CHAPTER - 2

COMMON FIXED POINT THEOREMS FOR SEMI-COMPATIBLE MAPPINGS

(17-38)
CHAPTER-2

COMMON FIXED POINT THEOREMS FOR

SEMI-COMPATIBLE MAPPINGS

2.1 In this chapter we prove some fixed point theorems for semi-compatible mappings in metric spaces. Our results extend, generalize and improve the results of [20, 28, 35, 36, 39, 40, 47, 56, 66, 67, 70, 90, 113, 115, 116, 127, 133, 134, 142, and 144].

2.2 There is a multitude of metrical fixed point theorems for mappings satisfying certain contractive type conditions. In each of these results sequences of iterates were considered which due to the contractive condition, becomes a Cauchy sequence limit of which is a fixed point of the mapping. In the case of common fixed point theorems, a joint sequence of iterates is usually suitable for the purpose.
It was Jungck [54] who replaced the identity mapping with a continuous function in order to generalize the celebrated Banach Contraction Principle and proved the following theorem:

**THEOREM J**: A continuous self-mapping $S$ of complete metric space $(X,d)$ has a fixed point in $X$ if and only if there exists a number $\alpha \in (0,1)$ and a mapping $T : X \to X$ which commutes with $S$ and satisfies the following

\begin{align*}
(2.2.1) \quad & T(X) \subset S(X) \\
\text{and} \quad & \text{for all } x, y \text{ in } X. \end{align*}

and

\begin{align*}
(2.2.2) \quad & d(Tx,Ty) \leq \alpha \cdot d(Sx, Sy) \\
\text{for all } x, y \text{ in } X. \end{align*}

In fact, $S$ and $T$ have a unique common fixed point in $X$.

Since then, a number of authors extended, generalized and unified this theorem in many ways (see for example Chang [18], Conserva [27], Das and
Naik [28], Fisher [37], Jungck [54,57], Naidu and Prasad [80], Prasad [92], Rhoades [98], Park and Moon [100], Sessa, Rhoades and Khan [117], Rhoades, Tiwari and Singh [142] and Yeh [144]).

It is important to note that, in most of the extensions and generalizations of Jungck's theorem a family of commuting mappings has been considered. Sessa [113], generalized the concept of commuting mappings by defining Weakly Commuting pair of mappings as below:

**DEFINITION 1** [113]. Let A and B be mappings from a metric space \((X,d)\) into itself. Then A and B are said to be weakly commuting if

\[
(2.2.3) \quad d(ABx,BAx) \leq d(Ax,Bx) \text{ for all } x \text{ in } X.
\]

Sessa and several authors have proved some fixed point theorems of weakly commuting mappings (see for example Baskaran and Subrahmanyam [9], Fisher and Sessa [40], Pathak [87], Rhoades and Sessa [101], Sessa
and Fisher [115], Sessa, Mukherjee and Som [116], Singh, Ha and Cho [129]. It is important to note that, commuting mappings are weakly commuting, but the converse is not true. In 1986, Jungck [55], proposed a generalization of the concepts of commuting mappings and weakly commuting mappings, which he called compatible mappings defined as:

**DEFINITION 2** [55]. Let A and B be mappings from a metric space \((X,d)\) into itself. Then A and B are said to be compatible if

\[
\lim_{n \to \infty} d(Ax_n, Bx_n) = 0,
\]

whenever \(\{x_n\}\) is a sequence in \(X\) such that,

\[
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = t,
\]

for some \(t\) in \(X\).

It is well known that weakly commuting pair of mappings are compatible but the converse is not true. To illustrate this we give the following example:
EXAMPLE 1. Let $X = \mathbb{R}$ (set of real numbers) with Euclidean metric $d$. Define $Ax = x^m$ and $Bx = 2x^m$ for all $x \in X$, where $m$ is an integer such that, $m > 1$.

