Chapter 2

The Time-dependent Canonical Perturbation Theory

2.1 Why is time dependence necessary?

In this chapter we discuss the reasons for failure of usual (time-independent) perturbation theory and the advantages of choosing the TCPT. We show theoretically as well as with some simple examples how the TCPT helps in getting rid of some of the problems in usual perturbation theory [20].

(1) The idea of canonical perturbation theory is to find a generator of canonical transformation $\mathcal{F}$ such that there is a one to one map between the orbits of the unperturbed Hamiltonian $H_0$ and that of the perturbed Hamiltonian $H$, i.e., the following diagram should commute.

![Figure 2.1: Commutation relation of $\mathcal{F}$ with evolution](image)

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![Figure 2.1: Commutation relation of $\mathcal{F}$ with evolution](image)
In the figure the symbol $H\tau$ denotes the canonical transformation given by the operator $e^{-H\tau}$ and similarly $H_0\tau$ denotes the corresponding time evolution generator. The symbol $\xi$ denotes phase-space coordinate vector and $\xi'$ denotes the transformed coordinates. The meaning of the diagram is that whether one first evolves a phase-space point $\xi(0)$ in time according to the unperturbed equations to get a new point $\xi'(\tau)$ and then apply a canonical transformation $\mathcal{F}$ to get the transformed point $\xi'(\tau)$ or first the canonical transformation $\mathcal{F}$ is applied to $\xi(0)$ to get $\xi'(0)$ and then this point is evolved using the perturbed Hamiltonian EOM, both the way one gets the same end result. The diagram is true for any canonical transformation which transforms $H_0$ to $H$ but we are interested in the special case where $H_0$ is an integrable Hamiltonian and $H$ is a chaotic Hamiltonian.

Suppose $\xi(0)$ is a phase-space point on a periodic orbit of $H_0$ with time period $\tau$, so that the action of the operator $e^{-H_0\tau}$ transforms it back to the same point, i.e., $\xi(0) = \xi(\tau)$. Now if the generator $\mathcal{F}$ is independent of time, i.e., $\mathcal{F}$ depends only on the phase-space coordinates then it will transform both the points $\xi(0)$ and $\xi(\tau)$ to the same point in the transformed coordinates, i.e., $\xi'(0) = \xi'(\tau)$. But this condition implies that if the unperturbed solution $\xi$ is periodic with period $\tau$ then the perturbed solution is also periodic with the same time period. In general this can not be true because as we know most of the phase-space of the perturbed Hamiltonian is covered by a single chaotic orbit which never comes back to it’s starting point. Thus if a canonical transformation exists which transforms $H_0$ to $H$ then it has to be either multiple valued function of phase-space so that the same point $\xi(0) = \xi(\tau)$ can be transformed to two different points $\xi'(0)$ and $\xi'(\tau)$, or it has to be time dependent in which case it will give different transformations at $t = 0$ and $t = \tau$ and so the transformed orbit need not be periodic.

(2) Equation 1.14 can be considered as an operator equation in a Hilbert space of periodic functions. Now as we know, an operator equation admits solutions iff the RHS of the equation is orthogonal to the null eigen vector of the adjoint of the operator. In our case the operator in the equation is $A_{op} = \omega \cdot \frac{\partial}{\partial \omega}$, which operates on $F_1$ to get $H_1$ in RHS, i.e.,

$$A_{op} F_1 = H_1$$

The operator $A_{op}$ is anti-self-adjoint, so the null eigen vectors of $A_{op}$ and $\text{adj} A_{op}$ are
The eigen vectors of $A_{op}$ are of the form $f(t)e^{i(m \cdot \omega)}$ and the eigen values are of the form $i(m \cdot \omega)$. Thus for equation 1.11 to be solvable, $H_{1m}$ should be zero whenever $m \cdot \omega = 0$

Above arguments show that by considering equation 1.11 as an operator equation and trying to solve it, the problem encountered is the same as that in the small denominator problem. In the small denominator problem also we required $H_{1m} = 0$ when $m \cdot \omega = 0$ for getting a non-singular solution. One possible reason for the failure is that the assumption with which we started, i.e. $F_1$ is Fourier expandable in terms of angle variables, may be wrong i.e. $F_1$ may be an aperiodic function of $\theta$.

