CHAPTER V

FIXED POINTS OF
ASYMPTOTICALLY REGULAR
MAPPINGS

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5.1 In this chapter, we prove some theorems on fixed points of a pair of asymptotically regular mappings in \( p \)-uniformly convex Banach space. We also prove convergence theorem for nonlipschitzian asymptotically nonexpansive mapping using weak asymptotic regularity. Our results extend, generalize and improve results of Kruppel [105], Görnicki [74-76], Bose [11] and others.

The concept of asymptotic regularity was introduced by Browder and Petryshyn [22]. Let \( E \) be a real Banach space with norm \( \| \cdot \| \). A mapping \( T : E \rightarrow E \) is said to be asymptotically regular if

\[
\lim_{n \rightarrow \infty} \| T^{n+1}x - T^n x \| = 0
\]

for all \( x \) in \( E \), where \( T^n x \) denotes the \( n \)th iterate of \( T \).
It is well known that if $T$ is nonexpansive, then

$$T_t = t \cdot I + (1 - t) \cdot T$$

is asymptotically regular for all $0 < t < 1$ (see Goebel and Kirk [71]).

SECTION - A :

5.2 In [105], Kruppel proved the following result:

**THEOREM K.** Let $E$ be a uniformly convex Banach space and $K$ a closed, convex and bounded subset in $E$, and let $T$ be a mapping from $K$ into itself. If $T$ is asymptotically regular and $\lim_{n \to \infty} \|T^n\| \leq 1$ (where $\|T^n\|$ denotes the Lipschitz norm of $T$), then $T$ has a fixed point in $K$.

At the same time Lin [112] constructed an asymptotically regular mapping acting on a compact subset of the Hilbert space $l^2$ with no fixed point. Gornicki [75], gave the sufficient condition for existence of fixed point of asymptotically regular
mapping in $L^p$ spaces. Recently Gornicki [76], extended this result in $p$-uniformly convex Banach space.

A point $z$ in $E$ is said to be a common fixed point of self mapping $S$ and $T$ on $E$ if $Sz = z = Tz$. It is useful to know when a pair of mappings have a common fixed point in $E$. The following example from Choudhary and Nanda [47], will show that if both the mapping $S$ and $T$ have a fixed point in $E$, they may not have a common fixed point.

**Example**: Let $E = [-1, 1]$, 

$$Tx = \frac{x + 1}{2}$$

and,

$$Sx = \frac{x - 1}{2}$$

for $x \in [-1, 1]$.

Then, 1 is the unique fixed point of $T$ and $-1$ is the unique fixed point of $S$. Hence $T$ and $S$ have no common fixed point.

It was conjectured by number of mathematicians
that two continuous commuting mappings on \([0,1]\) have a common fixed point. Later on, it was found that this conjecture was false. So it is important to find sufficient conditions for existence of a common fixed point for a class of self mappings.

So the following question is natural: when does a pair of asymptotically regular mappings have a common fixed point? In this section we give sufficient conditions for existence of a common fixed point of a pair of asymptotically regular mappings in uniformly convex Banach space. Our result generalize results of Görnicki [75-76], Kruppel [105-106] and others.

The normal structure coefficient \(N(E)\) of \(E\) is defined by (c.f. Bynum [33]):

\[
\left\{ \frac{\text{diam } K}{r_K(K)} : K \text{ a bounded convex subset of } E \text{ consisting of more than one point} \right\}
\]
where \( \text{diam } K = \sup \{ \|x - y\| : x, y \in K \} \) is the diameter of \( K \) and \( r_K(K) = \inf (\sup_{x \in K} \|x - y\|) \) is the Chebyshev radius of \( K \) relative to itself.

\( E \) is said to have uniformly normal structure if \( N(E) > 1 \). It is known that a uniformly convex Banach space has uniformly normal structure (c.f. Danes [49]) and for a Hilbert space \( H \), \( N(H) = 2^{1/2} \). Recently, Pichugov [134] (c.f. Prus [138]) calculated that

\[
N(L^p) = \min \{ 2^{1/p}, 2^{(p-1)/p} \}, \quad 1 < p < +\infty.
\]

Some estimates for normal structure coefficient in other Banach spaces may be found in Prus [139].

Let \( p > 1 \) and denote by \( \lambda \) the number in \([0,1]\) and by \( W_p(\lambda) \) the function \( \lambda \cdot (1 - \lambda)^p + (1 - \lambda) \cdot (1 - \lambda) \).

The functional \( \| \cdot \|^p \) is said to be uniformly convex (c.f. Zalinescu [188]) on the Banach space \( E \) if there exists a positive constant \( c_p \) such that for all \( \lambda \in [0,1] \) and \( x, y \in E \) the following inequality holds:

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Xu [183] proved that the functional $\| \cdot \|^p$ is uniformly convex on the whole Banach space $E$ if and only if $E$ is $p$-uniformly convex, i.e., there exists a constant $c_p > 0$ such that the moduli of convexity (see [71]), $\Delta_E(E) \leq c_p \cdot \varepsilon^p$ for all $0 \leq \varepsilon \leq 2$.

