CHAPTER 5

COINCIDENCE AND COMMON FIXED POINT THEOREMS FOR 2-COMPATIBLE MAPPINGS IN SAKS SPACES

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CHAPTER-5

COINCIDENCE AND COMMON FIXED POINT THEOREMS FOR

2-COMPATIBLE MAPPINGS IN SAKS SPACES

5.1 The first coincidence theorem was proved by Goebel [42] in 1968. After a gap of about thirteen years, Okada [81], Singh and Virendra [135] and Kulshestra [71] extended Goebel’s coincidence theorem to L-Space, 2-Metric Space and Metric Spaces.

It was first proved by Jungck [54] and improved by Singh [128], that a pair of commuting mappings S and T of a metric space into itself satisfying some conditions has a unique common fixed point. This result in some circles is called Jungck Contraction Principle (JCP). After Jungck’s Theorem, the contraction mapping theorem goton spurt and subsequently a multitude of fixed point theorems for a
pair of commuting mappings, a triplicate of mappings and a quadruplet of mappings were established (for example see \[18, 23, 24, 37, 64, 74, 78, 79, 82, 83, 119, 120, 128, 132, 133\] and 145].

In 1984, Cho and Singh [23], introduced the notion of Jungck Contraction Condition [54] to a pair of self mappings on a Saks space and proved some fixed point theorems for a commuting pair of mappings. They also obtained other extensions of certain well known fixed point theorems. In [24] they have further generalized above results for triplicate of mappings.

Some recent generalizations of these theorems were obtained by Murthy and Sharma [78], Sharma and Sahu [119] and Murthy, Sharma and Cho [79].

5.2 As discussed in the preceding chapters the condition of commutativity was replaced by weak commutativity defined by Sessa [113], and then replaced
by compatibility introduced by Jungck [55].

It is important to note that commutativity implies weak commutativity and weak commutativity implies compatibility but the converse is not true.

Now one can ask, whether this compatible condition can be replaced by any non compatible condition? This was answered in the affirmative by Sharma and Sahu [120], who defined 2-Compatibility. Motivated by this, this chapter deals with introductory concepts and properties of 2-compatible mappings and then proves coincidence and fixed point theorem for 2-compatible mappings in Saks space.

5.3 BASIC DEFINITIONS

Following [5, 23, 24, 42, 54, 79, 81 and 131], we define Saks space as below:

DEFINITION 1. Let X be a linear space. A real-valued function N defined on X will be called a B-norm if it
satisfies the following conditions:

\[(5.3.1) \quad N(x) = 0 \text{ if and only if } x = 0,\]

\[(5.3.2) \quad N(x + y) \leq N(x) + N(y),\]

\[(5.3.3) \quad N(ax) = |a| \cdot N(x), \text{ where } a \text{ is any real number.}\]

**DEFINITION 2.** Each real-valued function N satisfying the conditions (5.3.1), (5.3.2) and the following:

\[(5.3.4) \quad \text{If the sequence } \{a_n\} \text{ of real numbers converges to a real number } a \text{ and } N(x_n - x) \to 0 \text{ as } n \to \infty,\]

then \( N(a_n x_n - ax_n) \to 0 \text{ as } n \to \infty, \) will be called an F-norm.

A Two-norm Space is a Linear Space X with two norms, a B-norm \( N_1, \) and a F-norm \( N_2 \) are defined on X and \( x_n \in X, N_1(x_n) \to 0 \text{ as } n \to \infty \) implies \( N_2(x_n) \to 0, \) then the norm \( N_1 \) is called non-weaker than \( N_2 \) in X (denoted by \( N_2 \preceq N_1 \)). If \( N_2 \preceq N_1 \) and \( N_1 \preceq N_2 \), then the norms \( N_1 \) and \( N_2 \) are said to be equivalent.
**DEFINITION 3.** A sequence \(\{x_n\}\) in a two-norm space \((X,N_1,N_2)\) is said to be \(\gamma\)-convergent to a point \(x\) in \(X\) and denoted by \(x_n \xrightarrow{\gamma} x\) if,

\[
\sup_{n \to \infty} N_1(x_n) < \infty \quad \text{and} \quad \lim_{n \to \infty} N_2(x_n - x) = 0.
\]

**DEFINITION 4.** A sequence \(\{x_n\}\) in a two-norm space \((X,N_1,N_2)\) is said to be \(\gamma\)-Cauchy if \(N_2(x_{p_n} - x_{q_n}) \to 0\) as \(p_n,q_n \to \infty\).

