Chapter 3

Analyticity Properties of the Perturbation Series

In this chapter we discuss analyticity properties of the TCPT. This discussion is important in understanding convergence properties of the perturbation series and it also provides insight into results of other methods like the KAM theory and the Gustavson’s method. We shall consider two important aspects.

1. The finiteness of each term in the perturbation theory.
2. Analyticity of the total generator $\mathcal{F}$, defined by,

$$\mathcal{F} = \ldots \epsilon^{n} \epsilon^{w_{n}} \ldots \epsilon^{F_{1}}$$

3.1 Finiteness of the $F_{i}$

For a perturbation series to be convergent there are two requirements, (1) each term in the series should be finite and (2) the series while summed over should give a finite result. Now as we know from the small denominator problem, in the usual way of doing perturbation theory, the first order generator turns out to be singular (in fact the same problem persists at all orders). We require the canonical transformation given by different orders of the perturbation theory as well as the total transformation given by the composite of transformations at all the orders to be convergent. Thus our first step in the direction of a convergent perturbation
theory is to have the generating functions at different orders to be nonsingular. This result is shown in the following.

It is shown that if $F_1$ is an entire function of the two variables $t$ and $(n \cdot \omega)$ (and hence of I, if $\omega$'s and $H_{1n}$ are entire functions of $I$), and periodic in $\theta$ the same property will hold for generators at all orders [20].

Lemma: Assume that $H_0$ and $H_1$ are analytic in $I$. Define a sequence of canonical transformations through the equations 2.15:

\[ H^{(0)} = H_0, \]
\[ \{F_1, H_0\} = H_1, \]
\[ H^{(n+1)} = e^{(n+1)F_{n+1}H^{(n)}}, n \geq 0. \]
\[ -\{F_{n+1}, H_0\} = \text{coefficient of } e^{n+1} \text{ in } H^{(n)}, n \geq 1 \]

Then, $F_n$ is an entire function of $I$ and $t$ and is periodic in $\theta$ for all $n = 1, 2, \ldots$ (In brief, we shall say that $F^{(n)}$ is regular).

Proof: The proof follows from induction, using the following facts:

(1) If $H^{(n)}$ is regular and $F(n)$ is regular, so is $H^{(n+1)}$: this follows from the fact that the computation of $H^{(n+1)}$ involves the computation of derivatives, which preserves regularity.

(2) The PDE $\{A, H_0\} = G$ has the property that if $G$ is regular, then $A$ is regular: in fact, if $G = \sum G_x e^{(n \cdot \theta)}$, where the Fourier coefficients are entire functions of $I$ and $t$, then $A = \sum A_n e^{(n \cdot \theta)}$, where

\[ A_n = e^{-i(n \cdot \omega)t} \int_0^t e^{i(n \cdot \omega)\tau} G_n d\tau \quad (3.2) \]

In the RHS $(n \cdot \omega)$ and $t$ appear as arguments of the exponential function which can be singular only when the variable themselves become $\infty$. Further, as the integration is to be done for finite real time the integrand remains within the region of analyticity during the integration. Thus the RHS is an integral of an entire function of $t$ and $(n \cdot \omega)$ along a contour which lies within the region of analyticity; hence the conclusion.
The above result shows that the TCPT is successful in removing the problem of singularities in the generating function at each order of calculation, which was encountered in usual perturbation theory. The next question that remains is that of the convergence of the perturbation theory as a whole, i.e., when the total perturbation series is summed will it give convergent result? It should be noted that now the generating functions are also dependent on time. Hence a higher order calculation in \( \epsilon \) only does not suffice for high accuracy. Value of \( t \) also becomes important in determining convergence. We discuss this problem of convergence of the total TCPT series in the next section.

### 3.2 Convergence of the Perturbation Theory

In calculation of the generating functions at different orders, we have assumed that the total Lie generator \( \mathcal{F} \) is analytic in \( \epsilon \) and \( \mathcal{F} \) can be calculated as power series in \( \epsilon \) from the equation,

\[
\mathcal{F} = \ldots \epsilon^n F_n \ldots \epsilon F_1
\]

If \( \mathcal{F} \) has singularities in complex \( \epsilon \) plane then this power series approximation for \( \mathcal{F} \) will converge for that finite radius in \( \epsilon \) which is defined by the nearest singularity. Thus if \( \mathcal{F} \) has a singularity in complex \( \epsilon \) plane then the mapping and constant of motion approximation will be valid only inside the radius of convergence.

