CHAPTER - I

INTRODUCTION

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A fixed point of a mapping \( T : E \rightarrow E \) of a set \( E \) is an \( x \in E \) which is mapped onto itself, that is, \( Tx = x \), the image \( Tx \) coincides with \( x \). In other words, a fixed point of \( T \) is a solution of the functional equation \( Tx = x \), \( x \in E \). For example, translation has no fixed point, rotation of the plane has a single fixed point (i.e., the center of rotation), the mapping \( T : \mathbb{R} \rightarrow \mathbb{R} \) defined by \( Tx = x^2 \) has two fixed points \( 0 \) and \( 1 \).

Fixed point theorem is a combination of conditions on the set \( E \) and the mapping \( T : E \rightarrow E \) which, in turn, assures that \( T \) has at least one point of \( E \) fixed. Such theorems are important tools for finding solutions of nonlinear functional equations which can be formulated in terms of finding the fixed points.

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of a given nonlinear mapping of an infinite dimensional functional space $E$ into itself. For details, one can refer to [1,3,49,67,70,115,118,119-120,132,149,167, 172,174,179 and 190].

Before we state the fixed point problem in Banach space, let us discuss the linear case, which is where the whole theory originated: Possibly the most important result in this case is the theorem of Brouwer [13,14], which says that any continuous mapping taking the closed unit ball of $R^n$ to itself has a fixed point. This result was previously known to Poincare [136] in an equivalent form. An interesting proof which is not based on algebraic topology is given by Scarff [151]. The underlying causes behind Brouwer's theorem are the compactness and convexity of the unit ball of $R^n$. Thus in [152,153], Schauder has extended Brouwer's theorem to obtain the same conclusion for any compact convex set in any linear topological space
which is locally convex.

The fixed point theorem generally known as the Banach contraction principle, appeared in explicit form in Banach’s thesis [7] in 1922, where it was used to establish the existence of a solution for an integral equation. It has proved to be one of the most durable and fruitful methods in analysis. This theorem has its origins in Euler and Cauchy’s work [36] on the existence and uniqueness of a solution to the differential equation. An interesting generalization of the Banach contraction principle was given by Ekland [65]. For more on this theorems one can consult [86,167].

Banach contraction principle gives sufficient conditions for the existence and uniqueness of a fixed point for a class of mappings, called contraction. The definition is as follows:
DEFINITION C: Let $E$ be a normed space. A mapping $T : E \to E$ is called a contraction on $E$ if there exists a positive real number $k < 1$ such that for all $x, y \in E$,

$$\|Tx - Ty\| \leq k \|x - y\| \quad (k < 1).$$

Geometrically, this means that any points $x$ and $y$ have images that are closer together than those points $x$ and $y$. More precisely, the ratio $\|Tx - Ty\|/\|x - y\|$ does not exceed a constant $k$ which is strictly less than 1. Contraction is thus a Lipschitz mapping with Lipschitz constant $k < 1$.

The existence of a fixed point is often useless in applications without an algorithm for calculating its value. Hence from the point of view of application, it is essential not only to show the existence of fixed points of such mappings under suitable hypothesis but also to develop systematic techniques for the construction or calculation of such points.
Banach contraction principle also gives a constructive procedure for obtaining better and better approximations to the fixed point (the solution of the practical problem). This procedure is called an iteration. Iterative procedures are used in nearly every branch of applied mathematics.

Numerous iteration methods have been devised for finding fixed or periodic points of nonlinear operators and solution of variational inequalities in the papers of many authors such as [6, 23, 27, 28, 31, 57-59, 125, 127, 129-130, 145, 177-178, 182 and 189].

Many of the most researchers used following two iteration procedures for finding fixed points of nonlinear operators:

**Mann Iteration Process**: (See [116, 141]) which is defined as follows:
For $K$ a convex subset of a Banach space $E$ and $T$ a mapping of $K$ into itself, the sequence $\{x_n\}_{n=0}^{\infty}$ in $K$ is defined by:

$$
x_0 \in K, \\
x_{n+1} = (1 - c_n)x_n + c_nT(x_n), \quad n \geq 0
$$

where,

(i) $0 \leq c_n < 1, \quad n \geq 0$;

(ii) $\lim_{n \to \infty} c_n = 0$;

(iii) $\sum_{n=1}^{\infty} c_n = \omega$.

In some applications, condition (iii) is replaced by $\sum_{n=1}^{\infty} c_n (1 - c_n) = \omega$.