**DEFINITION 5.** A Two-norm Space \((X,N_1,N_2)\) is called \(\gamma\)-complete if for every \(\gamma\)-Cauchy sequence \(\{x_n\}\), there exists a point \(x\) in \(X\) such that \(x_n \xrightarrow{\gamma} x\).

**DEFINITION 6.** Let \(X\) be a linear set and suppose that \(N_1\) and \(N_2\) are a \(B\)-norm and a \(F\)-norm on \(X\), respectively.

Let \(X_s = \{x \in X : N_1(x) < 1\}\) and define \(d(x,y) = N_2(x-y)\) for all \(x,y\) in \(X_s\). Then \(d\) is a metric in \(X_s\) and the metric space \((X_s,d)\) will be called a Saks Set. If \((X_s,d)\) is complete then it will be called a Saks Space and will be denoted by \((X,N_1,N_2)\).
Now we recall the definition of compatible mappings on Saks space as follows.

**DEFINITION 7.** Let \((X, N_1, N_2) = (X_s, d)\) be a Saks space and \(N_1\) is equivalent to \(N_2\) on \(X\). Let \(S\) and \(T\) be self mappings of \(X\), then the pair \(\{S, T\}\) is called compatible if

\[
\lim_{n \to \infty} N_2(STx_n - TSx_n) \to 0,
\]

whenever \(\{x_n\}\) is a sequence in \(X\) such that

\[
\lim_{n \to \infty} Sx_n = t = \lim_{n \to \infty} Tx_n.
\]

Now, we define 2-compatible mappings as follows:

**DEFINITION 8.** Let \((X, N_1, N_2) = (X_s, d)\) be a Saks space and \(N_1\) is equivalent to \(N_2\) on \(X\). Let \(S\) and \(T\) be self mappings of \(X\), then the pair \(\{S, T\}\) is called 2-compatible if

\[
\lim_{n \to \infty} N_2(SSx_n - TTx_n) \to 0,
\]

whenever \(\{x_n\}\) is a sequence in \(X\) such that

\[
\lim_{n \to \infty} Sx_n = t = \lim_{n \to \infty} Tx_n \quad \text{for some } t \text{ in } X.
\]
It can be observed that 2-compatibility does not imply compatibility and vice versa. For an illustration, we consider the following examples:

**EXAMPLE 1.** Let \( X = [0,1] \) and the norm \( N_2 \) defined by \( N_2(x) = |x| \) for all \( x \in X \). Define \( T, S : [0,1] \to [0,1] \) by

\[
T_x = 1 - x \text{ for all } x \in [0,1] \\
S_x = \begin{cases} 
 x & \text{if } x \in [0,1/2) \\
 1 & \text{if } x \in [1/2,1] 
\end{cases}
\]

Then \( S \) is not continuous at \( x = 1/2 \).

If we consider a sequence \( \{x_n\} \) in \( X \) which converges to \( 1/2 \) and \( x_n < 1/2 \) for all \( n \in \mathbb{N} \). Then

\[ TT(x_n) = T(1-x_n) = x_n, \text{ and } SS(x_n) = S(x_n) = x_n. \]

Clearly then

\[ \lim_{n \to \infty} N_2(TT(x_n) - SS(x_n)) = 0 \text{ as } \lim_{n \to \infty} x_n \to 1/2. \]

Hence \( T \) and \( S \) are 2-compatible mappings.
But, since \( TS(x_n) = T(x_n) = 1 - x_n \),

and

\[ ST(x_n) = S(1 - x_n) = 1 \text{ for } 1 - x_n < 1/2. \]

Hence,

\[ \lim_{n \to \infty} N_2(TS(x_n) - ST(x_n)) \neq 0. \]

Thus 2-compatibility does not imply compatibility.