Now the total canonical transformation \( \mathcal{F} \) is equivalent to transforming backwards in time using \( H_0 \) and then transforming forwards in time using \( H \). Since the first step is independent of \( \epsilon \), the analyticity properties of \( \mathcal{F} \) is decided by the analyticity properties of the time evolution operator \( H \) as a function of \( \epsilon \). In general, of course, the analyticity properties of \( \mathcal{F} \) as a function of \( \epsilon \) will depend on \( t \). This is in contrast to the time independent case, where analyticity properties are independent of \( t \). It is not possible to derive general results regarding the analyticity properties of the transformation for any given perturbative Hamiltonian system.

There is no known analytical method to predict complex \( \epsilon \) singularities. We have developed a numerical method for this purpose. The idea of our method is as
follows. Solutions of the perturbed equations can be calculated from the unperturbed solutions using the equation 2.20. It is also known that the unperturbed solutions are analytic in $\epsilon$ (in fact they are independent of $\epsilon$). This implies that any $\epsilon$ singularity present in the solutions of the perturbed system has to come from the singularity in $\mathcal{F}$. Thus singularities in perturbed solutions are the singularities of the generating function. To investigate singularities in $\mathcal{F}$ in this manner a Fortran program was developed. We shall discuss this program in the chapter on the Henon-Heiles system.

Once the position of complex-$\epsilon$ singularity for a fixed real time is known, it is possible to overcome the convergence problem caused by this singularity, provided the singularity in question is an isolated singularity (which is almost always the case in Hamiltonian quasi-chaos). The way of avoiding convergence problem in perturbation series due to existence of isolated singularity is to analytically continue the generating series; we shall discuss this in detail later.

There are two special cases where definite predictions can be made about analyticity of $\mathcal{F}$.
1. The region in phase space where the KAM tori exist.
2. The case where $H_0$ and $H_1$ are homogeneous polynomials in the phase space variables.
We discuss these two cases in following two sections.

### 3.2.1 Relation of $\mathcal{F}$ with $\mathcal{F}_{KAM}$

According to the KAM theorem the perturbation theory converges for the irrational tori when the perturbation parameter is small. We show in the following that the TCPT will give convergent results in the cases where the KAM generator exists and is analytic in action angle coordinates.

We assume that there exists a time independent canonical transformation $\mathcal{F}_{KAM}$ (calculated using the KAM theory) for given irrational frequencies and for a certain value of $\epsilon$. The existence of the KAM generator for a torus implies that $\mathcal{F}_{KAM}$ intertwines between the time evolution generated by the unperturbed and the perturbed Hamiltonian systems, which implies that the following diagram is commutative.
This diagram shows relation of $\mathcal{F}$ with $\mathcal{F}_{KAM}$.

![Diagram showing relation of $\mathcal{F}$ with $\mathcal{F}_{KAM}$](image)

**Figure 3.1: Relation of $\mathcal{F}$ with $\mathcal{F}_{KAM}$**

This means that given a phase-space point (on the KAM torus) one can first apply the canonical transformation generated by $\mathcal{F}_{KAM}$ and then evolve the resultant point under $H$. Alternatively, one can first evolve the point under $H_0$ and then apply the canonical transformation generated by $\mathcal{F}_{KAM}$. In both cases, the final result will be the same. We also know that if $\mathcal{F}$ exists, it maps the solutions of $H_0$ to the solutions of $H$.

From the diagram one gets the following formula.

$$\mathcal{F} = \mathcal{F}_{KAM} e^{-iH_0} \mathcal{F}_{KAM}^{-1} e^{iH_0}$$

Equation 3.3 implies that if KAM theory gives convergent results, the usual canonical transformation generator $\mathcal{F}$ will have only isolated singularities in complex-$\epsilon$ real time space and so it can be analytically continued.  

Above relation provides us with insight into the results of the KAM theorem and the structure of the perturbation theory. KAM theorem guarantees convergence of perturbation theory under the conditions that the unperturbed frequencies are irrational and non-degenerate. Equation 3.3 implies that if KAM theory give convergent results, the usual canonical transformation generator $\mathcal{F}$ will have only isolated singularities in complex-$\epsilon$ real time space and so it can be analytically continued.

The KAM theorem does not provide us with any insight in case of rational tori. The method of overlapping resonances give good estimates of the critical energy for chaos,
to start but does not give any information about nature of the perturbation theory. We propose that in the case of chaos the singularity structure of $\mathcal{F}$ should be such that it does not allow for analytical continuation in the real-$t$ real-$\epsilon$ direction. In the following section we prove a result for a special class of Hamiltonian systems which shows that analytical continuation of the generating function may not be possible when the system is chaotic and its solutions have a natural boundary structure in complex time.