Before presenting our main result of this section we need the following definition and Lemmas:

We denote by $\| L^n \|$ the Lipschitzian norm of pair $\{ S^n, T^n \}, n = 1, 2, \ldots$.

i.e.,

$$L^n = \sup \left\{ \frac{\| S^n x - T^n y \|}{\| x - y \|} : x \neq y, x, y \in K \right\}.$$

Lemma 1 [183]: Let $p > 1$ and let $E$ be a $p$-uniformly convex Banach space, $K$ a nonempty closed convex subset of $E$ and let $\{ x_n \} \subset E$ be a bounded sequence. Then there exists a unique point $z$ in $K$ such that

$$(5.2.1) \quad \| \lambda x + (1 - \lambda) y \|^p \leq \lambda \| x \|^p + (1 - \lambda) \| y \|^p - \lambda \varepsilon(\lambda) \cdot c_p \cdot \| x - y \|^p.$$
Lemma 2 [105]: Let $K$ be a nonempty closed convex subset of a Banach space $E$ and let $\{n_i\}$ be an increasing sequence of natural numbers. Assume that $T : K \rightarrow K$ is an asymptotically regular mapping such that for some $m \in \mathbb{N}, T^m$ is continuous. If

$$r(x) = \limsup_{i \rightarrow \infty} \|x - T^{n_i}u\| = 0$$

for some $u \in K$ and $x \in K$, then $Tx = x$.

We are now in position to give our main result of this section:

Theorem 1: Let $p > 1$ and let $E$ be a $p$-uniformly convex Banach space, $K$ a nonempty closed convex and bounded subset of $E$ and $S, T : K \rightarrow K$ are asymptotically regular mappings. If

$$\limsup_{n \rightarrow \infty} \|x_n - z\|^p \leq \limsup_{n \rightarrow \infty} \|x_n - x\|^p - c_p \cdot \|x - z\|^p$$

for every $x$ in $K$, where $c_p$ is the constant given in (5.2.1).
(5.2.5) \[ \lim_{n \to \infty} \inf \| L^n \| = k < \left[ \frac{1}{2} (1 + \sqrt{1 + 4c_p N^p}) \right]^{1/p} \]

(where \( \| L^n \| \) is the Lipschitz constant (norm) of pair \( \{s^n, t^n\} \). \( N \) is the normal structure coefficient of \( E \) and \( c_p \) is the constant given in (5.2.1).

Then \( S \) and \( T \) have a common fixed point in \( K \).

**PROOF:** If \( k < 1 \), then we have done, so assume \( k \geq 1 \).

Let \( \{n_i\} \) be a sequence of natural numbers such that

\[(5.2.6) \quad \lim_{n \to \infty} \inf \| L^n \| = \lim_{i \to \infty} \inf_{i} \| L^{n_i} \| \]

\[= k < \left[ \frac{1}{2} (1 + \sqrt{1 + 4c_p N^p}) \right]^{1/p} \]

For an arbitrary \( x_0 \in K \) and by Lemma 2, we can inductively construct a sequence \( \{x_m\}_{m=1}^{\infty} \) in the following manner:

\[ x_m \text{ and } x_{m+1} \text{ are the unique asymptotic center of sequences } \{s^n x_{m-1}\}_{m=1}^{\infty} \text{ and } \{t^n x_m\}_{m=1}^{\infty} \text{ respectively.} \]

Now, we set

\[ d_m = \lim_{n \to \infty} \sup_{n} \| x_m - T^n x_m \| , \ m = 1, 2, \ldots . \]
and
\[ r_m = \limsup_{n \to \infty} \|x_{m+1} - S^n x_m\|, \quad m = 0, 1, 2, \ldots \]

By the result of Casini and Maluta [35] and the asymptotic regularity of \( T \), we have

\[
\begin{align*}
    r_m &= \limsup_{i \to \infty} \|S^i x_m - x_{m+1}\| \\
    &\leq \liminf_{n \to \infty} \left\{ \|S^i x_m - T^n \| : i, j \geq t \right\} \\
    &\leq \liminf_{n \to \infty} \left( \limsup_{i \to \infty} \|S^i x_m - T^n \| \right) \\
    &\leq \liminf_{n \to \infty} \left( \limsup_{i \to \infty} \left[ \|S^i x_m - T^n \| + \|S^i x_m - T^n \| \right] \right) \\
    &\leq \liminf_{n \to \infty} \left( \limsup_{i \to \infty} \left[ \|S^i x_m - T^n \| + \sum_{v=0}^{n_i-1} \|T^v x_m - T^n \| \right] \right) \\
    &\leq \liminf_{n \to \infty} \left( \limsup_{i \to \infty} \left[ \|S^i x_m - T^n \| \right] \right) \\
    &\leq \limsup_{n \to \infty} \|x_m - T^n x_m\| \\
    &\leq k \cdot \frac{1}{N} d_m.
\end{align*}
\]

i.e.

\[(5.2.7) \quad r_m \leq k \cdot \frac{1}{N} d_m.\]
where $N$ is the normal structure coefficient of $E$.

