CHAPTER 7

COMMON FIXED POINT THEOREMS FOR D-COMPATIBLE FAMILY OF FIRMLY NONEXPANSIVE MAPPINGS

(153-160)
CHAPTER-7

COMMON FIXED POINT THEOREMS FOR D-COMPATIBLE FAMILY
OF FIRMLY NONEXPANSIVE MAPPINGS

7.1 In this chapter we prove a common fixed point theorem for a D-compatible family of firmly nonexpansive mappings. Our result extends the results of Hong and Huang [51] and Smarzewski [137].

7.2 The Banach contraction principle has proved to be one of the most durable and fruitful methods in analysis. Lying just at the boundary of the Banach contractions is the set of nonexpansive mappings. The nonexpansive mappings are those which have Lipschitz Constant equal to one. The theory of nonexpansive mappings is fundamentally different from that of contraction mappings. It is important to note that nonexpansive mapping may be fixed point free and even
if a nonexpansive mapping has a nonempty set of fixed points, the Picard iteration may fail to converge. Also a fixed point set need not contain just one point.

Very little of interest can be said about the fixed point theory of nonexpansive mappings within the general metric space framework. Because of this, authors confine their attention to a Banach space setting (certainly a setting sufficiently general to include interesting applications).

Let $X$ be a Banach space with norm $\| \cdot \|$ and let $K$ denote a nonempty closed convex and bounded subset of $X$. In this context a mapping $T: K \to K$ is said to be nonexpansive if, $\| Tx - Ty \| \leq \| x - y \|$, $x, y \in K$.

A discussion of the role of nonexpansive mappings in nonlinear functional analysis and their relation to other important classes of mappings may be found in Browder [15].
Let $T:K \to K$ be any mapping and $x,y \in K$.

Consider the function $\phi_{x,y}$ defined by,

$$
\phi_{x,y}(\lambda) = \| (1-\lambda) (x-y) + \lambda (Tx-Ty) \|, \quad \lambda \in [0,1].
$$

Obviously, $\phi_{x,y}$ is a convex function of $\lambda$.

**DEFINITION 1** [43]. With $K$ as above, a mapping $T:K \to K$ is said to be firmly nonexpansive if for any $x,y \in K$, the function $\phi_{x,y}$ is nonincreasing on $[0,1]$.

Although any firmly nonexpansive mapping must be nonexpansive ($\phi_{x,y}(0) \leq \phi_{x,y}(1)$), the converse need not to be true (consider the mapping $Tx = -x$ in any space $X$). Also, since $\phi_{x,y}$ is convex, it is easy to check that a mapping $T:K \to K$ is firmly nonexpansive if and only if

$$
\|Tx-Ty\| \leq \| (1-\lambda) (x-y) + \lambda (Tx-Ty) \|
$$

for all $x,y \in K, \lambda \in [0,1]$. 

155
In view of the above, one might expect firmly nonexpansive mappings to exhibit better behavior than nonexpansive mappings in general. However, from the point of view of fixed point theory, the restriction is mild.

Goebel and Kirk [43, Theorem 11.3] have proved that one may study the structure of fixed point sets of nonexpansive mappings by restricting one's attention to only those sets which are also fixed point sets of firmly nonexpansive mappings.

In [137], Smarzewski has proved the following result:

**Theorem S:** Let $X$ be a uniformly convex Banach space. Let $C = \bigcup_{k=1}^{n} C_k$ be a union of nonempty, bounded, closed convex subsets $C_k$ of $X$ and suppose $T : C \rightarrow C$ is firmly nonexpansive for some $\lambda \in (0,1)$. Then $T$ has a fixed point in $C$.

If $C_1 = C_2 = \cdots = C_n = C$, then theorem

156
above is true for each nonexpansive mappings $T: C \to C$

which is the well known fixed point result of Browder [14], Gohde [45] and Kirk [69].

Recently Hong and Huang [51] have generalized the result of Smarzewski [137] for any weakly commuting family of firmly nonexpansive self mappings.

In the present chapter we extend above results for any D-compatible family of firmly nonexpansive self mappings.

7.3 As discussed in the preceding chapters Sessa [113] weakened the concept of commutativity by defining weak commutativity which was further weakened by Jungck [55] who defined compatibility. Recently Sharma and Sahu [121] have introduced the concept of D-compatibility and observed that, for a pair of mappings,
This is illustrated by examples 1 and 2. Before that we have to give the following definitions:

**DEFINITION 2** [12]. Let $A$ and $B$ be mappings from $X$ into itself. Then $A$ and $B$ are said to be weakly commuting if

$$\|ABx - BAx\| = \|Ax - Bx\| \quad \text{for all } x \in X.$$ 

**DEFINITION 3** [14]. Let $A$ and $B$ be mappings from a Banach space $X$ into itself, then $A$ and $B$ are said to be compatible if

$$\|ABx_n - BAx_n\| \to 0$$

whenever $\{x_n\}$ is a sequence in $X$ such that

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = t$$

for some $t$ in $X$.

