CHAPTER 2

FLOW OF A VISCO-ELASTIC FLUID AND HEAT TRANSFER IN A POROUS MEDIUM OVER A STRETCHING SHEET IN THE PRESENCE OF A TRANSVERSE MAGNETIC FIELD

2.1 INTRODUCTION

The study of boundary layer flow and heat transfer in a visco-elastic fluid over a continuous moving solid boundary has many applications in industries such as textile, paper, spinning of fibers and processes involving extrusion of plastic sheets. In addition, the transport of heat in a porous medium has practical applications in the fields of chemical engineering and geophysics.


The solutions obtained by most of the earlier researchers were however based on the analytical approach using essentially Kummer’s

It is proposed to study in this chapter, the MHD flow and heat transfer characteristics of a visco-elastic fluid (Walters’ liquid B model) in a porous medium over a stretching sheet. Both analytical and numerical techniques have been used for comparison of end results.

For the study of heat transfer problem, two standard cases namely, Prescribed Surface Temperature (PST) and Prescribed Heat Flux (PHF) have been considered by the earlier researchers. These cases are proposed to be considered in the present study too.

2.2 FORMULATION OF THE PROBLEM

A steady two-dimensional flow of an incompressible visco-elastic fluid (Walters’ liquid – B model) through a porous medium past a semi-infinite stretching sheet in the presence of a transverse magnetic field is considered. The $x$-axis is taken to be along the sheet and $y$-axis perpendicular to the sheet as shown in Figure 2.1. The fluid flow is confined to the half space $y > 0$. By applying two equal and opposite forces along the $x$-axis, the sheet is being stretched with a speed proportional to the distance from the fixed origin (slit).
Figure 2.1 Representation of a two dimensional stretching sheet problem

The continuity equation and the boundary layer equations governing the flow and heat transfer, in Walters’ Liquid B, in the presence of internal heat generation can be written in the form

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]  \hspace{1cm} (2.1)

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = v \frac{\partial^2 u}{\partial y^2} - k_0 \left( u \frac{\partial^3 u}{\partial x \partial y^3} + v \frac{\partial^3 u}{\partial y^3} + \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \partial y} \right) \]

\[-\frac{v}{k} u - \sigma \frac{B_0^2 u}{\rho} \]  \hspace{1cm} (2.2)

\[
u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = k \frac{\partial^2 T}{\partial y^2} + \frac{Q}{\rho c_p} \left( T - T_\infty \right) \]  \hspace{1cm} (2.3)

where \( u \) and \( v \) are the flow velocity components in the \( x \)-direction and \( y \)-direction respectively, \( T \) and \( T_\infty \) represent the fluid temperatures at any point within and at the far end of the boundary layer, \( \nu \) is the kinematic fluid...
viscosity, $k_o$ is the coefficient of viscoelasticity, $k'$ is the permeability of the porous medium, $\sigma$ is the electrical conductivity, $B_o$ is the uniform magnetic field strength, $\rho$ is the fluid density, $k$ is the thermal conductivity of the fluid, $c_p$ is the specific heat at constant pressure, $Q$ is the volumetric rate of heat generation in the source.

The corresponding boundary conditions are:

\[
y = 0: \quad u = \lambda x, \quad v = 0 \quad (\lambda > 0) \\
\text{PST case} \quad T = T_w \quad (= T_\infty + A x^r) \\
\text{PHF case} \quad -k \frac{\partial T}{\partial y} = q_w = D x^s \quad \quad (2.4)
\]

\[
y \to \infty: \quad u \to 0, u_y \to 0 \\
T \to T_\infty \quad \quad (2.5)
\]

where $\lambda$ is the stretching rate of the sheet in the $x$-direction, $r$ is the surface temperature parameter in the PST case, $s$ is the heat flux parameter in the PHF case and $A$ and $D$ are constants.

For the flow analysis, it has been assumed that the diffusion rate at the sheet results in a negligible normal velocity $v$. Besides, the condition $u_y \to 0$ as $y \to \infty$ in Equation (2.5) is an augmented condition since the flow is in an unbounded domain. This has been discussed by Rajagopal and Gupta (1984) and Rajagopal and Kaloni (1989).
2.3 TRANSFORMATION OF EQUATIONS

Equation (2.1) is satisfied by the stream function $\psi(x, y)$ defined by

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

(2.6)

The similarity transformation is defined as

$$\eta = y \sqrt{\frac{\lambda}{v}}$$

$$\psi(x, y) = \sqrt{\lambda v} x f(\eta)$$

(2.7)

The non-dimensional temperatures $\theta$ and $g$ for the two cases PST and PHF are written as follows:

PST case: $\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}$

(2.8)

where $T - T_\infty = Ax' \theta(\eta)$ and $T_w - T_\infty = Ax'$

PHF case: $g(\eta) = \frac{T - T_\infty}{T_w - T_\infty}$

(2.9)

where $T - T_\infty = \frac{D}{k} x' \sqrt{\frac{v}{\lambda}} g(\eta)$ and $T_w - T_\infty = \frac{D}{k} x' \sqrt{\frac{v}{\lambda}}$