(3) Alternatively let us look at this problem as that of solving the equation 1.11.

$$\{ F_1, H_0 \} = H_1$$

which is a first order partial differential equation. The equation can be solved using the method of characteristics (to be discussed in detail later). The formal solution is given by,

$$F_1(z) = \int H_1 \, dz \tag{2.1}$$

where $z$ is the characteristic direction of the differential equation. This direction is defined by solutions of unperturbed Hamiltonian equations and it turns out to be the same as the arc parameter $t$.

![Figure 2.2](image)

To write down the solution explicitly, one requires to give the initial value of $F_1$ at $t = 0$, i.e. $F_{10}$. Let us consider the situation shown in the figure 2.2 where $F_1$ is to be calculated for a periodic orbit. The boundary values of $F_1$ are given on a surface $A_0$. Now because $F_1$ is a function of phase-space coordinates, when the phase-space point takes it's initial value $F_1$ also should take it's starting value. As can be seen from the formal solution, if we insist $F_1$ to be periodic for a periodic unperturbed orbit,
integral of $H_1$ on that orbit should be zero. But an orbit being closed implies that
the corresponding frequencies are commensurate, i.e. there exists a non-zero number
vector $m$ such that $m \cdot \omega = 0$. Now considering the corresponding Fourier component
$H_{1m}$ and assuming it to be non-zero gives the old small denominator problem. Hence
for the generator to be finite $H_{1m}$ has to necessarily be zero. Analyzing the situation
differently, if integral of $H_1$ over the unperturbed orbit is non-zero then with increase
in time $\int H_1 dz$ over the closed orbit keeps increasing and in the limit $t \to 0$, the
integral becomes $\infty$. This explains appearance of singularities in usual canonical
perturbation theory when $H_{1m} \neq 0$ and $m \cdot \omega = 0$.

It is easy to see that the condition $H_{1m} = 0$ when $m \cdot \omega = 0$ is not satisfied in general.
One example is the case where $H_1$ is a function only of the action variables. In this
case the only non-zero Fourier component of $H_1$ which is the component with zero
number vector $m$ is non-zero, i.e., $H_{10} \neq 0$ but the corresponding $m \cdot \omega = 0$. In
such cases $F_1$ turns out to be singular. One can get rid of the problem by making
$F_1$ a multiple valued function of phase-space variables. We show in the following
that this property of $F_1$ being ill-defined can be removed by making it explicitly
time-dependent.

(4) Interpreting the above situation in a different manner; let us assume that $H_1$
contains a piece which is constant over the unperturbed trajectory, i.e.,

$$H_1 = f(I) + g(I, \theta)$$

then from the formal solution is,

$$F_1(z) = \int H_1 dz = f(I)z + \int g(I, \theta) dz$$

Thus $F_1$ will contain a piece which is linear in time. If we substitute back the
characteristic variable $z$ in terms of the phase-space variables from $\theta = \theta_0 + \omega z$, we
get a term which is linear in $\theta$. Now a function which is linear in $\theta$ is an aperiodic
function of $\theta$ and so it can not be Fourier expanded. This result is in agreement
with our analysis in (2).

Now as we have seen, if $m \cdot \omega = 0$ and $H_{1m} \neq 0$ then one encounters the small
denominator problem and it is exactly in such cases that we expect $F_1$ to be aperiodic.
in \( \theta \). This implies that the small denominator problem comes-up because we are working with Fourier expansion of an aperiodic function. If the phase-space is extended to include time then an unperturbed orbit never closes into itself because all the orbits travel forward in time. The time dependence of the transformation will arise naturally from the part of \( H_1 \) which is constant over the unperturbed orbits.

(5) A well defined canonical transformation which is not explicitly dependent on time is equivalent to a coordinate transformation of phase-space variables. One cannot expect such a transformation to change the dynamical features of the motion like topology of an orbit or that of an energy-surface. It is possible to change topology of an orbit by applying a time-dependent canonical transformation. As an example consider a free particle for which the phase-space diagram is straight lines with momenta \( p_i \) being constant. If for a given trajectory the coordinates are transformed as \( x_i \rightarrow x_i + p_i t \), then the corresponding straight line in old coordinates will now become a point in the transformed coordinates thus changing the topology of the phase space curve.

It is known that when a perturbation on an integrable Hamiltonian results in a non-integrable Hamiltonian there are drastic changes in the topology of the phase-space (e.g. break-down of resonant tori.). It is easy to see how a time-dependent canonical transformation can account for topology change of phase-curves by considering the extended phase-space. In the extended phase-space a fixed point in the usual phase-space diagram becomes a straight line whereas a straight line goes into another straight line. Thus topologically different phase-curves can have the same topology in extended phase-space.