It is easy to see that for two continuous mappings,

\[ \text{compatibility } \leftrightarrow \text{ 2-compatibility}. \]

We have then,

**PROPOSITION 1.** Let \( S, T : (X, d) \to (X, d) \) be mappings.

If \( S \) and \( T \) be 2-compatible mappings and \( S(x_n), T(x_n) \to t \)

for some \( t \in X \). Then

\[ \lim_{n \to \infty} TT(x_n) = St, \quad \text{if } S \text{ is continuous.} \]

**PROOF.** Since \( S \) and \( T \) are 2-compatible pair of mappings

and \( S \) is continuous we have

\[ \lim_{n \to \infty} N_2(TT(x_n) - SS(x_n)) = 0 \]

and hence

\[ \lim_{n \to \infty} TT(x_n) = St \quad \text{since } S \text{ is continuous.} \]

This completes the proof.  

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PROPOSITION 2. Let $S, T : (X, d) \to (X, d)$ be mappings.

If $S$ and $T$ are 2-compatible and $S(t) = T(t)$ for some $t \in X$, then $TS(t) = ST(t)$.

PROOF. Suppose that $\{x_n\}$ is a sequence in $X$ defined by $x_n = t$, $n = 1, 2, \ldots$, and $S(t) = T(t)$. Since $S$ and $T$ are 2-compatible pair, we have,

$$\lim_{n \to \infty} N_2(TT(x_n) - SS(x_n)) = 0,$$

and therefore $TT(t) = SS(t)$.

However $S(t) = T(t)$, then $TT(t) = TS(t)$ and $SS(t) = ST(t)$ and therefore $TS(t) = ST(t)$.

This completes the proof.

In [83], Orlicz has proved the following Lemma:

**Lemma 1.** Let $(X, d) = (X, N_1, N_2)$ be a Saks space. Then the following statements are equivalent.

(5.3.5) $N_1$ is equivalent to $N_2$ on $X$.

(5.3.6) $(X, N_1)$ is a Banach space and $N_1 \leq N_2$ on $X$. 

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(5.3.7) \((X, N_2)\) is a Frechet space and \(N_2 \leq N_1\) on \(X\).

For more details about Saks space please see [5, 82, 83 and 84].

5.4 Let \(\mathbb{N}\) and \(\mathbb{R}^+\) be the sets of all natural numbers and non-negative real numbers, respectively, and \(\Phi\) be the family of mappings \(\phi\) from \((\mathbb{R}^+)^9\) to \(\mathbb{R}^+\) such that \(\phi\) is upper semi-continuous, nondecreasing in each variable and for any \(t > 0\).

\[
\phi(t, t, t, t, 0, \alpha \cdot t, \alpha \cdot t, 0) < \beta \cdot t
\]

and

\[
\phi(t, t, t, t, \alpha \cdot t, 0, 0, 0, \alpha \cdot t) = \beta \cdot t
\]

where \(\beta = 1\) for \(\alpha = 2\) and \(\beta < 1\) for \(\alpha < 2\).

\[
\gamma(t) = \phi(t, t, t, t, a_1 \cdot t, a_2 \cdot t, a_3 \cdot t, a_4 \cdot t, a_5 \cdot t) < t.
\]

where \(\gamma : \mathbb{R}^+ \to \mathbb{R}^+\) and \(a_1 + a_2 + a_3 + a_4 + a_5 = 7\).