Introducing the following non-dimensional parameters,

$$k_i = \frac{k_w \lambda}{v}$$

(Visco-elastic parameter)
\[ k_z = \frac{\nu}{k'\lambda} \]  
(Permeability parameter)

\[ Mn = \frac{\sigma B_0^2}{\lambda \rho} \]  
(Magnetic parameter)

\[ Pr = \frac{\mu c_p}{k} \]  
(Prandtl number)

\[ \beta = \frac{Q \nu}{\lambda k} \]  
(Heat source/sink parameter)

Equations (2.2) and (2.3) reduce to the following form:

\[ f_{\eta} - \int f_{\eta} = f_{\eta\eta} - k_1 \{ 2f_{\eta} f_{\eta\eta} - \int f_{\eta\eta} - f_{\eta\eta}^2 \} - Mn f_{\eta} - k_z f_{\eta} \]  
\[ \theta_{\eta} + Pr f \theta_{\eta} - \left( r Pr f - \beta \right) \theta = 0 \]  
\[ g_{\eta} + Pr fg_{\eta} - \left( s Pr f - \beta \right) g = 0 \]

The boundary conditions (2.4) and (2.5) can be rewritten as

\[ \eta = 0: \quad f = 0, \quad f_{\eta} = 1 \]

PST case \( \theta = 1 \)

PHF case \( g_{\eta} = -1 \)  
\[ \eta \to \infty: \quad f_{\eta} = 0, \quad f_{\eta\eta} = 0 \]

PST case \( \theta \to 0 \)

PHF case \( g \to 0 \)
2.4 ANALYTICAL SOLUTION OF THE MOMENTUM EQUATION

The analytical solution of the differential Equation (2.10) satisfying the boundary conditions (2.13) and (2.14) is derived in this section.

New variable $z$ is introduced in the form

$$z = \alpha \eta \quad \text{and} \quad S(z) = \alpha f(\eta)$$

(2.15)

where $\alpha$ is a constant to be determined.

Equation (2.15) transforms Equation (2.10) to

$$S_z^2 - SS_{zz} = \alpha^2 S_{zzz} - k_1 \alpha^2 [2S_z S_{zzz} - SS_{zzzz} - S_{zz}^2] - MnS_z - k_2 S_z$$

(2.16)

The boundary conditions (2.13) and (2.14) are transformed as

$$z = 0: \quad S = 0, \quad S_z = 1$$

$$z \to \infty: \quad S_z \to 0, \quad S_{zz} \to 0$$

(2.17)

The following approximation satisfies the boundary conditions as $z \to \infty$

$$S_z = e^{-z}$$

(2.18)

On integrating Equation (2.18) with respect to $z$ and using Equation (2.17),

$$S(z) = 1 - e^{-z}$$

(2.19)
The following expression for $\alpha$ is obtained by substituting Equation (2.19) in Equation (2.16),

$$\alpha = \sqrt{\frac{1 + Mn + k_2}{1 - k_1}}$$

(2.20)

Thus, the analytical solutions for the transverse and axial velocity components are given by,

$$f(\eta) = \frac{1 - e^{-\alpha \eta}}{\alpha}$$

(2.21)

$$f_\eta(\eta) = e^{-\alpha \eta}$$

(2.22)

where $\alpha$ is given by Equation (2.20).

These expressions are identical to those of Abel et al (2004) when the porosity of the medium is neglected.

2.5 SKIN FRICTION COEFFICIENT

The sheet shear stress $\tau_w$, is given by

$$\tau_w = -\mu \left( \frac{\partial u}{\partial y} \right)_{y=0}$$

(2.23)

Substituting Equations (2.6) and (2.7) in Equation (2.23) and using Equation (2.22),

$$\tau_w = \mu \lambda x \alpha \sqrt{\frac{\lambda}{V}}$$

(2.24)
The skin friction coefficient \( c_f \), also called frictional drag coefficient is given by,

\[
c_f = \frac{\tau_w}{\mu \lambda x \sqrt{\frac{\lambda}{v}}} \tag{2.25}
\]

Using Equation (2.24) in Equation (2.25),

\[
c_f = \frac{\mu \lambda x \alpha}{\mu \lambda x \sqrt{\frac{\lambda}{v}}} = \alpha = |f_{\eta\eta}(0)| \tag{2.26}
\]

where \( \alpha \) is given by Equation (2.20).

### 2.6 ANALYTICAL SOLUTION FOR THE HEAT TRANSFER EQUATION

The non-dimensional form of temperature equations for the two cases (PST and PHF) are derived using a similarity transformation. These are given by Equations (2.11) and (2.12) respectively along with boundary conditions (2.13) and (2.14). These cases are now considered separately and solved analytically.