The concept of $D$-compatible mappings was
introduced by Sharma and Sahu [121]. Here we give its analogous definition in Banach space:

**DEFINITION 4.** Let $T_i$ and $T_j$ be mappings from a Banach space $X$ into itself, then the mappings $T_i$ and $T_j$ are said to be semi-compatible if they satisfy the following condition:

$$T_i p = T_j p \quad \text{for some } p \in X \text{ implies }$$

$$T_i T_j p = T_j T_i p,$$

**Example 1.** Let $X = [1, \infty)$ be endowed with the norm,

$$\| x - y \| = | x - y |$$

Let $T_i, T_j : X \to X, (i \neq j)$ defined by

$$T_i = 2x - 1 \text{ and } T_j = x^2 \quad \text{for all } x \in X.$$  

Clearly then $T_i$ and $T_j$ are compatible and D-compatible but they are not weakly commuting. Thus,

Weak commutativity $\Rightarrow$ Compatibility $\Rightarrow$ D-compatibility
Example 2. Let \( x = [0,1] \) be endowed with

\[ \| x-y \| = | x-y | \]

Let \( T_i, T_j : X \to X \) (\( i \neq j \)), defined by,

\[
T_i(x) = \begin{cases} 
  x & \text{if } x \in [0, 1/4) \\
  1 & \text{if } x \in [1/4, 1] 
\end{cases}
\]

\[
T_i(x) = \begin{cases} 
  1-3x & \text{if } x \in [0, 1/4) \\
  1 & \text{if } x \in [1/4, 1] 
\end{cases}
\]

Then clearly \( T_i \) and \( T_j \) are D-compatible but they are neither weakly commuting nor compatible. Thus

\[
\text{D-compatibility } \not\Rightarrow \begin{cases} 
  \text{weak commutativity} \\
  \text{or} \\
  \text{compatibility}
\end{cases}
\]

We need following Lemmas to prove our main result:

Lemma 1 [51]. Suppose \( C = \bigcup_{k=1}^{n} C_k \) is a finite union of nonempty, disjoint, bounded, closed convex subsets \( C_k \) of a Banach space \( X \), and suppose \( T : C \to C \) is
nonexpansive. For $k \in \{1,2,\ldots,n\}$ if there is an $x \in C_k$ such that $Tx \in C_r$ for some $r \in \{1,2,\ldots,n\}$, then $T(C_k) \subseteq C_r$.

**Lemma 2 [51].** Suppose $C = \bigcup_{k=1}^{n} C_k$ is a finite union of nonempty, disjoint, bounded, closed convex subsets $C_k$ of a uniformly convex Banach space $X$ and suppose $T:C \to C$ is firmly nonexpansive for some $\lambda \in (0,1)$. Then there exists some $k \in \{1,2,\ldots,n\}$ such that $T(C_k) \subseteq C_k$.

Let $X$ be a uniformly convex Banach space, $\{x_n\}$ be a bounded sequence in $X$, and $C$ be a closed convex subset of $X$, let us consider then the functional $r : X \to [0,\infty)$ defined by

$$r(x) = \limsup_{n \to \infty} \|x_n - x\|, \quad x \in X.$$ 

The infimum of $r(\cdot)$ over $C$ is called the radius of $\{x_n\}$ with respect to $C$ and is defined by $r(C, \{x_n\})$. A point $z$ in $C$ is called an asymptotic centre of the sequence $\{x_n\}$ with respect to $C$ if
\[ r(z) = \min \{r(x) : x \in C\}. \]

The set of all asymptotic centres is denoted by \( A(C,\{x_n\}) \).

**Lemma 3** [44]. Every bounded sequence \( \{x_n\} \) in a uniformly convex Banach space \( X \) has a unique asymptotic centre with respect to any closed convex subset \( C \) of \( X \).

### 7.4 MAIN RESULTS

Our main result of this chapter reads as follows:

**Theorem 1** [125]. Let \( C = \bigcup_{k=1}^{n} C_k \) be a finite union of nonempty, disjoint, bounded, closed convex subsets \( C_k \) of a uniformly convex Banach space \( X \), and let \( \{T_i\}_{i \in I} \) be any \( D \)-compatible family of firmly nonexpansive self-mappings on \( C \). If there is an \( x \) in \( C \) such that its orbit under \( \{T_i\}_{i \in I} \) is singleton then \( T_i', i \in I \) have a common fixed point in \( C \).

---

Proof: Let \( A = \{ x \in C : T_i x = T_j x, \text{ for all } i, j \in 0 \} \), then by assumption \( A \neq \emptyset \). For any \( x \) in \( A \) and for any \( T_i, T_j \), since \( T_i, T_j \) are \( D \)-compatible, we have

\[
T_i^2 x = T_i (T_j x) = T_j (T_i x) = T_j^2 x
\]

By induction we obtain that

\[(7.4.1)\quad T_i^q x = T_j^q x\]

for any \( i, j \) in 0 and for any \( q \) in \( N \).