**Ishikawa Iteration Process**: (See [84, 141]) which is defined as follows:

For $K$ a convex subset of a Banach space $E$ and $T$ a mapping of $K$ into itself, the sequence $\{x_n\}_{n=0}^{\infty}$ in $K$ is defined by:

$$
x_0 \in K.
$$
\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n, \]
\[ y_n = (1 - \beta_n)x_n + \beta_n T x_n, \quad n \geq 0, \]

where \( \{q_n\} \) and \( \{r_n\} \) are the sequences of positive numbers satisfying the conditions

(i) \( 0 \leq \alpha_n \leq \beta_n \leq 1, \quad n \geq 0. \)

(ii) \( \lim_{n \to \infty} \beta_n = 0, \) and

(iii) \( \sum_{n=1}^{\infty} \alpha_n \beta_n = \infty. \)

In its original form the Ishikawa iteration process does not include the Mann iteration process because of the condition:

\[ 0 \leq \alpha_n \leq \beta_n \leq 1. \]

For, if the Ishikawa process were to include the Mann process as a special case, then one would have to assign each \( \beta_n \) to be zero, forcing then each \( \alpha_n \) to be zero. In an effort to have an Ishikawa type iteration scheme which does include the Mann iteration process as a special case, some authors have modified the inequality condition to read \( 0 \leq \alpha_n, \beta_n \leq 1. \)
The above iteration processes have been studied extensively by several authors and have been successfully employed to approximate fixed points of nonlinear mappings (when these mappings are already known to have fixed points) and to approximate solutions of several nonlinear operator equations in Banach spaces (e.g., [23, 38-41, 45-46, 54, 80, 83, 124, 141-142, 144, 160-161, 180]).

Rhoades [141] compared the performance of the two schemes and showed that even though the processes are similar, they may exhibit different behavior for different classes of nonlinear mappings. It is, therefore, of interest to examine the behaviors of the two iteration process for any given class of mappings.

1.2 In the Banach fixed point theorem the Lipschitz condition \( k < 1 \) is crucial even for the existence part of the result, but within more restrictive setting
an amplified fixed point theorem exists for the case
$k = 1$. Mapping which satisfies the condition for $k = 1$
are known as nonexpansive mapping (i.e., $\|Tx-Ty\| \leq \|x-y\|
for each pair of points $x,y$ in the space) and the
theory of nonexpansive mapping is fundamentally
different from that of contraction mappings. It is
important to note that nonexpansive mapping may be
fixed point free (see [2,154,166]), and even if a
nonexpansive mapping $T$ has a nonempty fixed point set,
the Picard iterates may fail to converge (see [52]).
Again, fixed point set need not contain just one point.

One of the principal applications of the theory of
nonexpansive mapping in a functional analytic context
has been to the study of monotone and accretive
operators. A brief discussion of the fundamental way in
which the theory of nonexpansive and accretive mappings
are interwined may be found in [71].
Fixed point theory for nonexpansive mappings has its origin in the existence theorems of Browder [15], Gohde [73] and Kirk [94-95]. The principal results on approximation of fixed points of nonexpansive mappings in a uniformly convex setting are due to Kaniel [89]. These class of mappings have been studies by various researchers e.g. [17,24-26,29,61,64,70,78,81-82,85,93,97,99-104,107-108,121-122,126,128,140,147-148,156,158-159,163,173 and 186]. However, the attempt to classify those subsets of Banach spaces which have the fixed point property for such mappings has now became a study in its own right - one which has yielded many elegant results and let to numerous discoveries in Banach space geometry.

1·3 It is known that among all Banach spaces the Hilbert spaces are the ones with the best geometric structures. The reason for saying this is that certain geometric properties which characterized Hilbert spaces...
(e.g. the parallelogram law; the polarization identity) make problem posed in Hilbert spaces comparatively straightforward and relatively easy to solve. In several applications, however, there are now considerable research efforts have been made to find Banach spaces which are nearest to Hilbert spaces in the sense that their geometric structure can be characterized with relations similar to those that characterize Hilbert spaces. Several results have been obtained in this direction (e.g. [32, 34, 42, 51, 88, 92, 113-114, 137, 155, 161, 183 and 187]).

Set Valued Local Strictly Hemicontractive Mapping:

1.4 The class of pseudocontractive and strictly pseudocontractive mappings have introduced by Browder and Petryshyn [23] in Hilbert space and in Banach space by Browder [16]. This class of mappings, in single valued case, includes all nonexpansive mappings. In addition to generalizing the nonexpansive mappings, the
pseudocontractive mappings are characterized by its firm connection with the accretive operators. This class of nonlinear mappings have been studied by various researchers (e.g. [68,84,87,96]).

