CHAPTER - 3

COMMON FIXED POINT THEOREMS IN MENGGER SPACES

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COMMON FIXED POINT THEOREMS IN MENGER SPACES

3.1 In this chapter, we prove some common fixed point theorems for semi-compatible mappings in Menger Spaces. Results of Stojakovic [140], Dedeic and Sarapa [29] and Beg, Rahman and Sahzad [10] in Menger Space and the results of Kang and Rhoades [61] in metric spaces are obtained as corollaries. Illustrative examples are given to discuss the validity of our theorems.

3.2 In 1942, Menger [75] initiated the study of probabilistic metric space. A probabilistic metric space (briefly a PM space) is a space in which the "distance" between any two points is a probability distribution function. The axiomatic characterization of a probabilistic metric space is quite similar to
that of a metric space. In the theory of probabilistic metric space an area of active research is concerned with the study of fixed points.

The Banach Fixed Point Theorem guarantees a unique fixed point for a contraction mapping defined on a complete metric space. A similar theorem does not hold in a complete probabilistic metric space due to the fact that the triangle function in such spaces is often not strong enough to ensure the sequence of iterates of a point under a contraction mapping to be a Cauchy sequence (see Sherwood [126]).

Several authors have studied this problem, and they have essentially pursued two different approaches. One is to identify those triangle functions which are strong enough to guarantee that the sequence of iterates of a point is to be a Cauchy sequence. This was done by (Hadzic [48]) and the other one is to
modify the original definition of a contraction mapping. This was done by Sehgal [110], Sehgal and Bharuch-Reid [112] and Hicks [50].

The study of contraction mapping for PM-space was initiated by Sehgal [109] (c.f. also Schweizer and Sklar [108], Sehgal and Bharuch-Reid [112], Sherwood [126]), who obtained theorems analogous to the Banach Contraction Theorem and other fixed point theorems for contraction mapping. More recently, Cain and Karniel [16], Istratescu and Sacuiu [53] and many other authors have proved a number of interesting fixed point theorems for mappings on various classes of PM-Spaces. Such fixed point theorems have applications to Control Theory, System Theory and Optimization Problems.

3.3 Before further discussion, let us classify the terminology to be used and recall basic definitions and facts:
**DEFINITION 1.** A function $F : \mathbb{R} \to [0,1]$ is a distribution function if it is a nondecreasing, left continuous function with $\inf F = 0$ and $\sup F = 1$.

**DEFINITION 2.** A mapping $t : [0,1] \times [0,1] \to [0,1]$ which is commutative, associative, nondecreasing and $t(0,0) = 0$ and $t(a,1) = a$ for all $a \in [0,1]$ is called a $T$-norm.

A very important $T$-norm is the $T$-norm $t(a,b) = \min\{a,b\}$, $a,b \in [0,1]$ and we can easily show that this is unique $T$-norm such that $t(a,0) = a$ for all $a \in [0,1]$.

Throughout this chapter $H$ will denote the distribution function defined by

$$H(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0. \end{cases}$$

The set of all distribution function, satisfying $F(0) = 0$ will be denoted by $\mathcal{D}^+$. 

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DEFINITION 3. A Menger space is a triplet \((X,\mathcal{F},t)\)
(see,[107]), where \(X\) is any non empty set, \(\mathcal{F} : X \times X \rightarrow \mathbb{R}^+\)
such that \(\mathcal{F}(0) = (0)\) and \(t\) is a \(T\)-norm.