Furthermore, note that we have either

\[(7.4.2)\quad T_j^s (C_v) \not\subseteq C_u \text{ for any } s \in N \text{ and for any } u \in \{1,2,\ldots,p\} \text{ and for any } v \in \{p+1,p+2,\ldots,n\}\]

or

\[(7.4.3)\quad \text{There exists } s \in N, u \in \{1,2,\ldots,p\} \text{ and } v \in \{p+1,p+2,\ldots,n\} \text{ so that } T_j^s (C_v) \subseteq C_u .\]

To proceed further, we assume that \( T_j (C_{p+r}) \subseteq C_{p+r+1} \) for \( r = 1,2,\ldots,(n-p) \), where \( C_{n+1} \) denotes \( C_{p+1} \).
We first show that case (7.4.2) is impossible. Since \( T_j : \bigcup_{i=p+1}^{n} C_i \to \bigcup_{i=p+1}^{n} C_i \) from Lemma 2 and the Browder, Gohde and Kirk fixed point theorem it follows that there is \( y \) in \( \bigcup_{i=p+1}^{n} C_i \) so that \( T_j y = y \).

Choose \( q \in \{ p+1, p+2, \ldots, n \} \) so that \( y \in C_q \). Then Lemma 1 gives us \( T_j(C_q) \subseteq C_q \), which is impossible. Therefore only case (7.4.3) is possible.

Choose \( x \in A \cap C_v \). By (7.4.1) we have \( T_i^q x = T_j^q x \) for any \( i, j \in \mathbb{N} \) and \( q \in \mathbb{N} \). In particular the point \( z = T_i^s x = T_j^s x \in C_u \).

Moreover,

\[
T_j(z) = T_j(T_i^s x) = T_j(T_j^s x) = T_{j}^{s+1} x = T_i^{s+1} x = T_i(T_i^s x) = T_i(T_j^s x) = T_i(z).
\]

So we see that \( z \in A \cap C_u \), which again contradicts the assumption \( A \cap C_u = \emptyset \). Therefore we have shown that there is a \( T_j \)-invariant subset such that \( A \cap C_k = \emptyset \).
For $C_k$ as defined above, choose $x \in A \cap C_u$. Then for any $i \in I$, since $T_i x = T_j x \in C_k$, we see that $T_i : C_k \to C_k$. Let $x_q = T_j^q x$ ( $= T_i^q x$ for all $i \in I$) ($q \in \mathbb{N}$), then by Lemma 3 the sequence $\{x_q\}$ has a unique asymptotic centre $\zeta$ in $C_k$, i.e.

$$f(\zeta) = \inf_{y \in C_k} f(y), \text{ where } f(y) = \lim_{q \to \infty} \|y - x_q\|.$$  

But then, since $T_i$ is nonexpansive, we have

$$\|T_i \zeta - x_{q+1}\| = \|T_i \zeta - T_i^q x\| \\
\leq \|\zeta - T_i^q x\| \\
= \|\zeta - x_q\|.$$  

Thus $f(T_i \zeta) \leq f(\zeta)$. The uniqueness of asymptotic centre yields $T_i \zeta = \zeta$ for all $i \in I$.

This completes the proof.

If we take $\{T_i\}_{i \in I}$ as any weakly commuting family of firmly nonexpansive self mappings instead of $D$-compatible family of firmly nonexpansive self mappings we get the following result from Theorem 1:

165
COROLLARY 1 [51, THEOREM 2.4]. Let $C = \bigcup_{k=1}^{n} C_k$ be a finite union of nonempty, disjoint, bounded, closed convex subset $C_k$ of a uniformly convex Banach space $X$, and let $\{T_i\}_{i \in \mathbb{N}}$ be any weakly commutative family of firmly nonexpansive self-mappings on $C$, then if there is an $x$ in $C$ such that its orbit under $\{T_i\}_{i \in \mathbb{N}}$ is a singleton, $T_i$, $i \in \mathbb{N}$ have a common fixed point in $C$.

REMARK 1. For fixed $z \in C$, let $X_i$ be the unique asymptotic centre of the sequence $\{T^mz\}$ with respect to $C_i$, $1 \leq i \leq n$; Let $C'$ be the collection of all asymptotic centres of the same sequence $\{T^mz\}$ with respect to $C$; Obviously then $C'$ is a nonempty subset of $\{x_1, x_2, \ldots, x_n\}$ and is $T$ - invariant. For definiteness, let $C' = \{x_1, x_2, \ldots, x_p\}$, where $p \leq n$. Putting $C'_i = \{x_i\}$ and applying THEOREM 1 to $C' = \bigcup_{i=1}^{p} C'_i$ we get the result of Smarzewski [137, THEOREM 1].

* * *