For each fixed $m \geq 1$ and all $n_i, n_j$, we have

from (5.2.1):

$$
\| \lambda x_{m+1} + (1-\lambda) T^i x_{m+1} - S^n_i x_m \|_p
$$

$$
+ c_p \cdot \omega_p(\lambda) \cdot \| x_{m+1} - T^i x_m \|_p
$$

$$
\leq \lambda \| x_{m+1} - S^n_i x_m \|_p + (1-\lambda) \| T^i x_{m+1} - S^n_i x_m \|_p
$$

$$
\leq \lambda \| x_{m+1} - S^n_i x_m \|_p + (1-\lambda) \| T^i x_{m+1} - S^n_i x_m \|_p
$$

$$
+ \sum_{v=0}^{n_i-1} \| S^{n_i+v} x_m - S^n x_m \|_p.
$$

Taking the limit superior as $i \to +\infty$ on each side, by definition of $x_m$ and by the asymptotic regularity of $S$, we get

$$
r_m^p + c_p \cdot \omega_p(\lambda) \cdot \| x_{m+1} - T^i x_m \|_p \leq (\lambda + (1-\lambda) k^p) r_m^p.
$$

It then follows that,

$$
D_{m+1}^p \leq \frac{1-\lambda(k^p-1)}{c_p \cdot \omega_p(\lambda)} \cdot r_m^p \leq \frac{(1-\lambda)(k^p-1)}{c_p \cdot \omega_p(\lambda)} \cdot \frac{k^p}{p} \cdot b_m^p.
$$
Letting $\lambda \to 1$, we conclude that

$$D_{m+1} \leq \left[ \frac{k^p(k^p-1)}{c_p \cdot W_p(\lambda)} \right]^{1/p} D_m = A \cdot D_m, \quad m = 1, 2, \ldots,$$

where

$$\left[ \frac{k^p(k^p-1)}{c_p \cdot W_p(\lambda)} \right]^{1/p} \leq 1$$

by the assumption of the theorem.

Further, we set

$$D_n = \lim_{n \to \infty} \sup \|x_m - \sum_{n} x_m\|, \quad m = 0, 1, 2, \ldots,$$

and

$$r_m = \lim_{n \to \infty} \sup \|x_{m+1} - \sum_{n} x_m\|, \quad m = 1, 2, \ldots.$$ 

Repeating the above argument, we obtain

$$r_m \leq \frac{k^p}{n^p} D_n,$$

and hence

$$D_{m+1} \leq A \cdot D_m,$$

and hence

$$(5.2.8) \quad D_m \leq A \cdot D_{m-1} \leq A^2 \cdot D_{m-2} \leq \cdots \leq A^m \cdot D_0.$$
taking the limit superior as $i \to +\infty$,

$$
\|x_{n+1} - x_m\| \leq \|x_{n+1} - S^n x_m\| + \|S^n x_m - x_m\|,
$$

it follows that \(\{x_m\}\) is a Cauchy sequence.

Let $z = \lim_{m \to \infty} x_m$.

Then, we have

$$
\|z - S^n z\| \leq \|z - x_m\| + \|x_m - T^n x_m\| + \|T^n x_m - S^n z\|
$$

$$
\leq (1 + \|L^n\|) \|z - x_m\| + \|x_m - T^n x_m\|,
$$

taking the limit superior as $i \to +\infty$ on both sides, we get

$$
\lim_{i \to \infty} \|z - S^n z\| \leq (1 + k) \|z - x_m\| + \|x_m - T^n x_m\| \to 0
$$
as $m \to +\infty$.

Therefore by Lemma 2, $Sz = z$. By symmetry, we have

$$
z = Tz.
$$

This completes the proof.

If we put $S = T$ in theorem 1, then we will get...
Theorem 2 of Gornicki [76] as follows:

**COROLLARY 1** [76, Theorem 2]: Let $p > 1$ and let $E$ be a $p$-uniformly convex Banach space, $K$ a nonempty closed convex and bounded subset of $E$, $T : K \to K$ asymptotically regular mapping. If

$$\lim \inf_{n \to \infty} \| L^n \| = k < \left[ \frac{1}{2} \left( 1 + \sqrt{1 + 4c_p \cdot N^p} \right) \right]^{1/p}$$

(where $\| L^n \|$ is the Lipschitz constant (norm) of $T^n$. $N$ is the normal structure coefficient of $E$ and $c_p$ is the constant given in (5.2.1)).

Then $T$ has a fixed point in $K$.