Conclusion from the above discussion is that if a canonical transformation exists which transforms \( H_0 \) to \( H_1 \), it should have one of the following properties
a. Ill-defined by being singular (as in the small denominator problem).
b. Aperiodic in angle variables.
c. Time-dependent canonical transformation.

We have assumed \( \mathcal{F} \) to be time dependent.
2.2 The Formalism

In order to use the TCPT for chaotic Hamiltonian systems, for simplicity of calculations, we have introduced the following notation. We take time \( t \) as a new DOF and introduce a new phase-space variable \( T \) which is conjugate to \( t \). We redefine the initial and the perturbed Hamiltonian as \( H_0 \to H_0 + T \) and \( H \to H + T \) respectively. The Poisson bracket is now defined on the extended phase-space. Now let us solve equation 1.14 in this formalism using the Fourier expansion method. Writing equation 1.14 as a first order partial differential equation in the extended phase-space we get,

\[
\frac{\partial F_1}{\partial t} + \sum_i \frac{\partial F_1}{\partial \theta_i} \frac{\partial H_0}{\partial I_i} - \frac{\partial F_1}{\partial I_i} \frac{\partial H_0}{\partial \theta_i} = H_1 
\]

(2.2)

Here \( H_0 \) is integrable and so it is a function of action variables only, so \( \frac{\partial H_0}{\partial \theta_i} = 0 \) and derivative of \( H_0 \) with respect to the action variables gives the unperturbed frequencies \( \omega_i = \frac{\partial H_0}{\partial I_i} \). Thus the above equation reduces to,

\[
\frac{\partial F_1}{\partial t} + \sum_i \frac{\partial F_1}{\partial \theta_i} \omega_i = H_1 
\]

(2.3)

Inserting Fourier expansions of \( H_1 \) and \( F_1 \) and solving for Fourier components of \( F_1 \) we get the following,

\[
F_{1n} = e^{-\int_0^t (n \cdot \omega) d\tau} \int_0^t H_{1n} e^{\int_\tau^t i(n \cdot \omega) d\tau'} d\tau + F_{1n0} 
\]

(2.4)

which upon integration yields,

\[
F_{1n} = e^{-i(n \cdot \omega) t} H_{1n} \frac{e^{i(n \cdot \omega) t} - 1}{i(n \cdot \omega)} + F_{1n0} 
\]

(2.5)

On substitution \( \theta = \theta_0 + \omega t \), the Fourier sum for \( F_1 \) is given by,

\[
F_1 = \sum_u H_{1n} e^{iu \theta} \frac{1 - e^{-i(n \cdot \omega) t}}{i(n \cdot \omega)} + F_{10} 
\]

(2.6)

As can be easily checked, unlike the time independent first order generator, this generator of canonical transformation does not become singular when \( (n \cdot \omega) = 0 \) but in the limit \( (n \cdot \omega) \to 0 \) the generator becomes explicitly dependent on time \( t \).
Another way of solving the equation for $F_1$ which is sometimes more useful is the method of characteristics for solutions of first order partial differential equations. If we assume the characteristic direction for the equation to be $z$ then the partial differential equation for $F_1$ corresponding to equation 1.11 is

$$\frac{dF_1}{dz} = \frac{\partial F_1}{\partial t} \frac{dt}{dz} + \sum_k \left( \frac{\partial F_1}{\partial \theta_k} \frac{d\theta_k}{dz} + \frac{\partial F_1}{\partial l_k} \frac{dl_k}{dz} \right) = H_1$$

(2.7)

with the formal solution,

$$F_1(z) = \int H_1 dz$$

(2.8)

Comparing the coefficients of partial derivatives of $F_1$ in equation 2.7 with that in the equation 2.2 gives the characteristic equations,

$$\frac{dt}{dz} = 1$$

$$\frac{d\theta_k}{dz} = \omega_k$$

$$\frac{dl_k}{dz} = 0$$

which can be easily solved as they are the equations of motion for $H_0$.

$$t(z) = t(0) + z$$

$$\theta(z) = \theta(0) + \omega z$$

$$l(z) = l(0)$$

We shall select $t(0) = 0$ for convenience. Using the above expressions of phase-space variables in terms of characteristic direction in $H_1$ in equation 2.8 yields,

$$F_1(z) = \int_0^z \left[ H_1(\theta(0) + \omega \tau, l(0), \tau) d\tau + F_1(0) \right]$$