#### 2.6.1 Prescribed Surface Temperature (PST) case

Substituting Equations (2.21) and (2.22) in Equation (2.11), it is seen that,

\[
\theta_{\eta\eta} + \text{Pr} \theta_{\eta} \left(1 - e^{-\alpha \eta}\right) - \left(r \text{Pr} e^{-\alpha \eta} - \beta\right) \theta = 0 \tag{2.27}
\]
The following transformation is introduced,

\[ \zeta = -\frac{\Pr}{\alpha^2} e^{-\alpha \eta} \]  

(2.28)

Using Equation (2.28) in Equation (2.27), the governing non-dimensional heat equation for the PST case is obtained in the form

\[ \zeta \frac{d^2 \theta}{d \zeta^2} + \frac{d \theta}{d \zeta} \left( 1 - \zeta - b_0 \right) + \left( r + \frac{\beta}{\alpha^2} \zeta^{-1} \right) \theta = 0 \]  

(2.29)

where \[ b_0 = \frac{\Pr}{\alpha^2} \]  

(2.30)

The boundary conditions (2.13) and (2.14) may be rewritten in the form:

\[ \theta \left( \zeta = -\frac{\Pr}{\alpha^2} \right) = 1, \quad \theta \left( \zeta = 0 \right) = 0 \]  

(2.31)

Assuming a series solution of the homogeneous Equation (2.29) in the form,

\[ \theta \left( \zeta \right) = \sum_{z=0}^{\infty} a_z \zeta^{t+z} \]  

(2.32)

Equation (2.32) gives

\[ \theta \left( \zeta \right) = a_0 \zeta^t + a_1 \zeta^{t+1} + a_2 \zeta^{t+2} + \ldots \ldots \ldots \infty \]  

\[ \theta_{\zeta} \left( \zeta \right) = a_0 t \zeta^{t-1} + a_1 (t+1) \zeta^t + a_2 (t+2) \zeta^{t+1} + \ldots \ldots \ldots \infty \]  

\[ \theta_{\zeta \zeta} \left( \zeta \right) = a_0 t (t-1) \zeta^{t-2} + a_1 (t+1) t \zeta^{t-1} + a_2 (t+2) (t+1) \zeta^t + \ldots \ldots \ldots \infty \]  

(2.33)
Substituting Equation (2.33) in Equation (2.29),

\[
[a_0 t (t - 1) \zeta^{-1} + a_1 t^2 (t + 1) \zeta' + a_2 t (t + 2) (t + 1) \zeta^{t+1} + \ldots \ldots \infty] \\
+ (1 - b_0) \left[ a_0 t \zeta^{-1} + a_1 (t + 1) \zeta' + a_2 (t + 2) \zeta^{t+1} + \ldots \ldots \infty \right] \\
- \left[ a_0 a^t + a_1 (t + 1) \zeta^{t+1} + a_2 (t + 2) \zeta^{t+2} + \ldots \ldots \infty \right] \\
+ r \left[ a_0 \zeta'^t + a_1 \zeta'^{t+1} + a_2 \zeta'^{t+2} + \ldots \ldots \infty \right] \\
+ \frac{\beta}{\alpha^2} \left[ a_0 \zeta'^{t-1} + a_1 \zeta'^t + a_2 \zeta'^{t+1} + \ldots \ldots \infty \right] = 0
\]

Equating the coefficients of \( \zeta'^{-1} \),

\[
a_0 t (t - 1) + a_0 (1 - b_0) t + \frac{\beta a_0}{\alpha^2} = 0, \quad a_0 \neq 0
\]

\[
\Rightarrow t^2 - b_0 t + \frac{\beta}{\alpha^2} = 0
\]

\[
\Rightarrow t = a + b, \quad a = \frac{Pr}{2\alpha^2} \quad \text{and} \quad b = \frac{\sqrt{Pr^2 - 4\beta \alpha^2}}{2\alpha^2} \quad (2.34)
\]

Equating coefficients of \( \zeta'^t \),

\[
a_1 (a + b) (a + b + 1) + a_1 (1 - b_0) (a + b + 1) + a_1 \frac{\beta}{\alpha^2} = a_0 (a + b - r a_0)
\]

But \( b_0 = 2a \) and \( \frac{\beta}{\alpha^2} = a^2 - b^2 \) \quad (2.35)

\[
\Rightarrow a_1 \left\{ (a + b + 1) [b - a + 1] + a^2 - b^2 \right\} = a_0 [a + b - r]
\]

\[
\Rightarrow a_1 [2b + 1] = a_0 (a + b - r)
\]

\[
\Rightarrow a_1 = \frac{a + b - r}{2b + 1} a_0 \quad (2.36)
\]
Equating coefficients of $\zeta^{t+1}$ and using Equation (2.35),

$$a_2 \{ (a + b + 2)(b - a + 2) + a^2 - b^2 \} = a_1 (a + b - r + 1)$$

$$\Rightarrow a_2 (4b + 4) = a_1 (a + b - r + 1)$$

$$\Rightarrow a_2 = \frac{a + b - r + 1}{4(b + 1)} a_1$$

Using Equation (2.36),

$$a_2 = \frac{1}{2!} \frac{(a + b - r + 1)(a + b - r)}{(2b + 2)(2b + 1)} a_0$$

(2.37)