Now we give applications of the established inequalities analogous to (5.2.1) in some Banach spaces.

Let us first begin with the following:

**LEMMA 3**:

In a Hilbert space $H$, following inequality holds:
\[(5.2.9) \quad \| \lambda x + (1 - \lambda)y \|_2^2 = \lambda \|x\|_2^2 + (1 - \lambda) \|y\|_2^2 - \lambda(1 - \lambda) \|x - y\|_2^2 \]

for all \(x, y\) in \(H\) and \(\lambda \in (0,1]\).

If \(1 < p \leq 2\), then we have for all \(x, y\) in \(L^p\) and \(\lambda \in (0,1]\).

\[(5.2.10) \quad \| \lambda x + (1 - \lambda)y \|_2^2 \leq \lambda \|x\|_2^2 + (1 - \lambda) \|y\|_2^2 - \lambda(1 - \lambda)(p - 1) \|x - y\|_2^2.\]

(The inequality \((5.2.10)\) is contained in Lin, Xu and Xu [111] and Smarzewski [169]).

Assume \(2 < p < +\infty\) and \(t_p\) is the unique zero of the function \(g(x) = -x^{p-1} + (p - 1)x + p - 2\) in the interval \((1, +\infty)\).

Let \(c_p = (p - 1)(1 + t_p)^{2-p} = \frac{1 + t_p^{p-1}}{(1 + t_p)^{p-1}}\)

and we have the following inequality

\[(5.2.11) \quad \| \lambda x + (1 - \lambda)y \|_p^p \leq \lambda \|x\|_p^p + (1 - \lambda) \|y\|_p^p - \lambda \delta_p(\lambda) \cdot c_p \cdot \|x - y\|_p^p \]

for all \(x, y\) in \(L^p\) and \(\lambda \in (0,1]\).
(The inequality (5.2.11) is essentially due to \( \text{Liu} \), Liu, Xu and Xu \( [111] \) and Xu \( [183] \).)

By Lemma 3, we immediately obtain from Theorem 1 the following results:

**THEOREM 2**: Let \( K \) be a nonempty bounded closed convex subset of a Hilbert space \( H \). If \( S, T : K \rightarrow K \) are asymptotically regular mappings such that

\[
\lim_{n \to \infty} \inf \| L^n \| < \sqrt{2},
\]

(where \( \| L^n \| \) is the Lipschitz constant (norm) of pair \{\( s^n, t^n \)\}). Then \( S \) and \( T \) have a common fixed point in \( K \).

If we put \( S = T \) in the Theorem 2, we get the following result of Kruppel \( [105] \):

**COROLLARY 2** \( [105] \): Let \( K \) be a nonempty bounded closed convex subset of a Hilbert space \( H \). If \( T : K \rightarrow K \) is an asymptotically regular mapping such that

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\[ \lim \inf_{n \to \infty} \| L^n \| < \sqrt{2} , \]

(where \( \| L^n \| \) is the Lipschitz constant (norm) of \( T^n \)).

Then \( T \) has a fixed point in \( K \).

**THEOREM 3**: Let \( K \) be a nonempty bounded closed convex subset of \( L^p \) \((1 < p \leq 2)\). If \( S , T : K \to K \) are asymptotically regular mappings such that

\[ \lim \inf_{n \to \infty} \| L^n \| < \left[ \frac{1}{2} \left( 1 + \frac{1}{1 + 4 \cdot (p-1) \cdot 2^{(p-1)/p}} \right) \right]^{1/2} , \]

(where \( \| L^n \| \) is the Lipschitz constant (norm) of pair \( \{ S^n , T^n \} \)).

Then \( S \) and \( T \) have a common fixed point in \( K \).

If we put \( S = T \) in Theorem 3, then we have the following corollary:

**COROLLARY 3** [76, corollary 3]: Let \( K \) be a nonempty bounded closed convex subset of \( L^p \) \((1 < p \leq 2)\). If \( T : K \to K \) is an asymptotically regular mapping such that
\[
\lim_{n \to \infty} \inf \|L_n\| < \left[ \frac{1}{2} \left( 1 + \sqrt{1 + 4 \cdot (p-1) \cdot 2^{(p-1)/p}} \right) \right]^{1/2}.
\]

(where \(\|L_n\|\) is the Lipschitz constant (norm) of \(T_n\)).

then \(T\) has a fixed point in \(K\).

**THEOREM 4**: Let \(K\) be a nonempty bounded closed convex subset of \(L^p\) \((2 < p < +\infty)\). If \(S, T : K \to K\) are asymptotically regular mappings such that

\[
\lim_{n \to \infty} \inf \|L_n\| < \left[ \frac{1}{2} \left( 1 + \sqrt{1 + 8Cp} \right) \right]^{1/p},
\]

(where \(\|L_n\|\) is the Lipschitz constant (norm) of pair \(\{S^n, T^n\}\)), then \(S\) and \(T\) have a common fixed point in \(K\).