(2.9)

in which $\theta(0), l(0)$ and $\tau$ are to be substituted back in terms of $\theta, l$ and $t$. The final expression for $F_1$ becomes,

$$F_1(t) = \int_0^t \left[ H_1(\theta + \omega(\tau - t), l, \tau) d\tau + F_1(0) \right]$$

(2.10)

It can be easily seen that $F_1(0)$, which is a function of the phase-space variables $l, T, \theta$ and $t$ corresponds to an arbitrary canonical transformation which can be selected at $t = 0$. It is also easy to see from the following diagram that application of $F_1$ is equivalent to using the unperturbed Hamiltonian to go back in time and using the perturbed Hamiltonian to go forward in time, if $F_1(0)$ is selected to give identity transformation.
Figure 2.3: Commutation relation of $F_1$ with time evolution when $F_{10} = 0$

\[ \xi(t) \xrightarrow{\epsilon F_1} \xi'(t) \]

\[ H_0 t \]

\[ \xi(0) \]

In this thesis we have chosen $F_{10} = 0$ (The commutation relation shown in diagram is correct up to first order in $\epsilon$ only).

As we had seen earlier the first order generator $F_1$ can be calculated using the method of characteristics to be as in equation 2.10. The action of $F_1$ on $H_0$ gives rise to higher order terms in $\epsilon$ apart from the perturbation part $H_1$.

\[ \epsilon^{\epsilon F_1} H_0 = H_0 + \epsilon \{ F_1, H_0 \} + \frac{\epsilon^2}{2!} \{ F_1, \{ F_1, H_0 \} \} + \ldots \]  \hspace{1cm} (2.11)

To remove higher order terms form the right hand side of the equation one uses higher order generators. For example, equation for the second order generator is determined by,

\[ \epsilon^{\epsilon F_2} \epsilon^{\epsilon F_1} H_0 = H_0 + \epsilon \{ F_1, H_0 \} + \frac{\epsilon^2}{2!} \{ F_1, \{ F_1, H_0 \} \} + \ldots + \epsilon^2 \{ F_2, H_0 \} + \ldots \]  \hspace{1cm} (2.12)

Let us select $F_2$ such that the second order terms in $\epsilon$ from the RHS are removed, this gives an equation determining $F_2$.

\[ \{ F_2, H_0 \} = -\frac{1}{2!} \{ F_1, \{ F_1, H_0 \} \} = -\frac{1}{2!} \{ F_1, H_1 \} \]  \hspace{1cm} (2.13)

Similarly an equation determining $F_3$ is found by applying a third order canonical transformation $\epsilon^{\epsilon F_3}$ and selecting $F_3$ so that it cancels third order terms in $\epsilon$ from the RHS,

\[ \{ F_3, H_0 \} = -\frac{1}{3!} \{ F_1, \{ F_1, \{ F_1, H_0 \} \} \} - \{ F_2, \{ F_1, H_0 \} \} \]  \hspace{1cm} (2.14)

The transformation up to order $n$ transforms the Hamiltonian $H_0$ to the Hamiltonian $H$ correctly up to $n^{th}$ order in $\epsilon$. Notice that the equations determining $F_1$, $F_2$ and $F_3$ have the same form.

\[ \{ F_j, H_0 \} = RHS \]
where $j = 1, 2, 3$ and RHS is some function of phase-space variables. As the equations come in the same form, the procedure can be generalized for an arbitrary order transformation and all the equations can be solved using the same method. In general the partial differential equation determining the $(n+1)^{th}$ order generator can be determined using the following set of equations.

$$H^{(0)} = H_0,$$

$$\{F_1, H_0\} = H_1,$$

$$H^{(n+1)} = e^{\epsilon(n+1)F^{(n+1)}}H^{(n)}, n \geq 0,$$

$$- \{F_{(n+1)}, H_0\} = \text{coefficient of } \epsilon^{n+1} \text{ in } H^{(n)}, n \geq 1 \tag{2.15}$$

Here the $n^{th}$ order generator is such that it kills terms of order $\epsilon^n$ (which are generated by lower order generators) in the following equation.

$$\ldots \epsilon^n F_n \ldots \epsilon^{F_i} H_0 = H \tag{2.16}$$

### 2.2.1 Solutions and Invariants

Using the generators up to order $n$ one can calculate approximate constants of motion which are mutually involutive. These constants are given by,

$$I'_i = \epsilon^{nF_n} \ldots \epsilon^{F_i} I_i \tag{2.17}$$

where $I_i$ represent solutions for equations of action variables of $H$ and $I'_i$ are the constants of motion. The way to calculate $I'_i$ is to calculate the action of the generating functions on $I_i$ symbolically to get an expression in the unperturbed coordinates. In the resulting expression when $(I, \theta)$ are evolved according to the EOM of $H$, the $I'_i$ remain constant.