Thus, Equation (2.32) is rewritten of the form

$$\theta(\zeta) = 1 + \sum_{z=1}^{\infty} a_z \zeta^{z+r}$$

(2.38)

Substituting Equation (2.36) and Equation (2.37) in Equation (2.38),

$$\theta(\zeta) = 1 + \zeta \left[ \frac{a + b - r}{2b + 1} \zeta + \frac{(a + b - r)(a + b - r + 1)}{2!(2b + 1)(2b + 2)} \zeta^2 + \ldots \ldots \right]$$

Using Equation (2.34),

$$\theta(\zeta) = C_1 \zeta^{a+b} M[a + b - r, 2b + 1, \zeta]$$

(2.39)

where $C_1$ is a constant and $M[a, b, Z]$ is the Kummer’s function defined by
\[ M(a, b, Z) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n Z^n}{(b)_n n!} \]

\[
(a)_n = a(a+1)(a+2)\ldots (a+n-1) \]

\[
(b)_n = b(b+1)(b+2)\ldots (b+n-1) \]

Using boundary condition (2.31) in Equation (2.39), the value of the constant is obtained as,

\[
C_i = \left(\frac{-\alpha^2}{\text{Pr}}\right)^{a+b} M \left[ a + b - r, 2b + 1, \frac{-\text{Pr}}{\alpha^2} \right] \]  

(2.40)

Substituting Equation (2.40) in Equation (2.39),

\[
\theta(\zeta) = \left(\frac{-\alpha^2}{\text{Pr} \zeta}\right)^{a+b} \frac{M[a + b - r, 2b + 1, \zeta]}{M[a + b - r, 2b + 1, \frac{-\text{Pr}}{\alpha^2}]} \]  

(2.41)

Substituting Equation (2.28) in Equation (2.41), the solution becomes

\[
\theta(\eta) = e^{-\alpha(a+b)\eta} \frac{M[a + b - r, 2b + 1, \frac{-\text{Pr}}{\alpha^2} e^{-\alpha \eta}]}{M[a + b - r, 2b + 1, \frac{-\text{Pr}}{\alpha^2}]} \]  

(2.42)

The non-dimensional surface temperature gradient \((\theta_\eta(0))\) from Equation (2.42) becomes,

\[
\theta_\eta(0) = -\alpha(a+b) + \left(\frac{\text{Pr}}{\alpha}\right) \frac{M[a + b - r + 1, 2b + 2, \frac{-\text{Pr}}{\alpha^2}]}{M[a + b - r, 2b + 1, \frac{-\text{Pr}}{\alpha^2}]} \]  

(2.43)
and the local sheet heat flux \((q_w)\) can be expressed as

\[
q_w = -k \left( \frac{\partial T}{\partial y} \right)_{y=0} = -kA \sqrt{\frac{\lambda}{v}} x' \theta_q(0)
\]

Equations (2.42) and (2.43) give the analytical solution of the heat equation and the non-dimensional surface temperature gradient respectively for the case when the sheet is prescribed a certain temperature (PST case). These expressions are seen to be identical to those of Abel et al (2004) for the case when the porosity of the medium is ignored.

2.6.2 Prescribed Heat Flux (PHF) case

Substituting Equations (2.21) and (2.22) in Equation (2.12),

\[
g_{g\eta} + \Pr g_{\eta} \left( \frac{1 - e^{-\alpha \eta}}{\alpha} \right) - \left( s \Pr e^{-\alpha \eta} - \beta \right) g = 0
\]

(2.44)

The following transformation is introduced,

\[
\xi = -\frac{Pr}{\alpha^2} e^{-\alpha \eta}
\]

(2.45)

Using Equation (2.45) in Equation (2.44), the governing non-dimensional heat equation for the PHF case is obtained in the form

\[
\xi \frac{d^2 g}{d \xi^2} + \frac{d g}{d \xi} \left( 1 - \xi - b_0 \right) + \left( s + \frac{\beta}{\alpha^2} \xi^{-1} \right) g = 0
\]

(2.46)

where \( b_0 = \frac{Pr}{\alpha^2} \)
The boundary conditions (2.13) and (2.14) may be rewritten of the form:

\[
g(\xi) = C_1 \xi \frac{a+b}{\alpha} \frac{a+b}{\alpha - 1} M \left[a + b - s, 2b + 1, \xi \right]
\]

(2.47)

Proceeding exactly in the same manner as in the PST case,

\[
g(\xi) = C_1 \xi \frac{a+b}{\alpha} \frac{a+b}{\alpha - 1} M \left[a + b - s, 2b + 1, \xi \right]
\]

(2.48)

Differentiating Equation (2.48) with respect to \( \xi \),

\[
g(\xi) = C_1 (a+b) \xi \frac{a+b}{\alpha} \frac{a+b}{\alpha - 1} M \left[a + b - s, 2b + 1, \xi \right] + C_1 \xi \frac{a+b}{\alpha} \frac{a+b}{\alpha - 1} M' \left[a + b - s, 2b + 1, \xi \right]
\]