If we put \(S = T\) in Theorem 4, then we have the following corollary:

**COROLLARY 4** [76, corollary 4]: Let \(K\) be a nonempty bounded closed convex subset of \(L^p\) \((2 < p < +\infty)\). If \(T : K \to K\) is an asymptotically regular mapping such that
\[
\lim \inf_{n \to \infty} \| T^n \| < \left[ \frac{1}{2} \left( 1 + \sqrt{1 + 8c_0} \right) \right]^{1/p},
\]

(where \( \| T^n \| \) is the Lipschitz constant (norm) of \( T^n \)),
then \( T \) has a fixed point in \( K \).

**COROLLARIES IN OTHER BANACH SPACES**

Using the results of Prus and Smarzewski [137], Smarzewski [168] and Xu [183] we can obtain from Theorem 1 the fixed point theorems, for example for Hardy and Sobolev spaces.

Let \( H^p, 1 < p < +\infty \), denote the Hardy space [60] of all functions \( x \) analytic in the unit disc \( |z| < 1 \) of the complex plane and such that

\[
\| x \| = \lim_{r \to 1^-} \left( \frac{1}{2\pi} \int_0^{2\pi} | x(re^{i\theta}) |^p \, d\theta \right)^{1/p} < +\infty.
\]

Now, let \( \Omega \) be an open subset of \( \mathbb{R}^n \). Denote by \( H^{r,p}(\Omega), r > 0, 1 < p < +\infty \), the Sobolev space [8, p. 147].
of distributions \( x \) such that \( D^\alpha x \in L^p(\Omega) \) for all \( \alpha \in \mathbb{N} \) equipped with the norm

\[
\|x\| = \left( \sum_{|\alpha| \leq k} \int |D^\alpha x(\omega)|^p \, d\omega \right)^{1/p}
\]

Let \( (\Omega_\alpha, \Sigma_\alpha, \mu_\alpha) \), \( \alpha \in \Lambda \), be a sequence of positive measure spaces, where index set \( \Lambda \) is finite or countable. Given a sequence of linear subspaces \( \mathcal{X}_\alpha \) in \( L^p(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha) \), we denote by \( L_{q,p} \), \( 1 < p < +\infty \) and \( q = \max(2, p) \) [114], the linear space of all sequences

\[ x = \{x_\alpha \in \mathcal{X}_\alpha : \alpha \in \Lambda \} \]

equipped with the norm

\[
\|x\| = \left( \sum_{\alpha \in \Lambda} (\|x_\alpha\|_{q,p})^q \right)^{1/q}
\]

where \( \|\cdot\|_{q,p} \) denotes the norm in \( L^p(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha) \).

Finally, let \( L_p = L^p(S_1, \Sigma_1, \mu_1) \) and \( L_q = L^q(S_2, \Sigma_2, \mu_2) \),

where \( 1 < p < +\infty \), \( q = \max(2, p) \) and \( (S_1, \Sigma_1, \mu_1) \) are positive measure spaces.
Denote by $L_q(L_p)$ the Banach spaces [56,III.2.10] of all measurable $L_p$-value function $x$ on $S_2$ such that

$$
\|x\| = \left\{ \int_{S_2} (\|x(s)\|^q)^{\frac{1}{q}} \mu_2(ds) \right\}^{1/q}.
$$

These spaces are $q$-uniformly convex with $q = \max(2,p)$, [137,168] and the norm in these spaces satisfies

$$
\|\lambda x + (1 - \lambda)y\|^q \leq \lambda \|x\|^q + (1 - \lambda) \|y\|^q - d \cdot \omega_q(\lambda) \|x - y\|^q,
$$

with a constant

$$
d = d_p = \begin{cases} 
\frac{p-1}{2} & \text{if } 1 < p \leq 2, \\
\frac{1}{p \cdot 2^p} & \text{if } 2 < p < +\infty.
\end{cases}
$$

Hence, from Theorem 1, we have the following results:

**Theorem 5:** Let $K$ be a nonempty bounded closed convex subset of the space $E$, where $E = H^p$, or $E = \overset{\text{r}}{H} \cdot P(Q)$, or
E = L_{q,p}, or E = L_q(L_p), and 1 < p < +\infty, q = \max(2,p), r \geq 0. If S, T : K \rightarrow K are asymptotically regular mappings such that

$$\lim_{n \rightarrow \infty} \inf \|L_n\| < \left[ \frac{1}{2} \left( 1 + \sqrt{1 + 4 \cdot d \cdot N^q} \right) \right]^{1/q},$$

where q = \max(2,p) and \|L_n\| is the Lipschitz constant (norm) of pair \{S^n, T^n\}, then S and T have a common fixed point in K.