Then, $\lim_{n \to \infty} Ax_n = Bx_n = 0$ as $x_n \to 0$ and

$$d(ABx_n, BAx_n) = |2^m - 2| \cdot |x_n|^m 2^m \to 0$$

as $x_n \to 0$, hence $A$ and $B$ are compatible.

But since, $d(ABx, BAx) = 2^m (2^m - 2) > 2^m = d(Ax, Bx)$, for $x (=2)$ in $X$ where $m < 1$, hence $A$ and $B$ are not weakly commuting.

Since Jungck several authors have proved common fixed point theorems using this concept. (see for example. Jungck [56], Kang, Cho and Jungck [60], Kang and Rhoades [61], Sharma, Sahu and Thakur [123], Sharma and Thakur [124].

Recently, Cho, Sharma and Sahu [22], introduced the concept of semi-compatible mappings which is defined as below:
DEFINITION 3 [22]. Let $A$ and $S$ be mappings from a metric space $(X,d)$ into itself. Then $A$ and $S$ are said to be semi-compatible, if

\begin{equation}
A_p = S_p \text{ for some } p \in X \implies AS_p = SA_p.
\end{equation}

\begin{equation}
The continuity of $A$ at a point $p$ in $X$ implies that, $\lim_{n \to \infty} SAx_n = Ap$, whenever $\{x_n\}$ is a sequence in $X$ such that,

$$\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Ax_n = p$$

for some $p$ in $X$.

It is important to note that every compatible pair (and hence commuting and weakly commuting pair) of mappings is semi-compatible but the converse is not true.

2.3 On the other hand Delbosco [30] introduced $p$-contraction mappings which is defined as below:

Let $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ denote the set of real numbers, the non-negative real numbers, and the set of
positive integers respectively. In the sequel \( \text{NDUSK} \)
\( (\mathbb{R}_+, \mathbb{R}_+)^k \) denotes the class of all nondecreasing upper semi-continuous functions, \( \gamma : \mathbb{R}_+^k \rightarrow \mathbb{R}_+ \), \( k \in \mathbb{N} \).

In the sequel \( \text{O}(\mathbb{R}_+) \) denotes the class of all functions \( \gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \), \( \gamma \in \text{NDUSK} \) \( (\mathbb{R}_+, \mathbb{R}_+) \) satisfying the following conditions:

\begin{align*}
(2.3.1) & \quad \gamma(0) = 0 \text{ and } \gamma(t) > t \text{ for each } t > 0, \\
(2.3.2) & \quad \lim_{t \to 0} \gamma^n(t) = 0 \ldots \text{ for each } t > 0.
\end{align*}

Let \( \mathcal{F} \) be the set of real functions \( p : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+ \) satisfying the following properties:

\begin{align*}
(2.3.3) & \quad p \text{ is continuous in whole } \mathbb{R}_+^3 \\
(2.3.4) & \quad p(1,1,1) = h < 1 \text{ where } n \in \mathbb{R}_+ \\
(2.3.5) & \quad \text{let } a, b \in \mathbb{R}_+ \text{ be such that, either } \\
& \quad \text{0} \leq p(a,b,b) \text{ or } a \leq p(b,a,b) \text{ or } a \leq p(b,b,a). \\
\end{align*}

Then \( a \leq k \cdot b \) for some \( k \in [h,1) \)

**DEFINITION 4.** A self mapping \( T \) of a metric space \((X,d)\) is called \( p \)-contraction if there exists a function
\( p \in \mathcal{P} \) such that for any \( x, y \in X \),

\[(2.3.6) \quad d(Tx,Ty) \leq p(d(x,y), d(x,Tx), d(y,Ty)).\]

Now we generalize the definition of \( p \)-contraction as below:

Let \( \mathcal{P} \) be the set of all real functions \( p : \mathbb{R}_+^5 \to \mathbb{R}_+ \)

satisfying the following properties for any \( \gamma \in \mathcal{O}(\mathbb{R}_+) \):

\begin{enumerate}
\item[(2.3.7a)] \( p \) is continuous in \( \mathbb{R}_+^5 \).
\item[(2.3.7b)] \( p \) is nondecreasing in the last two coordinate variables,
\item[(2.3.7c)] \( p(t,t,t,2t,2t) = \gamma(t) < t \) for each \( t > 0 \),
\item[(2.3.7d)] let \( a, b \in \mathbb{R}_+ \) be such that,
\item[(2.3.8a)] \( a \leq p(a,b,b,a,a) = \gamma(b) \),
\item[(2.3.8b)] \( a \leq p(b,b,a,a,b) = \gamma(b) \),
\item[(2.3.8c)] \( a \leq p(b,a,b,b,a) = \gamma(b) \),
\item[(2.3.8d)] \( a \leq p(b,b,a,2c,0) = \gamma(b) \) and
\item[(2.3.8e)] \( a \leq p(b,a,b,0,2c) = \gamma(b) \),
\end{enumerate}

where \( c = \max \{a,b\} \).
DEFINITION 5. Let $A, B, S$ and $T$ be mappings from a metric space $(X, d)$ into itself. Then $A$ and $B$ are said to be generalized $p$-contractions relative to $S$ and $T$, if there exists a function $p \in \mathcal{F}$ such that for any $\gamma \in 0(\mathbb{R}_+)$ and $x, y \in X$

\begin{equation}
\text{(2.3.9)} \quad d(Ax, By) \leq \{p(d(Sx, Ty), d(Sx, Ax), d(Ty, By), d(Sx, By), d(Ty, Ax))\}.
\end{equation}

It is easy to verify that the class $\mathcal{P}$ of $p$-contraction mappings \text{(2.3.8)} defined by Delbosco [30] is in class $\mathcal{F}$.

2.4 MAIN RESULTS

Now we give main results of this chapter:

THEOREM 1. Let $\{A_i\}_{i \in \mathbb{N}}, \{B_i\}_{i \in \mathbb{N}}, S$ and $T$ be mappings from a complete metric space $(X, d)$ into itself satisfying the following conditions:

\begin{equation}
\text{(2.4.1)} \quad A_i \text{ and } B_i \text{ are generalized } p\text{-contractions}
\end{equation}
relative to $S$ and $T$, i.e.

$$d(A_i x, B_i y) \leq \{p(d(Sx, Ty), d(Sx, A_i x),$$

$$d(Ty, B_i y), d(Sx, B_i y), d(Ty, A_i x))\}.$$

(2.4.2) (a) either $A_i(X) \subset T(X)$ and $B_i(X) \subset S(X)$

for each $i$,

or

(b) $A_i(X) \cup B_i(X) \subset S(X) \cap T(X)$ for each $i$,

or

(c) $S$ and $T$ are surjective.

(2.4.3) The pairs $\{A_i, S\}$ and $\{B_i, T\}$ are semi-compatible,

(2.4.4) at least one of $A_i, B_i, S$ and $T$ is continuous.

Then $A_i, B_i, S$ and $T$ have a unique common fixed point in $X$.

Since (2.4.2) holds, we can choose for any arbitrary point $x_0$ in $X$, sequences $\{x_n\}$ and $\{y_n\}$ such that,
\[
\begin{align*}
Y_{2n+1} &= T x_{2n+1} = A_i x_{2n} \\
\text{and} \\
Y_{2n+2} &= S x_{2n+2} = B_i x_{2n+1}
\end{align*}
\]

(2.4.5)

for each \(n \in \mathbb{N}_0\), where \(\mathbb{N}_0 = \mathbb{N} \cup \{0\}\).