Weng [180], introduced the class of local strictly pseudocontractive mappings.

On the other hand using the idea of Trikomi [175] Dotson Jr. [55] considerably weaken the concept of nonexpansive mapping by defining quasi-nonexpansive mapping. The definition is as follows:

**DEFINITION Q:** Let $\text{Fix}(T)$ denotes the fixed point set of $T$. A mapping $T : E \to E$ is said to be quasi-nonexpansive if

$$\|Tx - p\| \leq \|x - p\|$$

for all $x \in E$ and $p \in \text{Fix}(T)$.

Importance of this class of mapping is that it does not requires the full force of nonexpansiveness.
but requires existence of at least one fixed point together with such nature. Detailed study about quasi-nonexpansive mappings may be found in Petryshyn and Williamson [133]. This idea has stimulated authors to define hemicontractive from pseudocontractive and then strictly hemicontractive mapping in the same way. These class of mappings have been studied by various researchers (e.g. [45,50,80].)

In Chapter - II, we introduced the class of set valued local strictly hemicontractive mappings and proved that both Mann and Ishikawa iteration process converges strongly to the fixed point of this class of mappings in general uniformly smooth Banach space. Our results extend, generalize and improve several known result concerning to convergence of pseudocontractive, strictly pseudocontractive, local strictly pseudocontractive, hemicontractive and strictly hemicontractive mappings.
Asymptotically Nonexpansive Mappings:

1.5 In 1972, Goebel and Kirk [69], introduced the class of asymptotically nonexpansive mappings as a generalization of nonexpansive mappings. The class of asymptotically nonexpansive mappings is wider than the class of nonexpansive mappings. They [69] proved that, every asymptotically nonexpansive self-mapping of a nonempty closed, bounded and convex subset of a uniformly convex Banach space has a fixed point.

In 1978, Bose [11] proved convergence theorem for this class of mappings in uniformly convex Banach space. After this, several authors (e.g., [74, 143, 184]) have been concerned with the iterative construction of a fixed point of an asymptotically nonexpansive mapping as the weak limit of the sequence \( \{T^n x\}_{n \in \mathbb{N}} \) of iterates, assuming that \( T \) is (weakly) asymptotically regular.

On the other hand, Passty [131] extended the definition of asymptotically nonexpansive mapping to
sequence of mappings. He [131] also proved weak convergence theorem for this class of mappings in a uniformly convex Banach space with Fréchet-differentiable norm.

In Chapter - III, we proved convergence theorems for asymptotically nonexpansive mapping in Banach space of type \((U, \alpha, m+1, m)\) using Mann and Ishikawa type iteration scheme. We also proved convergence theorem for Passty type asymptotically nonexpansive mappings in uniformly convex Banach space equipped with Fréchet differentiable norm. Our results extend and generalize several known results on convergence of nonexpansive and asymptotically nonexpansive mappings.

**Strongly H-Accretive Operator:**

1.6 The accretive operators were introduced independently by Browder [16] and Kato [91]. Main interest in such operators is stems mainly from their
firm connection with the existence theory for nonlinear equation of evolution in Banach space ( e.g. [20,53,91, and 117] ).

If $E = H$, a Hilbert space, one of the earliest problems in the theory of accretive operators was to solve the equation $x + Tx = f$ for $x$, given an element $f$ of $H$ and accretive operator $T$.

The firm connection between the pseudocontractive and the accretive operators is that a mapping $T$ is pseudocontractive if and only if $(I - T)$ is accretive.

Consequently, the mapping theory for accretive operators is closely related to the fixed point theory of pseudocontractive operators.

It is well known that ( e.g.[20]) that many physically significant problems can be modeled in terms of an initial value problem of the form

$$\frac{dx}{dt} = -Tx : x(0) = x_0$$
where \( I \) is either accretive or strongly accretive.

Typical examples of how such evolution equations arise are found in models involving either the heat, the wave or the Schrödinger equations.

The general study of accretive operators in Banach spaces having duality mapping was begun in Browder and Figueiredo [21]. Accretive operators in more general Banach spaces have been studied by Browder [18] and Vainberg [178]. Strongly accretive operators are sometimes also called strictly accretive. Strongly accretive operators have been studied by various authors (e.g. [10,19,77,123 and 125]).