### 3.2.2 Relation Between Complex-$\epsilon$ and Complex-$t$ Singularities

For a class of Hamiltonian systems it is possible to relate the complex-time analytical structure of fully perturbed Hamiltonian systems to the complex-$\epsilon$ analytical structure of the canonical transformation which transforms the Hamiltonian with a small $\epsilon$ value to the Hamiltonian with a large $\epsilon$ value [20].

Let us take a Hamiltonian system where the unperturbed part is homogeneous function of degree $m$ in phase space variables $(q_i, p_i)$ and the perturbation part is homogeneous function of degree $n$.

$$H(q_i, p_i) = H_0(q_i, p_i) + \epsilon H_1(q_i, p_i); \quad (3.4)$$

For such Hamiltonian systems, it is possible to do a scale transformation of the phase space variables so that the resulting Hamiltonian is the same Hamiltonian, with $\epsilon = 1$ and a constant multiplier. Assuming $n > m$ the transformation is

$$q'_i = q_i \epsilon^\alpha, \quad p'_i = p_i \epsilon^\alpha \quad (3.5)$$

where $\alpha = \frac{1}{n-m}$ and the Hamiltonian becomes

$$H'(q'_i, p'_i) = \epsilon^{-m\alpha}(H_0(q'_i, p'_i) + H_1(q'_i, p'_i)) \quad (3.6)$$

(Note that this transformation is not a canonical transformation, but it transforms the vector field of $H$ at some $\epsilon$ to the vector field of $H$ at $\epsilon = 1$.) It is possible to reparametrize the system with $\epsilon'$ defined by, $\epsilon = (1 + \epsilon')^{-\frac{1}{n-m}}$. This yields,

$$H''(q'_i, p'_i) = (1 + \epsilon')(H_0(q'_i, p'_i) + H_1(q'_i, p'_i)) \quad (3.7)$$
The above Hamiltonian is equivalent to perturbing the Hamiltonian,
\[ H_{\varepsilon=1}(q_i', p_i') = H_0(q_i', p_i') + H_1(q_i', p_i') \]
by itself, the perturbation parameter being \( \varepsilon' \), i.e.,
\[ H'(q_i', p_i') = H_{\varepsilon=1}(q_i', p_i') + \varepsilon'H_{\varepsilon=1}(q_i', p_i') \quad (3.8) \]
For such a perturbation the first order generator \( F_1 \) alone gives the total canonical transformation because \( F_1 = H_{\varepsilon=1}(q_i', p_i')t \) and all the higher order generators vanish. \( F_1 \) can also be treated as the Hamiltonian for \( \varepsilon' \) evolution (except at \( \varepsilon = 0 \) because there the scaling transformation is ill-defined) because it generates infinitesimal transformation in \( \varepsilon' \). The \( \varepsilon' \) evolution equations are,
\[ \frac{dq_i'}{d\varepsilon'} = \frac{\partial F_1}{\partial p_i'}; \frac{dp_i'}{d\varepsilon'} = -\frac{\partial F_1}{\partial q_i'} \quad (3.9) \]
Now \( F_1 \) is the same as \( H(q_i, p_i) \) at \( \varepsilon = 1 \), but for an extra multiplier \( t \). So the above equations can be rewritten as,
\[ \frac{q_i'}{\varepsilon't} = \frac{\partial H_{\varepsilon=1}(q, p)}{\partial p_i}; \frac{p_i'}{\varepsilon't} = -\frac{\partial H_{\varepsilon=1}(q, p)}{\partial q_i} \quad (3.10) \]
But these equations are the time evolution equations for the Hamiltonian \( H_{\varepsilon=1} \). Thus, if \( H_{\varepsilon=1} \) has singularities in the complex \( t \) plane, then a canonical transformation defined by \( F_1 \) also will have singularities in the complex \( \varepsilon't \) plane, and hence in the complex \( \varepsilon \) plane for a fixed \( t \). Also, the presence of a natural boundary in the complex \( t \) plane for the Hamiltonian \( H_{\varepsilon=1} \) will manifest itself as a natural boundary in the complex \( \varepsilon \) plane for \( F_1 \). For example consider the \( t = 1 \) case, where the evolution equations for \( \varepsilon' \) are the same as those for \( t \) and so natural boundaries in complex \( t \) plane will manifest itself into that for complex \( \varepsilon' \) plane and hence for complex \( \varepsilon \) plane. Hamiltonian systems with homogeneous perturbation and which have natural boundary structure in complex time plane can be easily found in literature, one such well-studied system is the Henon-Heiles system [21].