We shall denote the distribution function
\(\mathcal{F}(x,y)\) by \(F_{x,y}\) and the quantity \(F_{x,y}(\epsilon)\) is to
interpret the probability that the distance from \(x\) to \(y\)
is less than \(\epsilon\). The functions \(F_{x,y} ; x,y \in X\) are
assumed to satisfy the following conditions:

\begin{align}
(3.3.1) \quad & F_{x,y}(\epsilon) = H(\epsilon) \quad \text{for all } \epsilon > 0 \text{ iff } x = z \\
(3.3.2) \quad & F_{x,y} = F_{y,x} \quad \text{for all } x,y \in X, \\
(3.3.3) \quad & F_{x,y}(\epsilon + \delta) \geq t\{F_{x,z}(\epsilon), F_{z,y}(\delta)\} \\
& \text{for all } x,y,z \in X \text{ and for all } \epsilon, \delta \in \mathbb{R}^+ - \{0\}.
\end{align}

The concept of neighbourhoods in Menger space
was introduced by Schweizer and Sklar [107]. If \(x \in X\)
\(\epsilon > 0\) and \(\lambda \in \{0,1\}\), then an \((\epsilon,\lambda)\)-neighbourhood of \(x\),
called \(U_X(\epsilon,\lambda)\), is defined by
\[ U_X(\epsilon,\lambda) = \{ y \in X : F_{x,y}(\epsilon) > (1 - \lambda) \}. \]
If $t$ is continuous, then $(X, \mathcal{F}, t)$ is a Hausdorff space in the topology induced by the family

$$U_x(\varepsilon, \lambda) = \{ x \in X, \varepsilon > 0, \lambda \in (0, 1) \}$$

of neighbourhoods.

**DEFINITION 4.** A set $W \subseteq X$ is called probabilistically bounded if and only if $\sup_{\varepsilon > 0} \inf_{x,y \in W} F_{x,y}(\varepsilon) = H(\varepsilon)$.

**DEFINITION 5.** A sequence $\{x_n\}$ in $X$ is said to be convergent to $x \in X$, if $\lim_{n \to \infty} F_{x_n,x}(\varepsilon) = H(\varepsilon)$ for all $\varepsilon > 0$.

**DEFINITION 6.** A sequence $\{x_n\}$ in $X$ is said to be Cauchy sequence, if $\lim_{n \to \infty} F_{x_m,x_n}(\varepsilon) = H(\varepsilon)$ for all $\varepsilon > 0$.

**DEFINITION 7.** A Menger space $(X, \mathcal{F}, t)$ is said to be complete, if every Cauchy sequence in $X$ is convergent to some point in $X$.

Now we recall the definition of semi-compatible mappings.
DEFINITION 8 [22]. Let A and S be mappings from X into itself. The pair \{A,S\} is said to be semi-compatible, if

(3.3.4) \( Sp = Ap \) for some \( p \in X \) implies \( ASP = SAP \)

(3.3.5) The continuity of S at a point \( p \) in X implies \( ASx_n = Sp \), where \( \{x_n\} \) is a sequence in X such that,

\[
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Ax_n = p \quad \text{for some } p \in X.
\]

DEFINITION 9. Let A and S be mappings from X into itself. A sequence \( \{x_n\} \) in X is said to be asymptotically \((A,S)\) regular, if

\[
\lim_{n \to \infty} F_{Ax_n,Sx_n}(\epsilon) = H(\epsilon) \quad \text{for all } \epsilon > 0.
\]

For the rest of this chapter \((X,\mathcal{F},t)\) denotes a complete Menger space, \( t \), a T-norm where

\[
t(a,b) = \min \{a,b\} \quad \text{for all } a,b \in [0,1].
\]

3.4 MAIN RESULTS

We now prove main theorems of this chapter:

THEOREM 1. Let \( \{A_i\}_{i \in \mathbb{N}}, \{B_i\}_{i \in \mathbb{N}}, S \) and \( T \) be mappings from X into itself satisfying the following conditions:
(3.4.1) \hspace{1cm} (3.4.1a) \hspace{1cm} S \text{ and } T \text{ are surjective}

or

(3.4.1b) \hspace{1cm} A_i(X) \subseteq T(X) \text{ and } B_i(X) \subseteq S(X)

or

(3.4.1c) \hspace{1cm} A_i(X) \cup B_i(X) \subseteq S(X) \cap T(X).