We first prove the following lemma:

**Lemma 1.** The sequence \(\{y_n\}\) defined by (2.4.5) is a Cauchy sequence in \(X\).

**Proof.** Define \(d_n = d(y_n, y_{n+1})\) for each \(n \in \mathbb{N}\). By (2.4.1), we have

\[
d_{2n+1} = d(A_i x_{2n}, B_i x_{2n+1}) \\
\leq p(d_{2n}, d_{2n}, d_{2n+1}, d(y_{2n}, y_{2n+2}), 0)
\]

Since

\[
d(y_{2n}, y_{2n+2}) \leq d_{2n} + d_{2n+1} \leq 2 \cdot \max \{d_{2n}, d_{2n+1}\},
\]

we have,

\[
d_{2n+1} \leq p(d_{2n}, d_{2n}, d_{2n+1}, 2 \cdot \max \{d_{2n}, d_{2n+1}\}, 0).
\]

Then by Property (2.3.8d), we have

\[
d_{2n+1} \leq \gamma (d_{2n}).
\]
Similarly, we have,

\[ d_{2n+2} \leq \gamma (d_{2n+1}). \]

So, in general,

\[ d_n \leq \gamma (d_{n-1}) \leq \gamma^2 (d_{n-2}) \ldots \leq \gamma^{n-1} (d_1). \]

By condition (2.3.2), we obtain

\[ \lim_{n \to \infty} d_n = 0. \]

We wish to show that \( \{y_n\} \) is a Cauchy sequence. Suppose that \( \{y_n\} \) is not a Cauchy sequence, then there exists an \( \varepsilon > 0 \) such that for each even integer \( 2k \), there exist even integers \( 2m(k) \) and \( 2n(k) \) with \( 2m(k) > 2n(k) \geq 2k \) such that,

\[ d(y_{2m(k)}, y_{2n(k)}) > \varepsilon. \]

For every even integer \( 2k \), let \( 2m(k) \) be the least even integer exceeding \( 2n(k) \) satisfying (2.4.7), that is

\[ \begin{align*}
    d(y_{2n(k)}, y_{2m(k) - 2}) &\leq \varepsilon \\
    \text{and} \\
    d(y_{2n(k)}, y_{2m(k)}) &> \varepsilon
\end{align*} \]

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Then for each even integer $2k$,

$$
\varepsilon < d(Y_{2n(k)}, Y_{2m(k)}) \\
\leq \{d(Y_{2n(k)}, Y_{2m(k)}-2) + d(Y_{2m(k)}-2, Y_{2m(k)}-1) \\
+ d(Y_{2m(k)}-1, Y_{2m(k)})\}
$$

which from (2.4.6), implies that

$$
\lim_{n \to \infty} d(Y_{2n(k)}, Y_{2m(k)}) = \varepsilon.
$$

By triangle inequality, we have

$$
|d(Y_{2n(k)}, Y_{2m(k)}-1) - d(Y_{2n(k)}, Y_{2m(k)})| \\
\leq d(Y_{2m(k)}-1, Y_{2m(k)})
$$

and

$$
|d(Y_{2n(k)}, Y_{2m(k)}-1) - d(Y_{2n(k)}, Y_{2m(k)})| \\
\leq d(Y_{2m(k)}-1, Y_{2m(k)}) + d(Y_{2n(k)}, Y_{2m(k)}+1)
$$

Thus we obtain, as $k \to \infty$

$$
\begin{cases}
\quad d(Y_{2n(k)}, Y_{2m(k)}) \to \varepsilon \\
\quad \text{and} \\
\quad d(Y_{2n(k)+1}, Y_{2m(k)}-1) \to \varepsilon.
\end{cases}
$$

Therefore, from (2.4.1) we have
\[ d(Y_{2n}(k), Y_{2m}(k)) \leq \{d(Y_{2n}(k), Y_{2n}(k) + 1) + d(Y_{2n}(k) + 1, Y_{2m}(k)) \} \]

\[ \leq d_{2n}(k) + P \{d(Y_{2n}(k), Y_{2m}(k) - 1), \]

\[ d_{2n}(k), d_{2m}(k) - 1), d(Y_{2n}(k), Y_{2m}(k)), \]

\[ d(Y_{2m}(k) - 1, Y_{2n}(k) + 1) \}. \]

By continuity of \( p \), we have \( c = p(c, 0, 0, c, c) \)
which implies, from property (2.3.8a) that,
\[ c = \gamma(0) = 0, \]

a contradiction. Therefore \( \{y_n\} \) is a Cauchy sequence in \( X \).