(2.49)

where \( M'[a,b,Z] = \frac{a}{b} M[a,b,Z] \)

Substituting Equation (2.47) in Equation (2.49), the value of the constant is obtained as

\[
C_1 = \frac{1}{\alpha} \left( \frac{-\alpha}{\alpha^2} \right)^{a+b} \left\{ (a+b) M \left[a + b - s, 2b + 1, \frac{-\alpha}{\alpha^2} \right] + \frac{\alpha}{\alpha^2} M' \left[a + b - s, 2b + 1, \frac{-\alpha}{\alpha^2} \right] \right\}
\]

(2.50)

Substituting Equation (2.50) in Equation (2.48),

\[
g(\xi) = \frac{1}{\alpha} \left( \frac{\xi}{\alpha^2} \right)^{a+b} M \left[a + b - s, 2b + 1, \xi \right] \left\{ (a+b) M \left[a + b - s, 2b + 1, \frac{-\alpha}{\alpha^2} \right] + \frac{\alpha}{\alpha^2} M' \left[a + b - s, 2b + 1, \frac{-\alpha}{\alpha^2} \right] \right\}
\]

(2.51)
Substituting Equation (2.45) in Equation (2.51),

\[
g(\eta) = \frac{1}{\alpha} e^{-\alpha(a+b)\eta} M \left[ a + b - s, 2b + 1, \frac{-\Pr}{\alpha^2} e^{-\alpha\eta} \right] \] 
\[
\left\{ (a+b) M \left[ a + b - s, 2b + 1, \frac{-\Pr}{\alpha^2} \right] - \frac{\Pr}{\alpha^2} M^\prime \left[ a + b - s, 2b + 1, \frac{-\Pr}{\alpha^2} \right] \right\}^{-1}
\]

(2.52)

The sheet temperature is obtained from Equation (2.9) as

\[
T_w - T_\infty = \frac{D}{k} \sqrt{\frac{\nu}{\lambda}} g(0)
\]

where \( g(0) \) is obtained from Equation (2.52) as

\[
g(0) = \frac{1}{\alpha} M \left[ a + b - s, 2b + 1, \frac{-\Pr}{\alpha^2} \right] \] 
\[
\left\{ (a+b) M \left[ a + b - s, 2b + 1, \frac{-\Pr}{\alpha^2} \right] - \frac{\Pr}{\alpha^2} M^\prime \left[ a + b - s, 2b + 1, \frac{-\Pr}{\alpha^2} \right] \right\}^{-1}
\]

(2.53)

Equations (2.52) and (2.53) give the analytical solution of the heat equation and the sheet temperature respectively for the case when the sheet is prescribed a certain heat flux. The above expressions are seen to be identical to those of Abel et al (2004) for the particular case when porosity of the medium is neglected.

2.7 NUMERICAL SOLUTION

The shooting method for the solution of non-linear differential equations basically involves choosing certain initial values for the concerned derivatives in such a way that the end boundary conditions are satisfied within
a prescribed numerical tolerance value. In this study, the numerical tolerance value has been chosen as $10^{-6}$, a sufficiently low value to represent zero at the end of the range. The sequence of initial values is given by the secant method and the initial value problem is solved using the fourth order Runge-Kutta scheme. The value of $\eta$ at $\infty$ i.e. $\eta_{\text{max}}$ is so chosen that the solution yields insignificant change in prescribed boundary values for $\eta$ larger than $\eta_{\text{max}}$.

The system of differential equations (2.10), (2.11), (2.12) along with the boundary conditions (2.13) and (2.14) are solved numerically using the Runge-Kutta algorithm and the shooting technique with a systematic initial guessing of $f_{\eta}(0), f_{\eta\eta}(0), \theta(0)$ and $g(0)$ and proceeding until the boundary values of $f_{\eta}, f_{\eta\eta}, \theta$ and $g$ at $\eta_{\text{max}}$ gradually approach the expected end values. If the boundary conditions at $\eta_{\text{max}}$ are not satisfied, then the numerical routine uses the half-interval method to calculate the corrections required to yield the new estimated values of $f_{\eta}(0), f_{\eta\eta}(0), \theta(0)$ and $g(0)$. This process is repeated iteratively until the prescribed end values for $f_{\eta}, f_{\eta\eta}, \theta$ and $g$ are obtained. A computer program has been developed in C language using the above logic of the solution and the end results are presented in graphical form, using MATLAB package. For quantitative comparison of the end results, typical results are also presented in the form of tables.