(3.4.2) \hspace{1cm} \text{at least one of } A_i, B_i, S \text{ and } T \text{ is continuous}

(3.4.3) \hspace{1cm} \{A_i, S\} \text{ and } \{B_i, T\} \text{ are semi-compatible pairs},

(3.4.4) \hspace{1cm} F_{A_i X, B_i Y}(\varepsilon) = \min \{F_{S X, T Y}(\phi(\varepsilon)),

\hspace{2cm} F_{S X, A_i X}(\phi(\varepsilon)),

\hspace{2cm} F_{T Y, B_i Y}(\phi(\varepsilon)),

\hspace{2cm} F_{T Y, A_i X}(2\phi(\varepsilon)),

\hspace{2cm} F_{S X, B_i Y}(2\phi(\varepsilon))\},

\text{for all } x, y \text{ in } X \text{ and for all } \varepsilon > 0, \text{ where } \phi : \mathbb{R}^+ \to \mathbb{R}^+

\text{is an increasing function such that } \lim_{n \to \infty} \phi^n(\varepsilon) = \infty \text{ for all } \varepsilon > 0. \text{ If there exists } x_0 \in X \text{ such that a sequence}

\{y_n\}_{n \in \mathbb{N}_0} \text{ in } X \text{ defined by}

(3.4.5) \begin{cases}
   T x_{2n+1} = A_i x_{2n} = y_{2n} \text{ (say)} \\
   S x_{2n+2} = B_i x_{2n+1} = y_{2n+1} \text{ (say)}
\end{cases}
for all \( n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) is probabilistically bounded.

Then \( A_i, B_i, S \) and \( T \) have unique common fixed point in \( X \).

**Proof.** For each case (3.4.1a) or (3.4.1b) or (3.4.1c), we can form a sequence \( \{x_n\} \) as was noted above. By triangle inequality (3.3.3), we have

\[
(F_{Y_{n+2}, Y_{n+2}}(2\varepsilon)) = \min \{ F_{Y_n, Y_{n+1}}(\varepsilon), F_{Y_{n+1}, Y_{n+2}}(\varepsilon) \}
\]

for all \( n \in \mathbb{N}_0 \) and \( \varepsilon > 0 \).

From (3.4.4), (3.4.5) and (3.4.6), we have

\[
F_{Y_{2n+1}, Y_{2n+2}}(\varepsilon) = F_{A_i x_{2n+2}, B_i x_{2n+1}}(\varepsilon)
\]

\[
= \min \{ F_{S x_{2n+2}, T x_{2n+1}} \phi(\varepsilon), F_{S x_{2n+2}, A_i x_{2n+2}} \phi(\varepsilon), F_{T x_{2n+1}, B_i x_{2n+1}} \phi(\varepsilon), F_{T x_{2n+1}, A_i x_{2n+2}}(2\phi(\varepsilon)), F_{S x_{2n+2}, B_i x_{2n+1}}(2\phi(\varepsilon)) \}
\]

\[
= \min \{ F_{Y_{2n+1}, Y_{2n}} \phi(\varepsilon), F_{Y_{2n+1}, Y_{2n+2}} \phi(\varepsilon), \}
\]
\[ F_{Y_{2n}, Y_{2n+1}} \phi(c), \]

\[ F_{Y_{2n}, Y_{2n+2}} (2\phi(c)), \]

\[ F_{Y_{2n+1}, Y_{2n+1}} (2\phi(c)) \}

\[ = F_{Y_{2n}, Y_{2n+1}} (\phi(c)) \]

Similarly we have

\[ F_{Y_{2n+2}, Y_{2n+3}} (c) \geq F_{Y_{2n+1}, Y_{2n+2}} (\phi(c)) \]

for all \( n \in \mathbb{N}_0 \) and \( c > 0 \). So, in general,

\[ F_{Y_n, Y_{n+1}} (c) \geq F_{Y_{n-1}, Y_n} (\phi(c)) \geq \cdots \geq F_{Y_0, Y_1} (\phi(n(c)) \]

for all \( n \in \mathbb{N} \). Letting \( n \to \infty \), we obtain

\[ (3.4.7) \quad \lim_{n \to \infty} F_{Y_n, Y_{n+1}} (c) = H(c) \text{ for all } c > 0. \]