**PROOF OF THE THEOREM.** By Lemma 1, \( \{y_n\} \) is a Cauchy sequence and it converges to some point \( z \) in \( X \).

Consequently, the subsequences \( \{A_i x_{2n}\}, \{B_i x_{2n+1}\}, \{S x_{2n}\} \) and \( \{T x_{2n+1}\} \) of \( \{y_n\} \) also converge to \( z \). Now, suppose that \( T \) is continuous. Since \( B \) and \( T \) are semi-compatible, we have

\[ TT x_{2n+1} \text{ and } B_i T x_{2n+1} \to T z \text{ as } n \to \infty. \]

By (2.4.1), we have

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\[ d(A_i x_{2n}, B_i T x_{2n+1}) \leq p(d(x_{2n}, T T x_{2n+1}), d(x_{2n}, A_i x_{2n}),
\]
\[ d(T T x_{2n+1}, B_i T x_{2n}), d(x_{2n}, B_i T x_{2n+1}),
\]
\[ d(T T x_{2n+1}, A_i x_{2n})). \]

Letting, \( n \to \alpha \), we have,
\[ d(z, T z) \leq p(d(z, T z), 0, 0, d(z, T z), d(T z, z)). \]

Then, by Property (2.3.8a), we have
\[ d(z, T z) \leq \gamma(0) = 0, \text{ so that } T z = z. \]

Further,
\[ d(A_i x_{2n}, B_i z) \leq \{ p(d(x_{2n}, T z), d(x_{2n}, A_i x_{2n}),
\]
\[ d(T z, B_i z), d(x_{2n}, B_i z),
\]
\[ d(T z, A_i x_{2n}) \}. \]

Letting \( n \to \omega \), by Property (2.3.8b) we have
\[ d(z, B_i z) \leq p(0, 0, d(B_i z, z), d(B_i z, z), 0) = 0, \]
and therefore \( B_i z = z \). Which implies from (2.4.2),

that, there exists a point \( u \) in \( X \) such that,
\[ B_i z = S u = z. \]

Then we have,
\[(A_1u, z) \leq d(A_1u, B_1z)\]

\[\leq \{p(d(Su, Tz), d(Su, A_1u), d(Tz, B_1z),
\]

\[d(Su, B_1z), d(T, A_1u))\}\]

\[\leq p(0, d(z, A_1u), 0, 0, d(z, A_1u))\]

\[= \gamma(0) = 0, \text{ by Property (2.3.8c)}\]

which implies that, \(A_1u = z\).

Since \(A_1\) and \(S\) are semi-compatible,

\[Sz = SA_1u = A_1Su = Az\]

From (2.4.1),

\[d(A_1z, z) = d(A_1z, B_1z)\]

\[\leq \{p(d(Sz, Tz), d(Sz, A_1z), d(Tz, B_1z),
\]

\[d(Sz, B_1z), d(Tz, A_1z))\}\]

\[\leq p(d(A_1z, z), 0, 0, d(A_1z, z), d(A_1z, z))\]

\[= \gamma(0) = 0, \text{ by Property (2.3.8a)}\],

which implies that, \(A_1z = z\).

Therefore, \(z\) is a common fixed point of \(A_1, B_1, S\) and \(T\).

The uniqueness of the common fixed point follows from
(2.4.1). The theorem can be proved in a similar manner, when \( A_i \) or \( B_i \) or \( S \) is continuous.