2.8 RESULTS AND DISCUSSION

2.8.1 Velocity Profiles

The transverse and axial velocity components given by Equations (2.21) and (2.22) are decreasing functions of $\eta$ as they are exponential functions with negative arguments. Equation (2.20) gives $\alpha$ as a function of the visco-elastic parameter $k_i$, the permeability parameter $k_s$ and
the magnetic parameter $Mn$. This represents the slope of these exponentially decaying velocity profiles. The parameter $\alpha$ is also important as it represents the skin friction coefficient at the stretching sheet. From Equation (2.26), it may be inferred that that viscoelasticity, permeability and applied magnetic field all contribute to increase the skin friction coefficient.

Figure 2.2 shows the plot of the transverse velocity component $f(\eta)$ versus $\eta$ for selected group of values of $k_1, k_2$ and $Mn$. The transverse velocity $f(\eta)$ decreases with increasing values of $k_1, k_2$ and $Mn$.

![Figure 2.2](image)

**Figure 2.2** Transverse velocity component $f(\eta)$ for different values of $k_1, k_2, Mn$

Figures 2.3 and 2.4 represent plots of $f_s(\eta)$ versus $\eta$ for selected values of $k_i$ and $Mn$ respectively. The main effect of viscoelasticity is to gradually reduce the flow velocity within the boundary layer, as seen in Figure 2.3.
Figure 2.3 Axial velocity profiles \( (k_2 = 0.5, Mn = 1.0) \)

Figure 2.4 Axial velocity profiles \( (k_1 = 0.2, k_2 = 1.0) \)

Figure 2.4 further reveals that the increasing the magnetic parameter decreases the axial flow velocity \( f_q(\eta) \) significantly. This is in accordance with the fact that the increase of \( Mn \) signifies an increase in the Lorentz force that opposes the horizontal flow in the reverse direction.
Figure 2.5 gives the variation of axial velocity with $\eta$ for different values of $k_2$. It shows that as the permeability parameter $k_2$ increases, the flow velocity decreases. Physically this means that an increase in the porosity of the medium leads to an enhanced deceleration of the flow.

![Figure 2.5 Axial velocity profiles ($k_1 = 0.3, Mn = 1.0$)](image)

The above figures show further that the transverse velocity profiles decay faster than the axial velocity profiles for increasing values of $k_1, k_2$ and $Mn$.

2.8.2 Temperature Profiles under heat transfer

2.8.2.1 PST case

Figure 2.6 shows that an increase in the magnetic parameter $Mn$ results in an increase in the fluid temperatures in the boundary layer. Physically this means that the application of magnetic field introduces additional skin frictional heating which results in higher temperature on the
sheet with increase of thermal boundary layer thickness. This is contrary to the effect of $Mn$ on momentum transfer earlier cited.

Figure 2.6 Temperature Profiles for varying magnetic parameter ($Mn$)

Figure 2.7 Temperature profiles for varying permeability parameter ($k_2$)
Figure 2.7 shows that the temperature within the boundary layer increases with increase in the permeability parameter $k_z$. The effect of the visco-elastic parameter $k_i$ on the temperature profile is shown in Figure 2.8. It is noticed that as $k_i$ increases, $\theta$ also increases. This is in conformity to the fact that an increase in the non-Newtonian visco-elastic parameter leads to an increase of thermal boundary layer thickness.

![Graph showing temperature profiles for varying visco-elastic parameter ($k_i$) with $k_1 = 0.2, 0.4, 0.6$, $Mn = 1.0$, $Pr = 1.0$, $r = 1.2$](image)

**Figure 2.8** Temperature profiles for varying visco-elastic parameter ($k_i$)

Figure 2.9 shows the temperature profiles for varying values of $\beta$. These profiles show increasing trend with the increase in the heat source/sink parameter ($\beta$). It is observed further that the temperature distribution is relatively lower throughout the flow field for negative values of $\beta$ and higher for positive values of $\beta$. Physically $\beta > 0$ implies there will be supply of heat from the sheet to the flow region while $\beta < 0$ implies that there will be a transfer of heat from the flow region to the sheet. The latter is not desirable in a cooling process.
Figure 2.10 shows that as the surface temperature parameter \( r \) increases, the temperature profiles show a gradual decrease throughout.

**Figure 2.9** Temperature Profiles for varying heat source/sink parameter \( \beta \)

**Figure 2.10** Temperature Profiles for varying surface temperature parameter \( r \)
Figure 2.11 shows that the temperatures decrease with an increase in the Prandtl number (Pr) and thus reduces the thermal boundary layer thickness. The decrease in thermal boundary layer thickness with an increase of Prandtl number is apparent from this figure.

![Temperature Profiles for varying Prandtl number (Pr)](image)

**Figure 2.11 Temperature Profiles for varying Prandtl number (Pr)**

Table 2.1 gives a comparison of the end values of $\theta(\eta)$ for varying values of $\eta$ as obtained from the analytical and numerical methods for the PST case for the chosen values of $k_1 = 0.2, \Pr = 1.0, \beta = -0.5, r = 1.2, Mn = 0.5, k_2 = 1.0$.