First of all we shall show that \( \{Y_n\}_{n \in \mathbb{N}_0} \) is a Cauchy sequence i.e.,

\[ \lim_{m,n \to \infty} (F_{Y_m, Y_n} (c)) = H(c) \text{ for all } c > 0. \]

Putting \( x = x_{2p} \) and \( y = x_{2q+1} \) (let \( p < q \)) in \( (3.4.4) \), and then taking limit on both sides as \( p,q \to \infty \) and using \( (3.4.7) \), we obtain

\[ \lim_{p,q \to \infty} F_{Y_{2p}, Y_{2q+1}} (c) \geq \lim_{p,q \to \infty} F_{Y_{2p-1}, Y_{2q}} (\phi(c)) \]
as

\[ F_{Y_2q,Y_2p}(2\phi(\varepsilon)) \geq \min \{ F_{Y_2q,Y_2p-1}(\phi(\varepsilon)), F_{Y_2p-1,Y_2p}(\phi(\varepsilon)) \}, \]

and

\[ F_{Y_2p-1,Y_2q+1}(2\phi(\varepsilon)) \geq \min \{ F_{Y_2p-1,Y_2q}(\phi(\varepsilon)), F_{Y_2q,Y_2q+1}(\phi(\varepsilon)) \}. \]

Similarly, we have

\[ \lim_{p,q \to -\infty} F_{Y_2p+1,Y_2q}(\varepsilon) = \lim_{p,q \to -\infty} F_{Y_2p-2,Y_2p-1}(\phi(\varepsilon)). \]

Thus,

\[ \lim_{p,q \to -\infty} F_{Y_2p,Y_2q+1}(\varepsilon) \geq \lim_{p,q \to -\infty} F_{Y_2p-1,Y_2q}(\phi(\varepsilon)) \]

\[ \geq \lim_{p,q \to -\infty} F_{Y_0,Y_2p-2p+1}(\phi_{2p}(\varepsilon)) \]

\[ \geq \lim_{p,q \to -\infty} \sup_{t_1 < \phi_{2p}(\varepsilon)} \inf_{l,r \in \mathbb{N}_0} F_{Y_1,Y_r}(\epsilon_{t_1}) \]

\[ = \lim_{p,q \to -\infty} \{ Y_n \}_{n \in \mathbb{N}_0}(\phi_{2p}(\varepsilon)) = H(\varepsilon), \]

since \( \{ Y_n \}_{n \in \mathbb{N}_0} \) is probabilistically bounded.

If \( m \) and \( n \) are both even or both odd, we proceed as

follows:
\[ F_{Y_{2p}, Y_{2q}}(\varepsilon) = \min\{F_{Y_{2p}, Y_{2q+1}}(\varepsilon/2),\]
\[ \min\{F_{Y_{2q}, Y_{2q+1}}(\varepsilon/2), Y_{2q} \cdot Y_{2q+1}\} \]
\[ \Rightarrow \min\{H(\varepsilon), H(\varepsilon)\} \]
\[ = H(\varepsilon), \]
\[ F_{Y_{2p+1}, Y_{2q+1}}(\varepsilon) = \min\{F_{Y_{2p+1}, Y_{2p}}(\varepsilon/2), \]
\[ \min\{F_{Y_{2p}, Y_{2p+1}}(\varepsilon/2), Y_{2p} \cdot Y_{2p+1}\} \]
\[ \Rightarrow \min\{H(\varepsilon), H(\varepsilon)\} \]
\[ = H(\varepsilon), \]

if \( p, q \to \infty \), for all \( \varepsilon > 0 \).