This completes the proof of the theorem.

We have the following results from Theorem 1:

**Theorem 2.** Let \( \{A_i\}_{i \in \mathbb{N}} \), \( \{B_i\}_{i \in \mathbb{N}} \), \( S \) and \( T \) be mappings from a complete metric space \( (X,d) \) into itself satisfying the conditions (2.4.2), (2.4.3), and (2.4.4) and

\[
(2.4.8) \quad d(A_i x, B_i y) \leq [f(\max\{d(Sx,Ty), d(Sx,A_i x),
\]
\[
d(Ty,B_i y), d(Sx,B_i y),
\]
\[
d(Ty,A_i x)\}]]
\]

for all \( x,y \) in \( X \), where \( f \in \text{NDUSK} (\mathbb{R}_+, \mathbb{R}_+) \) and satisfies \( f(t) < t \) for each \( t > 0 \).

Then \( A_i \), \( B_i \), \( S \) and \( T \) have a unique common fixed point in \( X \).

**Corollary 1 [39].** Let \( S \) and \( I \) be commuting mappings and let \( T \) and \( J \) be commuting mappings of a complete metric
space \((X,d)\) into itself satisfying the following inequality

\[(2.4.9) \quad d(Sx,Ty) \leq c \cdot \max \{d(Ix,Jy), d(Ix,Sx), d(Jy,Ty)\}\]

for all \(x, y\) in \(X\) where \(0 < c < 1\).

If the range of \(I\) contains the range of \(S\) and if one of \(S, T, I\) and \(J\) is continuous then \(S, T, I\) and \(J\) have a unique common fixed point \(z\). Further, \(z\) is the unique common fixed point of \(S\) and \(I\) and of \(T\) and \(J\).

**COROLLARY 2 [40].** Let \(S\) and \(I\) be weakly commuting mappings and let \(T\) and \(J\) be weakly commuting mappings of a complete metric space \((X,d)\) into itself satisfying the following inequality

\[(2.4.10) \quad d(Sx,Ty) \leq g \{d(Ix,Jy), d(Ix,Sx), d(Jy,Ty)\}\]

for all \(x, y\) in \(X\) where \(g \in \mathbb{P}\).

If the range of \(I\) contains the range of \(S\) and if one of \(S, T, I\) and \(J\) is continuous then \(S, T, I\) and \(J\) have a unique common fixed point \(z\). Further, \(z\) is the
unique common fixed point of \( S \) & \( I \) and of \( T \) & \( J \).

**COROLLARY 3 [115].** Let \( S, T, I \) and \( J \) be four self mappings of \((X,d)\) satisfying,

\[(2.4.11a) \quad T(x) \in I(x) \quad \text{and} \quad S(x) \in J(x)\]

\[(2.4.11b) \quad d(Sx,Ty) \leq f \left( \max \{d(Ix,Jy), d(Ix,Sx), d(Jy,Ty)\} \right)\]

for all \( x,y \) in \( X \) and \( f \) is in \( \mathcal{W} \), where \( \mathcal{W} \) is the set of all functions \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) such that,

\[(2.4.12a) \quad f \text{ is isotone i.e., if } t_1 \leq t_2 \text{ then } f(t_1) \leq f(t_2) \text{ for all } t_1, t_2 \text{ in } \mathbb{R}_+.\]

\[(2.4.12b) \quad f \text{ is upper semi continuous}\]

\[(2.4.12c) \quad f(t) < t \text{ for all } t > 0.\]