It is easy to see that $I'_i$ are indeed mutually involutive and are constants of motion from the following.

$$\{I'_i, I'_j\} = \{\epsilon^n F_n \ldots \epsilon^{F_i} I_i, \epsilon^n F_n \ldots \epsilon^{F_j} I_j\} = \epsilon^{F_i} \ldots \epsilon^{F_n} \{I_i, I_j\} \tag{2.18}$$
as \( \{I_i, I_j\} = 0 \), the \( I_i' \) are mutually involutive. That the \( I_i' \) are constants of motion can be shown as follows,

\[
\{I_i', H\} = \{e^{\epsilon F_1} \ldots e^{\epsilon F_n} I_i, H\} = e^{\epsilon F_1} \ldots e^{\epsilon F_n} \{I_i, e^{-\epsilon F_n} \ldots e^{-\epsilon F_1} H\} = e^{\epsilon F_1} \ldots e^{\epsilon F_n} \{I_i, H_0\} = 0
\]

(2.19)

where we have used the fact that the action variables are constants of motion for the unperturbed Hamiltonian, i.e. \( \{I_i, H_0\} = 0 \).

The mapping from the unperturbed solutions to the perturbed solutions can be derived as follows. Suppose \( \xi_i' \) represent solutions of EOM of \( H_0 \) and \( \xi_i \) represent solutions of EOM for \( H \) (here \( \xi_1 = \theta_1 \) etc. for EOM for \( H \) and \( \xi_1' = \theta_1 \) etc. for EOM for \( H_0 \)), then

\[
\xi_i(t) = e^{-\epsilon H} \xi_i(0)
\]

Using the commutation relation \( e^{-\epsilon F_1} e^{-\epsilon H_0} = e^{-\epsilon H} e^{-\epsilon F_1} \), which holds because \( F_1 \) is a canonical transformation, we get the following equation.

\[
\xi_i(t) = e^{-\epsilon F_1} e^{-\epsilon H_0} e^{\epsilon F_1} \xi_i(0)
\]

Operating by \( e^{\epsilon F_1} \) on both the sides in the above equation yields.

\[
e^{\epsilon F_1} \xi_i(t) = e^{-\epsilon H_0} e^{\epsilon F_1} \xi_i(0)
\]

Now if we define the mapping relation as \( \xi_i' = e^{\epsilon F_1} \xi_i \), then.

\[
\xi_i'(t) = e^{-\epsilon H_0} \xi_i'(0)
\]

which is the evolution equation for the unperturbed solutions.

The general mapping equation can be given by.

\[
\xi_i = e^{-\epsilon F_1} e^{-\epsilon F_2} \ldots e^{-\epsilon F_n} \xi_i'
\]

(2.20)

where \( \xi_i' \) and \( \xi_i \) are solutions for unperturbed and perturbed Hamiltonian systems respectively. Equation 2.20 allows one to calculate the solutions for perturbed system accurate upto order \( n \) in \( \epsilon \) once the unperturbed solutions and the generating
function up to order \( n \) are known. It should be noted that higher order in \( \epsilon \) does not mean highly accurate numerical results as time \( t \) appears explicitly in the transformation equations. What can be ensured is that for small values of time and away from the singularity in complex \( \epsilon \) plane the above formulae provide very good approximations for the constants of motion as well as the solutions.

2.3 The TCPT and the Hamilton-Jacobi theory

One motivation for considering TCPT comes from the Hamilton-Jacobi theory. Poincare believed that the time dependent series solutions of the perturbed system should be convergent because the solution of the Hamilton-Jacobi equation exists [1]. Instead of working in Lie formalism one can as well have time dependent canonical transformation generator which is of Goldstein type. One such generator is the Hamilton’s principle function. This generator gives a transformation (in mixed notation) which transforms the phase-space coordinates at time \( t \) to phase-space coordinates at time zero and the inverse transformation equations are the solutions of the Hamilton’s equations of motion.