Thus \( \{y_n\} \) is an Cauchy sequence in \( X \) and it converges to some point \( z \) in \( X \). Since the space \( (X, \mathcal{J}, \tau) \) is complete we have,

\[ \lim_{n \to \infty} y_n = \lim_{n \to \infty} A_i x_{2n} = \lim_{n \to \infty} B_i x_{2n+1} \]
\[ = \lim_{n \to \infty} S x_{2n} = \lim_{n \to \infty} T x_{2n+1} = z \in X. \]

Now, suppose that \( T \) is continuous. Since \( \{B_i, T\} \) is semi-compatible pair, we have from (3.3.5)

\[ TT x_{2n+1}, B_i T x_{2n+1} \to Tz \text{ as } n \to \infty. \]
Putting \( x = x_{2n} \) and \( y = T_{2n+1} \) in inequality (3.4.4) we have,

\[
F_{Aix_{2n}, B_{i}T_{x_{2n+1}}} (\varepsilon) \geq \min \{ F_{S_{x_{2n}}, T_{x_{2n+1}}} (\phi(\varepsilon)),
\]
\[
F_{S_{x_{2n}}, A_{i}x_{2n}} (\phi(\varepsilon)),
\]
\[
F_{T_{x_{2n+1}}, B_{i}T_{x_{2n+1}}} (\phi(\varepsilon)),
\]
\[
F_{T_{x_{2n+1}}, A_{i}x_{2n}} (2\phi(\varepsilon)),
\]
\[
F_{S_{x_{2n}}, B_{i}T_{x_{2n+1}}} (2\phi(\varepsilon))
\]

letting \( n \to \infty \) on both sides we get

\[
F_{z, T_{z}} (\varepsilon) \geq \min \{ F_{z, T_{z}} (\phi(\varepsilon)), F_{z, z} (\phi(\varepsilon)),
\]
\[
F_{T_{z}, T_{z}} (\phi(\varepsilon)), F_{T_{z}, z} (2\phi(\varepsilon)),
\]
\[
F_{z, T_{z}} (2\phi(\varepsilon)) \}
\]
i.e.

\[
F_{z, T_{z}} (\varepsilon) \geq \min \{ F_{z, T_{z}} (\phi(\varepsilon)) \}, \text{for all } \varepsilon > 0,
\]

which is a contradiction, therefore \( T_{z} = z \).

Again putting \( x = x_{2n} \) and \( y = z \) in (3.4.4), we have

\[
F_{A_{i}x_{2n}, B_{i}z} (\varepsilon) \geq \min \{ F_{S_{x_{2n}}, T_{z}} (\phi(\varepsilon)),
\]
\[
F_{S_{x_{2n}}, A_{i}x_{2n}} (\phi(\varepsilon)),
\]
\[
F_{T_{z}, B_{i}z} (\phi(\varepsilon))
\]

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\[ \begin{align*}
F_{T_2, A_{i} x_{2n}} (2\phi(\epsilon)) ,
F_{S_{x_{2n}}, B_{i} z} (2\phi(\epsilon)) \}
\end{align*} \]

letting \( n \to \infty \) on both sides we get

\[ \begin{align*}
F_{z, B_{i} z} (\epsilon) &\geq \min \{ F_{z, T_{2}} (\phi(\epsilon)) , F_{z, z} (\phi(\epsilon)), \\
F_{T_{2}, B_{i} z} (\phi(\epsilon)) , F_{T_{2}, z} (2\phi(\epsilon)) \\
F_{S_{x_{2n}}, B_{i} z} (2\phi(\epsilon)) \} \}
\end{align*} \]

i.e. \( F_{z, B_{i} z} (\epsilon) = F_{z, B_{i} z} (\phi(\epsilon)) \) for all \( \epsilon > 0 \),

a contradiction, so that \( B_{i} z = z \).

From condition (3.4.1) there exists a point \( u \) in \( X \) such that, \( B_{i} z = S_{u} \). Then, we have

\[ \begin{align*}
F_{A_{i} u, B_{i} z} (\epsilon) &\geq \min \{ F_{S_{u}, T_{2}} (\phi(\epsilon)) , \\
F_{S_{u}, A_{i} u} (\phi(\epsilon)) , \\
F_{T_{2}, B_{i} z} (\phi(\epsilon)) , \\
F_{T_{2}, A_{i} u} (2\phi(\epsilon)) , \\
F_{S_{u}, B_{i} z} (2\phi(\epsilon)) \} ,
\end{align*} \]

which implies that

\[ F_{A_{i} u, B_{i} z} (\epsilon) = F_{A_{i} u, B_{i} z} (\phi(\epsilon)) \text{ for all } \epsilon > 0 \]
a contradiction, so that \( A_iu = B_iz = z \).