We need the following Lemma to prove our main results:

**Lemma 2 [74].** For any \(t > 0\), \(\gamma(t) < t\) if and only if \(\lim_{n \to \infty} \gamma^n(t) = 0\), where \(\gamma^n\) denotes the \(n\)-times
composition of \( r \). Let \((X, d) = (X, N_1, N_2)\) be a Saks space and \( N_1 \) be equivalent to \( N_2 \) on \( X \). Let \( A, B, S \) and \( T \) be self mappings of \( X \) satisfying the following conditions:

\[(5.4.1)\]
\[A(X) \cup B(X) \subset S(X) \cap T(X)\]

\[(5.4.2)\]
\[N_2^2(Ax - By) \leq \phi (N_2^2(Sx - Ty),\]
\[N_2^2(Sx - Ax) \cdot N_2^2(Ty - By), N_2(Sx - Ty) \cdot N_2(Sx - Ax),\]
\[N_2(Sx - Ty) \cdot N_2(Ty - By), N_2(Sx - Ty) \cdot N_2(Sx - By),\]
\[N_2(Sx - Ty) \cdot N_2(Ty - Ax), N_2(Sx - By) \cdot N_2(Ty - Ax),\]
\[N_2(Sx - Ax) \cdot N_2(Ty - Ax), N_2(Sx - By) \cdot N_2(Ty - By))\]

for all \( x, y \) in \( X \), where \( \phi \in \Phi \).

Thus by \((5.4.1)\), since \( A(X) \subset T(X) \) for any arbitrary point \( x_0 \in X \), there exists a point \( x_1 \in X \) such that \( Ax_0 = Tx_1 \), since \( B(X) \subset S(X) \), for this point \( x_1 \) we can choose a point \( x_2 \in X \) such that \( Bx_1 = Sx_2 \) and so on. Inductively we can define a sequence \( \{y_n\} \) in \( X \) such that:
\begin{align}
\begin{cases}
y_{2n} = T_{2n+1} = A_{2n} \\
y_{2n+1} = S_{2n+2} = B_{2n+1}
\end{cases}
\end{align}

for every \( n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \)

Let \( A, B, S \) and \( T \) be mappings from \((X,d)\) into itself

satisfying the conditions (5.4.1) and (5.4.2), then we have the following Lemmas:

**Lemma 3.** \( \lim_{n \to \infty} N_2(y_n - y_{n+1}) = 0 \), where \( \{y_n\} \) is a sequence in \( X \) defined by (5.4.3).

**Proof.** Let \( C_n = N_2(y_n - y_{n+1}) \), \( n = 0, 1, 2, \ldots \). Now, we shall show that the sequence \( \{C_n\} \) is nondecreasing in \( \mathbb{R}^+ \), that is \( C_n \leq C_{n-1} \), \( n = 1, 2, 3, \ldots \).

By (5.4.2), we have

\[
C_{2n}^2 = N_2^2(y_{2n} - y_{2n+1})
\]

\[
= N_2^2(A_{2n} - B_{2n+1})
\]

\[
\leq \phi \left( N_2^2(S_{2n} - T_{2n+1}) \right),
\]

\[
N_2(S_{2n} - A_{2n}) \cdot N_2(T_{2n+1} - B_{2n+1})
\]

\[
N_2(S_{2n} - T_{2n+1}) \cdot N_2(S_{2n} - A_{2n}),
\]

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$$N_2(Sx_{2n} - Tx_{2n+1}) \cdot N_2(Tx_{2n+1} - Bx_{2n+1}),$$