If we put S = T in Theorem 5, then we have the following corollary:

**COROLLARY 5** [76 corollary 5] : Let K be a nonempty bounded closed convex subset of the space E, where

E = H^p, or E = H^{p,q}(\Omega), or E = L_{q,p}, or E = L_q(L_p), and 1 < p < +\infty, q = \max(2,p), r \geq 0. If T : K \rightarrow K is an asymptotically regular mappings such that

$$\lim_{n \rightarrow \infty} \inf \|T_n\| < \left[ \frac{1}{2} \left( 1 + \sqrt{1 + 4 \cdot d \cdot N^q} \right) \right]^{1/q},$$

where q = \max(2,p) and \|T_n\| is the Lipschitz constant (norm) of T^n, then T has a fixed point in K.
5.3 It is important to note that asymptotic regularity is not only fruitful in proving the existence of fixed points of self mappings, but it is equally helpful in showing the convergence of sequence to a fixed point of nonlinear mappings.

Several authors have used asymptotic regularity to prove convergence theorems for different class of nonlinear mapping (e.g. [74, 97, 126]). The principal results on asymptotic regularity are found in [5, 64, 70, 85].

In [126], Opial proved a weak convergence theorem for nonexpansive mapping in Banach space by using asymptotic regularity. Bose [11] extended the result of Opial to asymptotically nonexpansive mapping, a class of mapping defined by Goebel and Kirk [69] as a generalization on nonexpansive mapping.
Bruck [29] weaken the concept of asymptotic regularity by defining weak asymptotic regularity. By using above concept, Gornicki [74] extended the result of Bose [11] and gave a necessary and sufficient condition for the weak convergence of iterates of such mappings.

On the other hand, in [98], Kirk introduced nonlipschitzian asymptotically nonexpansive mapping and shown that every asymptotically nonexpansive mapping in essentially nonlipschitzian asymptotically nonexpansive but converse is not true. Hence the following question arises naturally: Is it possible to give necessary and sufficient condition for the weak convergence of iterates of nonlipschitzian asymptotically nonexpansive mappings?

In this section we answer in affirmative the above question.
Before further discussion, let us recall some basic definitions:

**DEFINITION 1**: A mapping $T : K \to K$ is called nonexpansive if for any $x, y \in K$

\[(5.3.1) \quad \| Tx - Ty \| \leq \| x - y \|.\]

holds.

**DEFINITION 2** [69]: A mapping $T : K \to K$ is called asymptotically nonexpansive if for each $x, y \in K$ and $i = 1, 2, 3, \ldots$

\[(5.3.2) \quad \| t^i x - t^i y \| \leq k_i \| x - y \|,\]

holds, where $\{k_i\} \in [0, +\infty)$ such that $k_i \to 1$ as $i \to +\infty$.

The class of asymptotically nonexpansive mappings is essentially wider than the class of nonexpansive mappings.

**DEFINITION 3** [98]: A mapping $T : K \to K$ is called nonlipschitzian asymptotically nonexpansive if for
It is important to note that every asymptotically nonexpansive mapping is necessarily nonlipshitzian asymptotically nonexpansive but converse is not true [98] as given below:

If a mapping $T: K \to K$ is asymptotically nonexpansive, then there exists a sequence $\{k_i\}$ of constants such that $k_i \to 1$ as $i \to +\infty$ and for which

$\|T^i x - T^i y\| \leq k_i \|x - y\|,$

for all $x, y \in K$ and $i = 1, 2, \ldots$.

Thus, we have

$\|T^i x - T^i y\| - \|x - y\| \leq (k_i - 1) \|x - y\|$

$\leq |k_i - 1| \delta(K)$

and

$\lim \{\sup_{i \to \infty} \{\|T^i x - T^i y\| - \|x - y\|\}\} \leq \lim_{i \to \infty} |k_i - 1| \delta(K) = 0,$

where $\delta(K)$ denotes the diameter of $K$. 

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Let us first recall Edelstein's definition [62,63] of the asymptotic center of a bounded sequence \( \{x_n\} \) in a uniformly convex Banach space \( E \), let \( K \) be a closed convex subset of \( E \).

Consider a functional

\[
r : E \rightarrow [0, +\infty),
\]

defined by

\[
r(x) = \limsup_{n \to \infty} \| x_n - x \|, \quad x \in E.
\]

The infimum of \( r(\cdot) \) over \( K \) is called radius of \( \{x_n\} \) with respect to \( K \) and is denoted by \( r(K,\{x_n\}) \). A point \( z \) in \( K \) is said to be an asymptotic center of the sequence \( \{x_n\} \) with respect to \( K \) if

\[
r(z) = \min \{ r(x) : x \in K \}.
\]

The set of all asymptotic centers is denoted by \( A(K,\{x_n\}) \).