Since \( A_i \) and \( S \) are semi-compatible mappings therefore we have, \( A_i z = A_i S u = SA_i u = S z \).

Moreover by \( (3.4.4) \), we have

\[
F_{Aiz, z}(\epsilon) = F_{Aiz, Biz}(\epsilon)
\]

\[
\leq \min \{ F_{Sz, Tz}(\phi(\epsilon)), F_{Sz, Aiz}(\phi(\epsilon)), F_{Tz, Biz}(\phi(\epsilon)), F_{Tz, Aiz}(2\phi(\epsilon)), F_{Sz, Biz}(2\phi(\epsilon)) \},
\]

which implies that

\[
F_{Aiz, z}(\epsilon) = F_{Aiz, z}(\phi(\epsilon)) \quad \text{for all } \epsilon > 0
\]

a contradiction, so that \( Aiz = z \).

Thus \( z \) is a common fixed point of \( A_i, B_i, S \) and \( T \).

The uniqueness of \( z \) follows easily form \( (3.4.4) \).

The proof of the theorem remains similar if we take any one of the mappings \( A_i, B_i \) or \( S \) as continuous.

This completes the proof.

The following example illustrates the theorem above.
EXAMPLE 1. Let $X = \{0, 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots \}$. Define

$\mathcal{F} : X \times X \to \mathbb{R}^+$, via

$$F_{x,y}(\varepsilon) = \begin{cases} 
0, & \varepsilon \leq 0 \\
\varepsilon^{-\frac{1}{2}}|x-y|, & 0 < \varepsilon < \infty \\
1, & \varepsilon = \infty
\end{cases}$$

It is easy to verify that $(X, \mathcal{F}, t)$ with $t = \min$ is a Menger space.

Define each $A_i = A$, $B_i = B$ and $T : X \to X$ by

$$A(\frac{1}{2^n}) = \begin{cases} 
\frac{1}{2^n+4}, & \text{if } n \text{ is even} \\
\frac{1}{2^n+6}, & \text{if } n \text{ is odd},
\end{cases}$$

$$B(\frac{1}{2^n}) = \begin{cases} 
\frac{1}{2^n+5}, & \text{if } n \text{ is even} \\
\frac{1}{2^n+4}, & \text{if } n \text{ is odd};
\end{cases}$$

$$S(\frac{1}{2^n}) = \begin{cases} 
\frac{1}{2^n+3}, & \text{if } n \text{ is even} \\
\frac{1}{2^n+4}, & \text{if } n \text{ is odd},
\end{cases}$$

and
\[ T(1/2^n) = \begin{cases} 
1/2^{n+5}, & \text{if } n \text{ is even} \\
1/2^{n+3}, & \text{if } n \text{ is odd}; 
\end{cases} \]

where \( A(0) = B(0) = S(0) = T(0) = 0. \)

Consider \( \phi(c) = c/K \) for all \( c \in \mathbb{R}^+ - \{0\} \), where \( k \in (0, 1) \). Since \( \{A, S\} \) and \( \{B, T\} \) are semi-compatible on \( X \) but \( AS \neq SA \) and \( BT \neq TB \) (i.e. \( \{A, S\} \) and \( \{B, T\} \) are not commuting) one of \( A, B, S \) and \( T \) is continuous and satisfy \( A(X) \leq T(X) \) and \( B(X) \leq S(X) \). Moreover, for \( x_n = 1/2^n \); \( n = 0, 1, 2, \ldots \)

\[ Ax_{2n+1} = Tx_{2n+1} = 1/[2^{2n+4}] = Y_{2n} \quad \text{(say)} \]

and

\[ Bx_{2n+1} = Sx_{2n+2} = 1/[2^{2n+5}] = Y_{2n+1} \quad \text{(say)} \]

Thus the sequence \( \{Y_n\} \) converges to 0, which is the only common fixed point of \( A, B, S \) and \( T \).

