^{\infty} \alpha^{(i)}(zR^{(i)})^{(n-1)} 1' - \sum_{n=0}^{\infty} \alpha^{(i)}(zR^{(i)})^n 1' + \alpha^{(i)} 1'
\]

\[
= [z - 1] \sum_{n=1}^{\infty} \alpha^{(i)}(zR^{(i)})^{(n-1)} 1' + 1 = [z - 1] \alpha^{(i)} \left[ I - zR^{(i)} \right]^{-1} 1' + 1
\]

Working on the same line as in (ii) we can prove (iii)

\[
\phi_{N(2)}(z) = [z - 1] \alpha^{(2)} \left[ I - zR^{(2)} \right]^{-1} 1' + 1
\]

**Remark 5.1:** We can find the expected number of trials in the test and in each phase of the test by taking the derivative of the pgfs with respect to \( s \) and setting \( s = 1 \). Hence we have from the above generating function

1. \( E(N) = \alpha^{(i)} \left[ I - R^{(i)} \right]^{-1} 1' + \alpha^{(i)} \left[ I - R^{(i)} \right]^{-1} Q^{(i)} \left[ I - R^{(i)} \right]^{-1} 1' \)

2. \( E(N^{(i)}) = \alpha^{(i)} \left[ I - R^{(i)} \right]^{-1} 1' \)

3. \( E(N^{(2)}) = \alpha^{(2)} \left[ I - R^{(2)} \right]^{-1} 1' \)

It will be interesting to find the expected number of trials before we accept or reject the product.

**Theorem 5.2:** Expected number trials given that accept or reject the product is

\[
E(N|\text{Accept}) = \alpha^{(i)} \left[ I - R^{(i)} \right]^{-2} Q_a^{(i)} 1' + \alpha^{(i)} \left[ I - R^{(i)} \right]^{-1} \left\{ Q^{(i)} 1' \alpha^{(2)} \left[ I - R^{(2)} \right]^{-1}
\right.
\]

\[
+ \left[ I - R^{(i)} \right]^{-1} Q^{(i)} 1' \alpha^{(2)} \left[ I - R^{(2)} \right]^{-1} Q_a^{(2)} 1'
\]

63
\[ E(N|\text{Reject}) = \alpha^{(i)} \left[ I - R^{(i)} \right]^{-2} Q^{(i)}_t 1' + \alpha^{(i)} \left[ I - R^{(i)} \right]^{-1} \left\{ Q^{(i)}_t 1' \alpha^{(2)} \left[ I - R^{(2)} \right]^{-1} + \left[ I - R^{(1)} \right]^{-1} Q^{(i)}_t 1' \alpha^{(2)} \left[ I - R^{(2)} \right]^{-1} Q^{(2)}_t 1' \right\} \]

**Proof:**

We have

\[ E(N|\text{Accept}) = \sum_{n=1}^{\infty} n \alpha^{(i)} \left( R^{(i)} \right)^{(n-1)} Q^{(i)}_a 1' + \sum_{n=1}^{\infty} n \sum_{\mu=1}^{n-1} \alpha^{(i)} \left( R^{(1)} \right)^{\mu-1} Q^{(i)}_t 1' \alpha^{(2)} \left( R^{(2)} \right)^{(n-\mu-1)} Q^{(2)}_a 1'. \]

First let us consider

\[ \sum_{n=1}^{\infty} n \alpha^{(i)} \left( R^{(i)} \right)^{(n-1)} Q^{(i)}_a 1' = \alpha^{(i)} \left[ I - R^{(i)} \right]^{-2} Q^{(i)}_a 1'. \]

Also we have

\[ \sum_{n=1}^{\infty} n \sum_{\mu=1}^{n-1} \alpha^{(i)} \left( R^{(1)} \right)^{\mu-1} Q^{(i)}_t 1' \alpha^{(2)} \left( R^{(2)} \right)^{(n-\mu-1)} Q^{(2)}_a 1' = \sum_{\mu=1}^{\infty} \alpha^{(i)} \left( R^{(i)} \right)^{(\mu-1)} Q^{(i)}_t 1' \alpha^{(2)} \sum_{n=\mu+1}^{\infty} n \left( R^{(2)} \right)^{(n-\mu-1)} Q^{(1)}_a, \]