In case the singularities of the time evolution operator \( e^{-Ht} \) in the complex \( \varepsilon \) plane are isolated (which almost always happens in the case of Hamiltonian quasi-chaos), it is possible to use analytic continuation to define \( F' \) beyond the radius of convergence. However, as the above study shows, the existence of natural boundaries in the complex \( t \) plane may imply the existence of similar boundaries in the complex \( \varepsilon \) plane...
plane also. It is possible to study the analyticity properties of $F$ using standard tools for determining the analyticity properties of the solutions of the equations of motion of $H$. We discuss one such technique in the appendix A-1 in the next chapter.

### 3.3 Analytical Continuation of the Transformation

A perturbation theory gives convergent results provided the value of the perturbation parameter lies inside the radius of convergence determined by the position of nearest singularity in complex perturbation parameter plane. Thus the TCPT becomes divergent beyond this radius. A way out of this problem is to analytically continue the generator of the transformation.

A two step perturbation theory was worked out. In this theory the range of $\epsilon$ for which calculations are to be done is divided in two small steps $\epsilon = \epsilon_1 + \epsilon_2$. The first step of the canonical transformation transforms the initial unperturbed Hamiltonian to the Hamiltonian with perturbation parameter value $\epsilon_1$. In the next step the Hamiltonian resulting from first step is transformed to get the final Hamiltonian with perturbation parameter value $\epsilon_1 + \epsilon_2$.

The transformation equation for a $(2+1)$ perturbation theory (second order transformation in first step and first order transformation in second step) is,

$$e^{\epsilon_2 F_2}e^{\epsilon_1 F_1} H_0 = H$$

Here $F_{1p}$ is the first order generator for the second step, which can be calculated as follows,

$$\{ F_{1p}, e^{\epsilon_2 F_2} e^{\epsilon_1 F_1} H_0 \} = H_1$$

$$\Rightarrow \{ \epsilon^{-\epsilon_1 F_1} e^{-\epsilon_2 F_2} F_{1p}, H_0 \} = \epsilon^{-\epsilon_1 F_1} e^{-\epsilon_2 F_2} H_1$$

$$\Rightarrow F_{1p}(z) = e^{\epsilon_2 F_2} e^{\epsilon_1 F_1} \int_0^z \epsilon^{-\epsilon_1 F_1} e^{-\epsilon_2 F_2} H_1 d\tau + F_{1p}(0)$$

The single step perturbation theory diverges if value of $\epsilon$ is beyond the radius of convergence but the two-step perturbation theory will give convergent results if the
two steps are correctly chosen. Inside the radius of convergence a third order single-step perturbation theory is supposed to give better results compared to a \((2 + 1)\) order two-step perturbation theory.

We applied the double-step perturbation theory to calculate the canonical transformation from \(H_0 = \frac{v^2 + x^2}{2}\) to \(H = H_0 + \epsilon \frac{x^2}{1}\). The perturbed Hamiltonian \(H\) has one-degree-of-freedom and so it is integrable. Further, for positive values of \(\epsilon\) the motion is compact for all energies. When \(\epsilon\) is negative the motion for \(H\) become noncompact at large enough energy values. We calculated radius of convergence in \(\epsilon\) in real time at the point \(\epsilon = (0, 0)\) for specific initial conditions and applied the two-step theory taking both steps in \(\epsilon\) to be positive so that the resulting system has compact dynamics. The results were compared at two different initial conditions and compared with the single-step theory results. Graph 3.1a shows constant of motion where the final \(\epsilon\) lies outside the radius of convergence. Plot \(O(3)\) is the result of the single-step third order perturbation theory and \(O(2 + 1)\) is the result of the two-step theory. Graph 3.1b is the same as graph 3.1a with the final \(\epsilon\) being inside the radius of convergence.

As can be clearly seen from the graphs, in both the cases the third order predictions are better for time values which are well within the radius of convergence whereas the two-step perturbation theory results turn out to be better for larger time values. Graph 3.1a shows that when the radius of convergence is small the analytically continued generating function gives much better results than the single-step theory.
Graph: 3.1a: Relative variation in predicted time-dependent constant of motion $\dot{r}'$. The nearest singularity is at $t \approx 2.67$. $O(2 + 1)$ shows the two-step perturbation theory results and $O(3)$ shows a single-step third order perturbation theory result.
Graph: 3.1b: Same as graph 3.1a. In this case the singularity is at $t \approx 9.63$. 