The following theorem follows easily from Theorem 1:
THEOREM 2. Let \( \{A_i\}_{i \in \mathbb{N}}, \{B_i\}_{i \in \mathbb{N}}, \) and \( T : X \to X \) be the mappings satisfying the conditions (3.4.1), (3.4.2), (3.4.3) and

(3.4.8) \[ F_{A_i x, B_i y}(\varepsilon) \geq F_{S x, T y}(\phi(\varepsilon)) \]

for all \( x, y \in X \) and for all \( \varepsilon > 0 \), then \( A_i, B_i, S \) and \( T \) have a unique common fixed point in \( X \).

The following example shows that some of the assumptions of Theorem 1 and 2 cannot be dropped:

EXAMPLE 2. Let \( X = \{p, q, r\} \) and define \( F : X \times X \to \mathbb{D}^+ \) via,

\[ F_{x, y}(\varepsilon) = \begin{cases} 
0, & \varepsilon \leq 0 \\
\varepsilon^{-\Gamma(|x-y|)}, & 0 < \varepsilon < \infty \\
1, & \varepsilon = \infty
\end{cases} \]

Define each \( A_i = B_i = A, S \) and \( T : X \to X \) by \( x = q \) for all \( x \in X \); \( Sp = r, S q = p, S r = q \) and \( p = p, T q = r, T r = q \). Consider \( \phi(\varepsilon) = \varepsilon/k \) for all \( \varepsilon \in \mathbb{R}^+ - \{0\} \), where \( k \in (0,1) \).
It can be easily seen that all the hypothesis of Theorem 1 and hence of Theorem 2 are satisfied except condition (3.4.3), but there is no common fixed point of $A_i, B_i, S$ and $T$. Indeed, we have, for all $\varepsilon > 0$

$$F_{ASr, SAr}(\varepsilon) = F_{p, q}(\varepsilon) < H(\varepsilon)$$

and

$$F_{ATr, TAR}(\varepsilon) = F_{q, r}(\varepsilon) < H(\varepsilon),$$

where $\{r\}$ is a constant sequence in $X$.

Clearly, $\{A, S\}$ and $\{B, T\}$ are not semi-compatible on $X$.

Now, consider the following generalized contractive type condition:

\[
(3.4.9) \quad F_{A_ix, B_iy}(\varepsilon) = \min \{F_{Sx, Ty}(\phi(\varepsilon)), F_{Sx, A_ix}(\phi(\varepsilon)), F_{Ty, B_iy}(\phi(\varepsilon))\}
\]

\[
(3.4.10) \quad F_{A_ix, B_iy}(\varepsilon) = \min \{F_{Sx, Ty}(\phi(\varepsilon)), F_{Sx, A_ix}(\phi(\varepsilon)), F_{Ty, B_iy}(\phi(\varepsilon))\}
\]
We remark that condition (3.4.4) \Rightarrow (3.4.9), i.e., mappings satisfying (3.4.4) will satisfy (3.4.9).

In view of the remark above, we have the following results:

**THEOREM 3.** Let $A_i, B_i, S$ and $T : X \to X$ ($i = 1, 2, \ldots$) be the mapping satisfying condition (3.4.1), (3.4.2), (3.4.3) and (3.4.9). Then $A_i, B_i, S$ and $T$ have a unique common fixed point in $X$.

Further, we remark that Theorem 1 will be false, if condition (3.4.4) is replaced by (3.4.10). In this case we need an additional condition such as asymptotic regularity (see Definition 9).

Following Theorem concerns with asymptotic regularity of sequence $\{x_n\}$ defined by (3.4.5):

\[ F_{Ty, A_i x}(\phi(\varepsilon)), \]
\[ F_{Sx, B_i y}(\phi(\varepsilon)). \]
THEOREM 4. Let \( \{A_i\}_{i=1}^n, \{B_i\}_{i=1}^n, S \) and \( T \) satisfies conditions of Theorem 1, and the condition (3.4.4) is replaced by (3.4.10) and additionally, the sequence \( \{x_n\} \) is asymptotically \( (A_i,S) \)-regular and is also asymptotically \( (B_i,T) \)-regular, the \( A_i, B_i, S \) and \( T \) have unique common fixed point in \( X \).