The generating function for Hamilton-Jacobi theory, \( S \) is given by,

\[
\frac{\partial S}{\partial t} + H(\frac{\partial S}{\partial \theta}, \theta) = 0
\]  

(2.21)

where \( S \) is a function of the old angle variables \( \theta \) and the new momenta \( \alpha \). If the Hamiltonian \( H \) does not explicitly depend on time then one assumes \( S \) to be linear in time. Assuming \( S = W - Et \), the Hamilton-Jacobi equation becomes,

\[
H(\frac{\partial W}{\partial \theta}, \theta) = E
\]  

(2.22)

Instead of taking \( S \) to be linearly dependent on time let us assume a general time dependence of \( S \) and write power series expansion of \( S \) in terms of \( \epsilon \).

\[
H(I, \theta) = H_0(I) + \epsilon H_1(I, \theta)
\]

\[
S(\theta, t) = S_0(\theta, t) + \epsilon S_1(\theta, t) + \epsilon^2 S_2(\theta, t) + \ldots
\]

\[
I = \frac{\partial S}{\partial \theta}
\]
(We have not shown the constant transformed action variables in the arguments of $S$). Using these equations in the Hamilton-Jacobi equation 2.21 and equating equal powers of $\epsilon$ we get a series of equations determining the $S_i$. It can be seen that by making $S$ explicitly dependent on time the resulting canonical transformation turns out to be equivalent to the transformation given by the TCPT.

The relation between the Goldstein type generator $S$ and the Lie generating function $F$ which generates translation in $\epsilon$ is given by,

$$\frac{\partial S}{\partial \epsilon} + F(\theta, \frac{\partial S}{\partial \theta}) = 0 \quad (2.23)$$

For the case where $S$ is the solution of Hamiltonian-Jacobi equation this relation is,

$$\frac{\partial S}{\partial t} + H = 0 \quad (2.24)$$

which is the Hamilton-Jacobi equation itself. In H-J equation $S$ generates the canonical transformation which connects the phase-space coordinates at time $t$ to the phase-space coordinates at time $t = 0$. The Hamiltonian generates the same transformation when used as a Lie generating function.

Here it will be worth-while to show a relationship which makes the connection between the TCPT, the Hamilton-Jacobi theory and the time evolution of the system clear. Suppose that an operator $F_{op}$ defines a canonical transformation (either Lie or Goldstein), then according to the definition of canonical transformations the Poisson bracket should be invariant, i.e.,

$$F_{op}\{H, \phi\} = \{F_{op}H, F_{op}\phi\}$$

where $\phi$ is some function over phase-space. Now if we write the transformed Hamiltonian as $H' = F_{op}H$ and write $adH$ for the operator $\{H, \}$ then the following operator relation holds,

$$F_{op}adH = adH'F_{op}$$

This implies,

$$F_{op}(t) \sum_{n=0}^{\infty} \frac{t^n}{n!} adH^n = \sum_{n=0}^{\infty} \frac{t^n}{n!} adH'F_{op}(0)$$

$$\Rightarrow F_{op}(t)e^{-tH} = e^{-tH'}F_{op}(0)$$
where the arguments 0 and \( t \) of \( F_{op} \) indicate that the two operators are to be considered at different times. Now let us select \( F_{op}(0) \) to be an identity transformation, then

\[
F_{op}(t) = e^{-itH'}e^{itH}
\]

Thus a canonical transformation which takes EOM of \( H \) to that of \( H' \) can be written as above mentioned composition of the time evolution of the two Hamiltonian systems. It's known that the H-J generator provides transformation equations which are the same as solutions of EOM. Hence the H-J theory, \( F_{op} \) and the time evolution of the Hamiltonian systems are closely related. The TCPT generator is effectively the same as \( F_{op} \) when at \( t = 0 \) the TCPT transformation is selected to be identity.

In general one expects the TCPT generator to be singular in \( \epsilon \). As is the usual case the unperturbed evolution is always independent of \( \epsilon \). Thus all the \( \epsilon \) singularities of \( \mathcal{F} \) should be coming from the time evolution given by the perturbed Hamiltonian system. But the transformation given by \( \mathcal{F} \) is composition of two transformations \( \mathcal{F} = e^{-H't}e^{Hst} \). Hence a non-trivial \( \epsilon \) singularity structure in \( \mathcal{F} \) corresponds to a similar nontrivial singularity structure of \( S \), the H-J generating function.