In Chapter IV - Section A, we introduced the class of strongly \( H \)-accretive operators. Justification of this class of operator is that it does not assert the full force of strongly accretiveness but requires existence of at least one solution together with such nature. We also proved convergence theorems for this.
class of operators using both Mann and Ishikawa iteration process. Our results extend, generalize and improve several known result concerning to convergence of accretive and strongly accretive operators.

**H - Dissipative Operator**:

1.7 It is well known that if \( T \) is accretive, then \((-T)\) is known as dissipative. Following dissipative type operator is defined by Browder and Petryshyn [23]:

**DEFINITION** D : Let \( K \) be a nonempty closed convex subset of a Hilbert space, \( H \), with inner product \(< \cdot , \cdot >\). Then the mapping \( T : K \rightarrow K \) is said to be dissipative type if

\[
\text{Re} \, <Tx - Ty, x - y> \leq C \|x - y\|^2
\]

holds for some \( C < 1 \), and for all \( x, y \in K \).

They [23] proved that if \( T \) is also Lipschitz continuous on \( K \), then \( T \) has precisely one fixed point.
in K. Dunn [57], proved that, if \( \{c_n\} \subset (0,1] \) satisfies the following conditions

\[
\lim_{n \to \infty} c_n = 0, \quad \sum_{n=0}^{\infty} c_n = \infty
\]

then, the recursion

\[
x_{n+1} = (1 - c_n)x_n + c_n T x_n, \quad x_0 \in K
\]

will converge to the fixed point of \( T \).

Further, Dunn [59] introduced weaker version of dissipative type operator which is set valued. Recently Weng [181], extended this definition to the general complex uniformly smooth Banach spaces. He [181] also proved convergence theorem for this class of operator by using Mann iteration process.

In Chapter IV - Section 8, we introduced the concept of \( H \)-dissipative operator. The class of \( H \)-dissipative operator includes all dissipative type operators with nonempty fixed point set. We claim that \( H \)-dissipative operator is weaker than dissipative type operator considered by Weng [181].
i.e. any dissipative type operator \( T \) with \( \text{Fix}(T) \neq \emptyset \) is \( H \)-dissipative and converse is not true in general.

It was also proved in this section that both Mann and Ishikawa iteration process converges strongly to the fixed point of this class of mappings.

\[ p - \text{Uniformly Convex Banach Space} : \]

1.8 It is well known that many problems in a Hilbert space \( H \) are solved by applying the following identity:

\[
(1.8.1) \quad \| \lambda x + (1 - \lambda)y \|^2 = \lambda \| x \|^2 + (1 - \lambda) \| y \|^2 - \lambda(1 - \lambda) \| x - y \|^2
\]

for all \( x, y \) in \( H \) and \( 0 \leq \lambda \leq 1 \).

Therefore, one natural method to solve problems in a Banach space \( E \) is to establish equalities and (usually) inequalities in \( E \) analogous to (1.8.1). And it can be found in the some recent works of [109, 111, 135, 137, 145-146 and 170].
Xu [183], established some inequalities analogous to (1.8.1) in uniformly convex Banach spaces. He also obtain new characteristics of $p$-uniformly convexity of a Banach space $E$ via the functional $\| \cdot \|^p$. He proved that the functional $\| \cdot \|^p$ is uniformly convex on the Banach space $E$ if and only if $E$ is $p$-uniformly convex.

Gorniacki [76], proved some fixed point theorems for asymptotically regular mappings in $p$-uniformly convex Banach space. In Chapter V - Section A, we proved some common fixed point theorems for a pair of asymptotically regular mappings. Our results extend, generalize and improve several known results concerning to fixed points of asymptotically regular mappings.

**Nonlipschitzian Asymptotically Nonexpansive Mapping:**

1.9 In [98] Kirk introduced the nonlipschitzian asymptotically nonexpansive mapping, as a generalization of asymptotically nonexpansive mapping. It is
important to note that every asymptotically nonexpansive mapping is necessarily nonlipschitzian asymptotically nonexpansive but converse is not true.

He [98], also proved existence theorem for this class of mappings. Xu [184], extended above theorem to nearly uniformly convex Banach space.

In Chapter V-Section B, we prove weak convergence theorem for nonlipschitzian asymptotically nonexpansive mapping using weak asymptotic regularity. Our result extend, generalize and improve several results which are related to convergence of nonexpansive and asymptotically nonexpansive mappings.

At the end of the thesis, we have given a well arranged bibliography containing 190 references and a list of research publications.

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