PROOF. First we shall show that the sequence \( \{y_n\} \) is a Cauchy sequence i.e.

\[
\lim_{m,n \to \infty} F_{y_m,y_n}(\epsilon) = H(\epsilon) \text{ for all } \epsilon > 0.
\]

If \( m = 2p \) and \( n = 2q+1 \), then

\[
F_{y_{2p},y_{2q+1}}(\epsilon) = F_{A_i x_{2p}, B_i x_{2q+1}}(\epsilon)
\]

\[
= \min \{ F_{S x_{2p}, Tx_{2q+1}}(\phi(\epsilon)), \ldots \}
\]

which implies that,
\begin{align*}
F_{Y_{2p}, Y_{2q+1}}(\varepsilon) & \geq \min \left\{ F_{A_iX_{2p}, S_{x2p}}(\phi(\varepsilon)/3), \right. \\
& \quad \left. F_{B_iX_{2q+1}, T_{A_iX_{2q+1}}} (\phi(\varepsilon)/3), \right. \\
\end{align*}

as

\begin{align*}
F_{S_{x2p}, T_{x2q+1}}(\phi(\varepsilon)) & \geq \min \left\{ F_{S_{x2p}, A_{iX_{2p}}}(\phi(\varepsilon)/3), \\
& \quad \left. F_{A_{iX_{2p}}, B_{iX_{2q+1}}} (\phi(\varepsilon)/3), \right. \\
& \quad \left. F_{B_{iX_{2q+1}, T_{x2q+1}}} (\phi(\varepsilon)/3) \right\}, \\
\end{align*}

and

\begin{align*}
F_{S_{x2p}, B_{iX_{2q+1}}} (\phi(\varepsilon)) & \geq \min \left\{ F_{S_{x2p}, S_{x2p}}(\phi(\varepsilon)/2), \\
& \quad \left. F_{A_{iX_{2p}}, B_{iX_{2q+1}}} (\phi(\varepsilon)/2), \right. \\
\end{align*}

Letting \( p, q \to \infty \), we have

\[
\lim_{p, q \to \infty} F_{Y_{2p}, Y_{2q+1}}(\varepsilon) = H(\varepsilon) \text{ for all } \varepsilon > 0.
\]

Thus, we can easily show that \( \{y_n\} \) is a Cauchy sequence in \( X \). The rest of the proof follows from the proof of Theorem 1.

**Corollary 1 [29].** If we put \( A_i = B_i = R \) (i = 1, 2, \ldots),

In Theorem 3, with \( S \) and \( T \) continuous ; \( \phi(\varepsilon) = \varepsilon/k \),
where $k \in (0,1)$, and replace semi-compatibility by commutativity, then we get the result of Dedeic and Sarapa [29].

COROLLARY 2 [140]. In Theorem 2 with condition (3.4.1b)

if we put $A_i = f$, $B_i = g$ ($i = 1, 2, \ldots$), $S = k$, $T = h$,
where $f, g, h$ and $k$ are continuous and semi-compatibility is replaced by commutativity, then we get a result of Stojakovic [140].

COROLLARY 3 [10]. In Theorem 1 with (3.4.1b) and (3.4.4) if we put $A_i = A$, $B_i = B$, ($i = 1, 2, 3, \ldots$) and $\phi(\varepsilon) = \varepsilon/k$ where $k \in (0,1)$, and semi-compatibility is replaced by compatibility then we get the result of Beg, Rahman and Sahzad [10, Theorem 4.4].

COROLLARY 4. [61]. In Theorem 2 with (3.4.1a) and (3.4.8) if we put $A_i = S$, $B_i = T$, ($i = 1, 2, 3, \ldots$), $S = A$ and $T = B$ and semi-compatibility is replaced by
compatibility, then we get the result of Kang and Rhoades [61].

***