Relative variation in $I'$
3.4 Exact Calculation of the TCPT for Some Special Cases

Though it is not possible to calculate the full TCPT transformation in general. There are some special cases where exact calculation of the generating function is possible.

Consider an integrable Hamiltonian $H_0$ which is perturbed by a constant perturbation $H_1$,

$$\{H_0, H_1\} = 0$$

Examples of such systems are where:

1. $H_1$ is a function only of the unperturbed action variables.
2. The angle dependence of $H_1$ is such that $H_1$ remains constant over unperturbed orbits.

In these cases only the first order generating function $F_1$ is non-zero, all the higher order generators vanish. This can be shown as follows. $H_1$ being constant on the unperturbed orbits means that $F_1 = \int H_1 d\tau = H_1 t$. Calculation of $F_2$ requires calculation of $\{F_1, \{F_1, H_0\}\} = \{F_1, H_1\}$. Now $F_1 = H_1 t$ implies $\{F_1, H_1\} = 0$ which in turn implies $F_2 = 0$. In the same manner generators at all the orders are seen to be zero. In other words, since $F_1$ does not create terms of $O(\epsilon^2)$ or higher orders, there is no need to have higher order generating functions which were required to kill higher order terms in $\epsilon$ in the transformed Hamiltonian.

It should be noted that in both the above cases (1) and (2) the CPT is singular and so cannot be applied. Now in the cases considered since only $F_1$ is nonzero and higher order generating function vanish. $F_1$ can be treated as the Hamiltonian for $\epsilon$ evolution. This follows from the fact that if the unperturbed Hamiltonian is considered to be $H_0 + \epsilon_1 H_1$ with the perturbation $\epsilon_1 H_1$, for all finite values of $\epsilon_1$. $F_1$ remains the same. In other words at all values of $\epsilon$,

$$\frac{dI_i}{d\epsilon} = \{I_i, F_1\}$$
$$\frac{d\theta_i}{d\epsilon} = \{\theta_i, F_1\}$$
which are the coefficients of the first order term in $\epsilon$ in the equations for mapping.

Let us consider an example where,

$$H_0 = I_1 + I_2; \quad H_1 = 2\sqrt{I_1I_2}\cos(\theta_1 - \theta_2)$$

It is easier to solve for the canonical transformation after the following coordinate change,

$$
\begin{align*}
I_1 &\rightarrow \frac{p_1^2 + q_1^2}{2} \\
I_2 &\rightarrow \frac{p_2^2 + q_2^2}{2} \\
\theta_1 &\rightarrow \tan^{-1} \frac{p_1}{q_1} \\
\theta_2 &\rightarrow \tan^{-1} \frac{p_2}{q_2}
\end{align*}
$$

under which the perturbed Hamiltonian becomes,

$$H = \frac{q_1^2 + q_2^2 + p_1^2 + p_2^2}{2} + (q_1q_2 + p_1p_2)$$

Now,

$$F_1(t) = \int_0^t H_1(\phi + \omega(\tau - t), \ldots, \tau)d\tau + F_1(0)$$

with $F_1(0) = 0$ and because $H_1$ is constant over unperturbed orbits,

$$F_1 = H_1/l = (q_1q_2 + p_1p_2)/l$$

$F_2$ is calculated by integrating $-\frac{F_1H_1}{2}$ over unperturbed orbits but $\{F_1, H_1\} = 0$. Thus $F_2$ vanishes and similarly all the higher order generating functions also vanish.

The equations defining the canonical transformation are,

$$
\begin{align*}
\frac{dq_1}{d\epsilon} &= p_1/l \\
\frac{dq_2}{d\epsilon} &= p_2/l \\
\frac{dp_1}{d\epsilon} &= -q_2/l \\
\frac{dp_2}{d\epsilon} &= -q_1/l
\end{align*}
$$
These can be solved as,

\[ q_1(\epsilon) = q_{10} \cos \epsilon + p_{10} \sin \epsilon \]
\[ q_2(\epsilon) = q_{20} \cos \epsilon + p_{10} \sin \epsilon \]
\[ p_1(\epsilon) = -q_{20} \sin \epsilon + p_{10} \cos \epsilon \]
\[ p_2(\epsilon) = -q_{10} \sin \epsilon + p_{20} \cos \epsilon \]

where \( q_{10} \) denotes solution of the \( q_1 \) equation at \( \epsilon = 0 \), i.e. the unperturbed solution etc.. It is straight-forward to see that these mapped solutions satisfy the differential equations given by the perturbed Hamiltonian system.