Table 2.2 gives a comparison of the end values of $\theta_\eta(0)$ for varying values of $\eta$ as obtained from the analytical and numerical methods for the PST case for the chosen values of $k_1 = 0.2, \Pr = 1.0, \beta = -0.5, r = 1.2, Mn = 0.5, k_2 = 1.0$. 
Table 2.1 Comparison of values of $\theta(\eta)$ from the two methods in PST case

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>Numerical solution $\theta(\eta)$</th>
<th>Analytical solution $\theta(\eta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$h = 0.1$</td>
<td>$h = 0.05$</td>
</tr>
<tr>
<td>0.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>2.0</td>
<td>0.102347</td>
<td>0.102710</td>
</tr>
<tr>
<td>4.0</td>
<td>0.011498</td>
<td>0.011541</td>
</tr>
<tr>
<td>6.0</td>
<td>0.001297</td>
<td>0.001302</td>
</tr>
<tr>
<td>8.0</td>
<td>0.000146</td>
<td>0.000147</td>
</tr>
<tr>
<td>12.0</td>
<td>0.000002</td>
<td>0.000002</td>
</tr>
<tr>
<td>14.0</td>
<td>0.000001</td>
<td>0.000001</td>
</tr>
</tbody>
</table>

Table 2.2 Comparison of values of $\theta_{\eta}(0)$ from the two methods in PST case

<table>
<thead>
<tr>
<th>$s, \beta$</th>
<th>$k_1$</th>
<th>$k_2$</th>
<th>$Mn$</th>
<th>$Pr$</th>
<th>Numerical value $\theta_{\eta}(0)$</th>
<th>Analytical value $\theta_{\eta}(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s = 1.2, \beta = -0.5$</td>
<td>0.2</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>-1.1861</td>
<td>-1.1847</td>
</tr>
<tr>
<td>$s = 2.0, \beta = -0.5$</td>
<td>0.2</td>
<td>2.0</td>
<td>1.0</td>
<td>1.0</td>
<td>-1.3373</td>
<td>-1.3337</td>
</tr>
<tr>
<td>$s = 1.2, \beta = 0.0$</td>
<td>0.2</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>-1.3787</td>
<td>-1.3762</td>
</tr>
<tr>
<td>$s = 2.0, \beta = 0.0$</td>
<td>0.2</td>
<td>1.0</td>
<td>2.0</td>
<td>1.0</td>
<td>-1.0034</td>
<td>-0.9959</td>
</tr>
<tr>
<td>$s = 1.2, \beta = -0.5$</td>
<td>0.4</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>-1.1532</td>
<td>-1.1513</td>
</tr>
</tbody>
</table>

It may be inferred from Tables 2.1 and 2.2 that results of the numerical solution obtained with the choice of the incremental value ($h$) as 0.01 agree closely with those of the analytical solution throughout the boundary layer in the PST case.
2.8.2.2 PHF case

The last column in Table 2.3 contains the values of the surface temperature $g(0)$ deduced from the numerical analysis for selected values of visco-elastic parameter($k_1$), permeability parameter($k_2$) and magnetic Parameter($Mn$), Prandtl number($Pr$), heat source/sink parameter ($\beta$) and the heat flux parameter($s$).

From this table, it may be observed that the surface temperature increases with increase in $k_1, k_2$ and $Mn$. It is also noticed that the temperature at the surface decreases with increase in the heat flux ($s$) and with the Prandtl number($Pr$). On the other hand, the temperatures at $\eta = 0$(sheet level) increase with increase in the heat source/sink parameter($\beta$).

Table 2.3 Values of $g(0)$ from numerical solution in the PHF case for the chosen parameters

<table>
<thead>
<tr>
<th>$s = 1.2, \beta = -0.5$</th>
<th>$k_1$</th>
<th>$k_2$</th>
<th>$Mn$</th>
<th>$Pr$</th>
<th>$g(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.2</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>0.8452</td>
</tr>
<tr>
<td></td>
<td>0.4</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>0.8700</td>
</tr>
<tr>
<td>$s = 2.0, \beta = -0.5$</td>
<td>0.2</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>0.7279</td>
</tr>
<tr>
<td></td>
<td>0.2</td>
<td>2.0</td>
<td>1.0</td>
<td>1.0</td>
<td>0.7518</td>
</tr>
<tr>
<td>$s = 1.2, \beta = 0.0$</td>
<td>0.2</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.2030</td>
</tr>
<tr>
<td></td>
<td>0.2</td>
<td>1.0</td>
<td>1.0</td>
<td>2.0</td>
<td>0.7280</td>
</tr>
<tr>
<td>$s = 2.0, \beta = 0.0$</td>
<td>0.2</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>0.9355</td>
</tr>
<tr>
<td></td>
<td>0.2</td>
<td>1.0</td>
<td>2.0</td>
<td>1.0</td>
<td>1.0067</td>
</tr>
</tbody>
</table>

Table 2.4 gives a comparison of the end values of $g(\eta)$ for varying values of $\eta$ as obtained from the analytical and numerical methods for the
PHF case for the chosen values of $k_1 = 0.2, \text{Pr} = 1.0, \beta = -0.5, r = 1.2, Mn = 0.5, k_2 = 1.0$. 