If one of \( S, T, I \) and \( J \) is continuous and if \( S \) and \( T \) weakly commute respectively with \( I \) and \( J \), then \( S, T, I \) and \( J \) have a common fixed point \( z \). Further \( z \) is the unique common fixed point of \( S \) & \( I \) and of \( T \) & \( J \).
COROLLARY 4 [20]. Let $I$ and $J$ be two continuous mappings of a complete metric space $(X,d)$ into itself. Then $I$ and $J$ have a common fixed point in $X$ if and only if there exists a function $\phi(t) : \mathbb{R}_+ \to \mathbb{R}_+$ which is nondecreasing, upper semi-continuous and $\phi(t) < t$ for all $t > 0$ and mappings $S,T : X \to X$ commute respectively with $I$ and $J$ and satisfy the following condition,

\begin{align*}
(2.4.13a) & \quad S(X) \cup T(X) \subset I(X) \cap J(X) \\
(2.4.13b) & \quad d(Sx,Ty) \leq \phi(\max\{d(Ix,Jy),d(Ix,Sx), \\
& \quad \quad \quad \quad \quad \quad \quad \quad d(Jy,Ty), 1/2[d(Ix,Ty) + d(Jy,Sx)]\})
\end{align*}

for all $x,y \in X$. Then $S,T,I$ and $J$ have a unique common fixed point.

COROLLARY 5 [90]. Let $S,T,I$ and $J$ be four self mappings of $(X,d)$ satisfying,

\begin{align*}
(2.4.14a) & \quad T(X) \subset I(X) \text{ and } S(X) \subset J(X) \\
(2.4.14b) & \quad d(Sx,Ty) \leq \phi(\max\{d(Ix,Jy),d(Ix,Sx), \\
& \quad \quad \quad \quad \quad \quad \quad \quad d(Jy,Ty), 1/2[d(Ix,Ty) + d(Jy,Sx)]\})
\end{align*}

for any $x,y \in X$. 

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If one of $S$, $T$, $I$ and $J$ is continuous and if $S$ and $T$ weakly commute respectively with $I$ and $J$, then $S$, $T$, $I$ and $J$ have a common fixed point $z$. Further, $z$ is the unique common fixed point of $S$ & $I$ and of $T$ & $J$.

**Remark 1.** Theorem 1, with condition (2.4.8),
 generalize, improve and unify the results of [28, 35, 36, 39, 47, 56, 66, 67, 70, 113, 116, 127, 133, 134, 142, and 144].

**Remark 2.** If $f$ is replaced by a function $\phi \in O(R_+)$ then
Theorem 1, with condition (2.4.8), extends and improves Theorem 1 of Kang and Rhoades [61].

**Remark 3.** If $f$ is replaced by a function $Q : R_+ \to R_+$ satisfying the following conditions:

(2.4.15a) $Q(t)$ is nondecreasing,

(2.4.15b) $0 < Q(t) < t$ for each $t > 0$ and $Q(0) = 0$,

(2.4.15c) $g(t) = t/(t-Q(t))$ is non-increasing on $(0, \infty)$

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\[ (2.4.15d) \int_{0}^{t_1} g(t) \, dt < \infty \text{ for each } t_1 > 0. \]

Then Theorem 1 improves and generalizes the result of Carbone, Rhoades and Singh [17].

**Remark 4.** Theorem 1, with the following condition,

\[ (2.4.16) \quad d(A_1x, B_1y) \leq \{a_1 \cdot d(Sx, Ty) + a_2 \cdot d(Sx, A_1 x) + a_3 \cdot d(Tx, B_1y) + a_4 \cdot d(Sx, B_1y) + a_5 \cdot d(Ty, A_1 x) \} \]

for all \( x, y \) in \( X \), where \( a_h = a_h(x, y) \), \( h = 1, \ldots, 5 \), are non-negative functions from \( X^2 \) into \( \mathbb{R}_+ \) such that,

\[
\sup_{x, y \in X} (a_1 + a_2 + a_3 + 2a_4 + 2a_5) < 1.
\]

generalizes, improves and unifies the results of Hardy and Rogers [49], Kaneko [59], Mukherjee [76], Naidu and Praisad [80] and also extends and improves the results of Fisher [35] and Hadzic [47].

\[ \star \star \star \]