$$N_2(Sx_{2n} - Tx_{2n+1}) \cdot N_2(Sx_{2n} - Bx_{2n+1}),$$

$$N_2(Sx_{2n} - Tx_{2n+1}) \cdot N_2(Tx_{2n+1} - Ax_{2n}),$$

$$N_2(Sx_{2n} - Bx_{2n+1}) \cdot N_2(Tx_{2n+1} - Ax_{2n}),$$

$$N_2(Sx_{2n} - Ax_{2n}) \cdot N_2(Tx_{2n+1} - Ax_{2n}),$$

$$N_2(Sx_{2n} - Bx_{2n+1}) \cdot N_2(Tx_{2n+1} - Bx_{2n+1}),$$

$$\phi^2(N_2(Y_{2n-1} - Y_{2n}),$$

$$N_2(Y_{2n-1} - Y_{2n}) \cdot N_2(Y_{2n-1} - Y_{2n+1}),$$

$$N_2(Y_{2n-1} - Y_{2n}) \cdot N_2(Y_{2n-1} - Y_{2n}),$$

$$N_2(Y_{2n-1} - Y_{2n}) \cdot N_2(Y_{2n-1} - Y_{2n+1}),$$

$$N_2(Y_{2n-1} - Y_{2n}) \cdot N_2(Y_{2n-1} - Y_{2n+1}),$$

$$N_2(Y_{2n-1} - Y_{2n}) \cdot N_2(Y_{2n} - Y_{2n}),$$

$$N_2(Y_{2n-1} - Y_{2n}) \cdot N_2(Y_{2n} - Y_{2n}),$$

$$N_2(Y_{2n-1} - Y_{2n}) \cdot N_2(Y_{2n} - Y_{2n}),$$

$$N_2(Y_{2n-1} - Y_{2n+1}) \cdot N_2(Y_{2n} - Y_{2n+1})$$

$$(5.4.4) = \left( \frac{2}{C_{2n-1} \cdot C_{2n-1} \cdot C_{2n}}, \frac{2}{C_{2n} \cdot C_{2n-1} \cdot C_{2n-1} \cdot C_{2n}}, \frac{2}{C_{2n-1} \cdot C_{2n}}, \frac{2}{C_{2n-1} \cdot C_{2n}} \right).$$
Suppose that \( C_{n-1} < C_n \) for some \( n \). Then, for some \( \alpha < 2 \),

\[
C_{n-1} + C_n = \alpha \cdot C_n.
\]

Since \( \phi \) is nondecreasing in each variable and \( \beta < 1 \) for some \( \alpha < 2 \), by (5.4.4)

\[
\gamma^2 \leq \phi(C_{2n}^2, C_{2n'}^2, C_{2n}^2, C_{2n'}^2) \leq \beta \cdot C_{2n}^2 < C_{2n}^2.
\]

Similarly, we have

\[
C_{2n+1}^2 \leq \phi(C_{2n+1}^2, C_{2n+1}^2, C_{2n+1}^2, C_{2n+1}^2) \leq \beta \cdot C_{2n+1}^2 < C_{2n+1}^2.
\]

Hence, for every \( n \),

\[
C_n^2 < \beta \cdot C_n^2 < C_n^2
\]

a contradiction.

Therefore \( \{C_n\} \) is nonincreasing in \( \mathbb{R}^+ \).

Again by (5.4.2), we have

\[
C_1^2 = N_2(y_1 - y_2) = N_2(Ax_2 - Bx_1)
\]

\[
\leq \phi(N_2(Sx_2 - Tx_1), N_2(Sx_2 - Ax_2) \cdot N_2(Tx_1 - Bx_1)
\]

\[
= N_2(Sx_2 - Tx_1) \cdot N_2(Sx_2 - Ax_2)
\]

\[
= N_2(Sx_2 - Tx_1) \cdot N_2(Tx_1 - Bx_1).
\]
In general we have, \( C_n^2 \leq \gamma^n(C_0^2) \), which from Lemma 3 implies that, if \( C_0 > 0 \), then

\[
\lim_{n \to \infty} C_n^2 \leq \lim_{n \to \infty} \gamma^n(C_n^2) = 0.
\]

Therefore, we have \( \lim_{n \to \infty} C_n = 0 \), for \( C_0 = 0 \), since \( \{C_n\} \) is non-increasing, we have clearly \( \lim_{n \to \infty} C_n = 0 \).

which implies \( \lim_{n \to \infty} N_2(y_{2n-1}y_{2n+1}) = 0 \)
also \[ \lim_{n \to \infty} \delta_n(y_{2n} - y_{2n+1}) = 0 \]
by Lemma 1.

This completes the proof.

**Lemma 4.** The sequence \( \{y_n\} \) defined by (5.4.3) is a Cauchy sequence in \( X \).

**Proof.** Since by Lemma 3 \( \lim_{n \to \infty} C_n = 0 \), it is sufficient to prove that a subsequence \( \{y_{2n}\} \) of \( \{y_n\} \) is a Cauchy sequence in \( X \). Suppose that the sequence \( \{y_{2n}\} \) is not a Cauchy sequence in \( X \). Then there exists an \( \varepsilon > 0 \) and strictly increasing sequence \( \{m_k\}, \{n_k\} \) of positive integers such that, \( m_k > n_k \geq k \) satisfying

\[
\begin{align*}
N_{2n}(y_{2n_{k}} - y_{2m_k}) &\geq \varepsilon \\
N_{2n}(y_{2n_{k}} - y_{2m_{k-2}}) &< \varepsilon
\end{align*}
\]

(5.4.5)

for all \( k = 1, 2, \ldots \).