We need following lemmas to prove our main result:
LEMMA 4 [72]: Every bounded sequence \( \{x_n\} \) in a uniformly convex Banach space \( E \) has a unique asymptotic center with respect to any closed convex subset \( K \) of \( E \), i.e. \( A(K,\{x_n\}) = \{z\} \) and for all \( x \neq z \)

\[
\limsup_{n \to \infty} \| x_n - z \| < \limsup_{n \to \infty} \| x_n - x \|.
\]

LEMMA 5 [12]: Let \( \{x_n\} \) be a bounded sequence in a closed convex subset \( K \) of a uniformly convex Banach space \( E \), and \( A(K,\{x_n\}) = \{z\} \). If \( \{Y_m\} \subseteq K \) and \( r(Y_m) \to r(K,\{x_n\}) \) as \( m \to +\infty \), then \( Y_m \to z \) as \( m \to +\infty \).

In the rest of the section the weak convergence of sequence will be denoted by \( x_n \rightharpoonup x \), while the strong convergence by \( x_n \to x \). The set of fixed points of a mapping \( T \) will be denoted by \( \text{Fix}(T) \).

THEOREM 6: Let \( K \) be a closed convex (but not necessarily bounded) subset of a uniformly convex Banach space. If \( T \) be a nonlipschitzian asymptotically
nonexpansive mapping, then the following statements are equivalent:

(a) \( T \) has a fixed point;

(b) There is a point \( x_0 \in K \) such that the sequence \( \{T^n x_0\} \) of iterates is bounded.

**PROOF:** (a) \( \Rightarrow \) (b) follows easily.

Conversely to prove (b) \( \Rightarrow \) (a), assume \( x_0 \in K \) is such that the sequence \( \{x_n\} \) defined by \( x_n = T^n x_0 \) is bounded, and let \( A(K,\{x_n\}) = \{z\} \), and let \( \{y_m\} \) be a sequence in \( K \) defined by \( y_m = T^m z \) for \( m = 1, 2, \ldots \).

We shall show that

\[
r(y_m) = \lim_{n \to \infty} \sup_{m \leq n} \| x_n - y_m \| \to r(K,\{x_n\}) \text{ as } m \to +\infty.
\]

By Lemma 5, this would imply \( y_m \to z \) as \( m \to +\infty \), and because \( T \) is continuous, we have

\[
Tz = T \left( \lim_{n \to \infty} T^m z \right) = \lim_{n \to \infty} T^{m+1} z = z.
\]

For two integers \( n > m \geq 1 \), we have

\[
\| x_n - y_m \| = \| T^n x_0 - T^m z \| = \| T^m (T^{n-m} x_0) - T^m z \|.
\]
and

\[ r(y_n) \leq \sup_{y \in K} \left[ \|x_n^m - x_n^m z\| + \|y - z\| \right] + r(z). \]

This shows that \( r(y_m) \to r(K, \{x_n\}) \) as \( m \to +\infty \).

This completes the proof.

We say that a Banach space \( E \) satisfies the Opial's condition [126], if for each sequence \( \{x_n\} \subset E \) weakly convergent to a point \( z \), and for all \( y \in E \) with \( y \neq z \)

\[ \lim \inf_{n \to \infty} \|x_n - z\| < \lim \inf_{n \to \infty} \|x_n - y\|. \]

It is easy to observe that the above inequality can be given an equivalent form in terms of \( \lim \sup_{n \to \infty} \).

Examples of Banach spaces which satisfy the Opial's condition are Hilbert spaces and all \( L^p[0, 2\pi] \) with \( 1 < p < +\infty \). On the other hand \( L^p[0, 2\pi] \) with
1 \leq p \neq 2$ fails to satisfy the Opial's condition [126]. Despite this deficiency, Opial's condition remained the principal tool for proving weak convergence.

**Lemma 6** ([11]): Let $K$ be a closed convex subset of a uniformly convex Banach space satisfying the Opial's condition. If a sequence $\{x_n\} \subset K$ converges weakly to a point $x$, then $x$ is the asymptotic center of $\{x_n\}$ in $K$.

**Lemma 7**: Let $K$ be a closed convex subset of a uniformly convex Banach space satisfying the Opial's condition and $T: K \to K$ a non-lipschitzian asymptotically nonexpansive mapping. Suppose that $x_0$ is the asymptotic center of the bounded sequence $\{T^n x\}$ for some $x \in K$. If the weak limit $z$ of a subsequence $\{T^{n_i} x\} \subset \{T^n x\}$ is a fixed point of $T$, then $x_0$ coincides with $z$.

**Proof**: Clearly $r(K, \{T^{n_i} x\}) \leq r(K, \{T^n x\})$. Since $T^{n_i} x \rightharpoonup z$, by Lemma 6, $A(K, \{T^{n_i} x\}) = \{z\}$ and so, for any $\epsilon > 0$, we can choose an integer $i_0$ such that
\[ \| z - T^{n_0} z \| \leq r(K,\{T^n x\}) + \epsilon/2. \]

Since \( z \) is a fixed point of \( T \) and \( T \) is nonlipschitzian asymptotically nonexpansive, we can choose an integer \( J \)
such that for all \( j \geq J \)

\[ \| z - T^{n_0 + j} z \| \leq \sup_{v \in K} [ \| T^j z - T^j v \| - \| z - v \| ] + r(K,\{T^n x\}) + \epsilon/2 \]

\[ \leq r(K,\{T^n x\}) + \epsilon \]

\[ \leq r(K,\{T^n x\}) + \epsilon. \]

It follows therefore that

\[ \limsup_{n \to \infty} \| z - T^n x \| = r(K,\{T^n x\}) \]

and therefore, \( x_0 \) being the unique point with this

property, we have \( z = x_0 \).