Interchanging the order of summations

\[ = \sum_{\mu=1}^{\infty} \alpha^{(i)} \left( R^{(i)} \right)^{(\mu-1)} Q^{(i)}_t 1' \alpha^{(2)} \left\{ \sum_{n=\mu+1}^{\infty} (n-\mu) \left( R^{(2)} \right)^{(n-\mu-1)} Q^{(2)}_a 1' + \mu \sum_{n=\mu+1}^{\infty} \left( R^{(2)} \right)^{(n-\mu-1)} Q^{(2)}_a 1' \right\} \]

\[ = \sum_{\mu=1}^{\infty} \alpha^{(i)} \left( R^{(i)} \right)^{(\mu-1)} Q^{(i)}_t 1' \alpha^{(2)} \left\{ \left[ I - R^{(2)} \right]^{-2} Q^{(2)}_a 1' + \mu \left[ I - R^{(2)} \right]^{-1} Q^{(2)}_a 1' \right\} \]

\[ = \alpha^{(i)} \left[ I - R^{(i)} \right]^{-1} \left\{ Q^{(i)}_t 1' \alpha^{(2)} \left[ I - R^{(2)} \right]^{-1} + \left[ I - R^{(1)} \right]^{-1} Q^{(i)}_t 1' \alpha^{(2)} \left[ I - R^{(2)} \right]^{-1} Q^{(2)}_a 1' \right\} \]

64
\[ E(N|\text{Accept}) = \alpha^{(i)} \left[ I - R^{(i)} \right]^{-1} Q_a^{(i)} \mathbf{1} + \alpha^{(i)} \left[ I - R^{(i)} \right]^{-1} \left\{ Q_i^{(i)} \left[ I - R^{(2)} \right]^{-1} \left[ I - R^{(2)} \right]^{-1} + \left[ I - R^{(1)} \right]^{-1} Q_i^{(i)} \left[ I - R^{(2)} \right]^{-1} \left[ I - R^{(2)} \right]^{-1} Q_a^{(2)} \right\} \]

Proceeding as in (i) we can prove (ii).

5.4 Numerical Illustrations

In this section we will consider examples for each model to validate the above-discussed results. Authors had developed a MATLAB® program that generates, for given values of \( k, d, c, k, \) and \( d, \) the involved sub matrices and computes various probabilities and the expected number of trials under different conditions. Here we will consider a \( l \) dependent trials discussed by Aki and Hirano (1999). The constants under the given conditions are given by

\[
\begin{align*}
k_1 &= 12; p_{nu} (t) = 1 - 1/(t+1); p_{ff} (t) = 1 - 1/(t^{1/2} + 1); \\
k_2 &= 5; d_2 = 8; p_{nu} (t) = 1 - (1/(t+1))^2; p_{ff} (t) = 1 - (t^{1/3} / (t+2));
\end{align*}
\]

with the initial distribution, in both phases, as equally likely in the states \((0,s,0)\) and \((0,f,0)\).

5.4.1 Model 1

First we consider the first model and obtain different probabilities for values of \( c_1 \) less than \( d_1 = 18 \). Figure 1 probability of acceptance in the test, probability of acceptance in phase one, phase two are depicted. Probabilities of repair for the product for various values of \( c_1 \) are also given.
It can be seen from the figure 5.1 that as $c_1$ increases the probability of repair converges to zero and the Probability of acceptance of the product in the test and in phase one stabilize to the same limit.

In figure 5.2 expected number of trials required to terminate the test and in phase one are depicted for various values of $c_1$. 
Figure 5.2 Values of different Expected value measures for various values of \( c_1 \) in the case of Model 1

The expected number of trials in the test and in phase one, even though distant apart for small values of \( c \) will converge to the same limit as \( c \) increases. Also it can be seen that the expected number of trials will converge to a limit as the values of \( c_1 \) increases.

5.4.2 Model 2

Here we will obtain different probabilities for values of \( d_1 \) ranging from six to 50. For the given constant value of \( c = 6 \).

In figure 5.3 as in figure 1, plots of various probabilities against corresponding \( d_1 \) value is shown. The probability of interest include probability of acceptance, probability of acceptance in phase one, phase two and the probability of repair.
As the value of $d_1$ increase the probability of repair as well as probability of acceptance in phase two converges to zero. Also Probability of acceptance in the test and in phase one converges to the same limit.
It can be seen that as the value of $d_i$ increases the expected number of trials becomes convergent. It asserts the logical results that the number of trials in phases one becomes equal to the number of trials in the test.

5.5 Conclusion

In this chapter we proposed two new models of start-up demonstration having two phases with the condition for the corrective action in the first model being specified number of consecutive failures and in the second model being specified number of random failures.