Table 2.5 gives a comparison of the end values of $g(0)$ for varying values of $\eta$ as obtained from the analytical and numerical methods for the PHF case for the chosen values of $k_1 = 0.2, \text{Pr} = 1.0, \beta = -0.5, r = 1.2, Mn = 0.5, k_2 = 1.0$. 

The results of the numerical solution shown in Tables 2.4 and 2.5 agree closely with those of the analytical solution throughout the boundary layer for the PHF case too.

Table 2.4  **Comparison of values of $g(\eta)$ from the two methods in PHF case**

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>Numerical solution $g(\eta)$</th>
<th>Analytical solution $g(\eta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$h = 0.1$</td>
<td>$h = 0.05$</td>
</tr>
<tr>
<td>0.0</td>
<td>0.807044</td>
<td>0.810004</td>
</tr>
<tr>
<td>2.0</td>
<td>0.082598</td>
<td>0.083196</td>
</tr>
<tr>
<td>4.0</td>
<td>0.009280</td>
<td>0.009348</td>
</tr>
<tr>
<td>6.0</td>
<td>0.001047</td>
<td>0.001055</td>
</tr>
<tr>
<td>8.0</td>
<td>0.000118</td>
<td>0.000119</td>
</tr>
<tr>
<td>10.0</td>
<td>0.000013</td>
<td>0.000014</td>
</tr>
<tr>
<td>12.0</td>
<td>0.000002</td>
<td>0.000002</td>
</tr>
<tr>
<td>14.0</td>
<td>0.000001</td>
<td>0.000001</td>
</tr>
</tbody>
</table>
Table 2.5  Comparison of values of $g(0)$ from the two methods in PHF case

<table>
<thead>
<tr>
<th></th>
<th>$k_1$</th>
<th>$k_2$</th>
<th>$Mn$</th>
<th>$Pr$</th>
<th>Numerical value $g(0)$</th>
<th>Analytical value $g(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s = 1.2, \beta = -0.5$</td>
<td>0.2</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>0.8452</td>
<td>0.8441</td>
</tr>
<tr>
<td>$s = 2.0, \beta = -0.5$</td>
<td>0.2</td>
<td>2.0</td>
<td>1.0</td>
<td>1.0</td>
<td>0.7518</td>
<td>0.7498</td>
</tr>
<tr>
<td>$s = 1.2, \beta = 0.0$</td>
<td>0.2</td>
<td>1.0</td>
<td>1.0</td>
<td>2.0</td>
<td>0.7280</td>
<td>0.7266</td>
</tr>
<tr>
<td>$s = 2.0, \beta = 0.0$</td>
<td>0.2</td>
<td>1.0</td>
<td>2.0</td>
<td>1.0</td>
<td>1.0067</td>
<td>1.0041</td>
</tr>
<tr>
<td>$s = 1.2, \beta = -0.5$</td>
<td>0.4</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>0.8700</td>
<td>0.8686</td>
</tr>
</tbody>
</table>

2.9 CONCLUSION

Following conclusions are drawn from the numerical and analytical methods for the range of values considered for the involved parameters. These values have been selected from the earlier works dealing with this problem.

1. The transverse velocity profile decays faster than the axial velocity profile for increasing values of $k_1, k_2$ and $Mn$.

2. The flow velocity decreases with increase in visco-elastic parameter ($k_1$), permeability parameter ($k_2$) and magnetic parameter ($Mn$). The individual influence of $k_2$ and $Mn$ is however, more marked than $k_1$.

3. A reduced momentum boundary layer thickness can be obtained by choosing large values for the visco-elastic parameter $k_1 (< 1)$, the permeability parameter $k_2$ and the magnetic parameter $Mn$ (both around 2.0).
4. For the PST case, the temperature distribution at any point in the boundary layer increases with increase in parameters $k_1, k_2, Mn$ and the heat source/sink parameter ($\beta$). The same however is found to decrease with increase in temperature parameter ($r$) and the Prandtl number ($Pr$).

5. For the PHF case, the surface temperature increases with increase in parameters $k_1, k_2, Mn$ and the heat source/sink parameter $\beta$. The same however decreases with increase in the heat flux parameter ($s$) and the Prandtl number ($Pr$).

6. Higher the value of the temperature parameter ($r$) in the PST case or the heat flux parameter ($s$) in the PHF case, the lower is the temperature in the flow field.

7. The presence of the porous medium decreases the momentum boundary layer thickness significantly and increases the thermal boundary layer thickness in both the PST and PHF cases.

8. A comparison of the end results obtained from the numerical and analytical methods used herein, for the chosen set of values of the parameters considered, in both the PST and PHF cases, has shown good agreement. As a further check on the correctness of the work done herein, the results of this study are also compared with those published by Abel et al (2004) for the particular case when the porosity of the medium is ignored. These are also found to be in good agreement. The study thus demonstrates that the numerical technique adopted, can be used to yield to the same precision of end results, as that of the analytical solution.