This completes the proof.

The concept of asymptotic regularity was introduced by Browder and Petryshyn [22] and Weak

asymptotic regularity by Bruck [29].
**DEFINITION 4**: A mapping $T : K \to K$ is said to be (weakly) asymptotically regular at $x \in K$ if

$$T^{n+1}x - T^n x \to 0 \text{ (weakly)}$$

as $n \to +\infty$.

Now we give our main result of this section:

**THEOREM 7**: Let $E$ be a uniformly convex Banach space satisfying the Opial's condition and $K$ be a closed convex (but not necessarily bounded) subset of $E$, and $T : K \to K$ is a nonlipschitzian asymptotically nonexpansive mapping and $x \in K$, then $\{T^n x\}$ converges weakly to a fixed point of $T$ if and only if $T$ is weakly asymptotically regular at $x$.

**PROOF**: Let us assume that $T^n x \overset{w}{\to} p$ as $n \to +\infty$. We can show that $p \in \text{Fix}(T)$. By Lemma 6, $A(K, \{T^n x\}) = \{p\}$ and analogously as in theorem 6, $p \in \text{Fix}(T)$. From $T^n x \overset{w}{\to} p$ as $n \to +\infty$, it follows that

$$T^{n+1}x - T^n x \overset{w}{\to} 0 \text{ as } n \to +\infty.$$
Conversely, now we are going to show the implication in the opposite way.

From the assumption \( T^{n+1}x - T^n x \xrightarrow{w} 0 \) as \( n \to +\infty \), we have

\[
T(n_i + m)x \xrightarrow{w} y \quad \text{as} \quad i \to +\infty
\]

for \( m = 0, 1, 2, \ldots \). By Lemma 6, \( A(K, \{T(n_i + m)x\}) = \{y\} \) for \( m = 0, 1, 2, \ldots \). Let \( \{y_s\} \) be a sequence in \( K \) defined by \( y_s = T^s y \) for \( s = 1, 2, \ldots \). For integers \( m > s \geq 1 \), we have

\[
\| Y_s - T^{n_i + m}x \| = \| T^s y - T^s (T^{n_i + m - s}x) \|
\]

\[
= [ \| T^s y - T^s (T^{n_i + m - s}x) \| - \| y - (T^{n_i + m - s}x) \| ]
\]

\[+ \| y - (T^{n_i + m - s}x) \|,\]

which implies that

\[
r(y_s) \leq \sup_{u \in K} [ \| T^s y - T^s u \| - \| y - u \| ] + r(y).
\]

By Lemma 5, \( T^s y \to y \) as \( s \to +\infty \) and by the continuity of \( T \), \( Ty = y \). Let \( w_w(x) \) denote the set of weak limits of subsequences of a sequence \( \{T^n x\} \). From this part
of the proof, we have $w_w(x) \subset \text{Fix}(T)$. By Lemma 7, $y \in w_w(x)$. This proves that $w_w(x) = \{z\}$ and, so

$$T^nx \rightarrow z \text{ as } n \rightarrow +\infty.$$

This completes the proof.

As a immediate consequence of Theorem 7, we have the following results:

**THEOREM B [165]**: Let $E$ be a uniformly convex Banach space satisfying Opial's condition and $K$ be a nonempty bounded closed convex subset of $E$. Suppose $T : K \rightarrow K$ is nonlipschitzian asymptotically nonexpansive and asymptotically regular. Then for any $x \in K$, the sequence $\{T^nx\}$ converges weakly to a fixed point of $T$.

**COROLLARY 6 [76]** : Let $E$ be a uniformly convex Banach space satisfying the Opial's condition and $K$ be a closed convex (but not necessarily bounded) subset of $E$. If $T : K \rightarrow K$ is an asymptotically nonexpansive mapping and $x \in K$, then $\{T^nx\}$ converges weakly to a
fixed point of $T$ if and only if $T$ is weakly asymptotically regular at $x$.

**COROLLARY 7** [11]: Let $K$ be a closed convex subset of a uniformly convex Banach space $E$ satisfying the Opial's condition. Assume that $T : K \rightarrow K$ is an asymptotically nonexpansive, weakly asymptotically regular and $\text{Fix}(T) \neq \emptyset$. Then for any $x \in K$, the sequence of iterates $\{T^n x\}$ is weakly convergent to a fixed point of $T$.

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