Then for each even integer \( 2k \), we have

(5.4.6) \[ \varepsilon \leq N_{2n}(y_{2n_k} - y_{2m_k}) \leq N_{2n}(y_{2n_k} - y_{2m_{k-2}}) + N_{2n}(y_{2n_k-2} - y_{2m_{k-1}}) \]
Using Lemma 3, it follows that

\[(5.4.7) \quad N_{2n}(y_{2n_k} - y_{2m_k}) \to \varepsilon \quad \text{as} \quad k \to \infty.\]

Again we have

\[|N_{2n}(y_{2n_{k-1}} - y_{2m_{k-1}}) - N_{2n}(y_{2n_k} - y_{2m_k})| \leq N_{2n}(y_{2m_{k-1}} - y_{2m_k})\]

and

\[|N_{2n}(y_{2n_{k+1}} - y_{2m_{k+1}}) - N_{2n}(y_{2n_k} - y_{2m_k})| \leq N_{2n}(y_{2m_k} - y_{2m_{k-1}}) + N_{2n}(y_{2n_k} - y_{2n_{k+1}}).\]

Form Lemma 3 and inequality (5.4.6), we have as \(n \to \infty\)

\[(5.4.8) \begin{cases} 
N_{2n}(y_{2n_k} - y_{2m_{k-1}}) \to \varepsilon \\
N_{2n}(y_{2n_{k+1}} - y_{2m_{k-1}}) \to \varepsilon.
\end{cases}\]

Therefore, by using (5.4.2) and (5.4.3), we have

\[N_{2n}(y_{2n_k} - y_{2m_k}) \leq N_{2n}(y_{2n_k} - y_{2n_{k+1}}) + N_{2n}(y_{2n_{k+1}} - y_{2m_k})\]

\[= N_{2n}(y_{2n_k} - y_{2n_{k+1}}) + N_{2n}(Ax_{2m_k} - Bx_{2n_{k+1}})\]

\[\leq N_{2n}(y_{2n_{k+1}} - y_{2n_{k+1}}) + \phi(N_{2}^{2}(Sx_{2m_k} - Tx_{2n_{k+1}}),\]

\[N_{2n}(Sx_{2m_{k-1}} - Ax_{2m_k}) \cdot N_{2n}(Tx_{2n_{k+1}} - Bx_{2n_{k+1}}),\]

\[N_{2n}(Sx_{2m_{k+1}} - Tx_{2n_k}) \cdot N_{2n}(Sx_{2m_k} - Ax_{2m_k}).\]
Since \( \phi \) is upper semi-continuous, as \( k \to \infty \), by Lemma 3, (5.4.7), (5.4.8) and (5.4.9), we have
\[ e \leq \sqrt{\phi(e^2,0,0,0,e^2,e^2,0,0)} \leq \sqrt{y(e^2)} < e, \]

which is a contradiction.

Therefore, \( \{y_{2n}\} \) is Cauchy sequence in \( X \) and so is \( \{y_n\} \). This completes the proof.

\[ \bullet \]

5.5 COINCIDENCE POINT THEOREMS

**Theorem 1.** Let \((X_s,d) = (X,N_1,N_2)\) be a Saks space and \(N_1\) be equivalent to \(N_2\) on \(X\). Let \(A,B,S\) and \(T\) be self mappings of \(X\) satisfying the conditions (5.4.1), (5.4.2) and the following

(5.5.1) \( S(X) \cap T(X) \) is closed subspace of \(X\) with respect to \(N_1\) then

(i) \( A \) and \(S\) have a coincidence point,

(ii) \( B \) and \(T\) have a coincidence point.

**Proof.** By Lemma 4, the sequence \( \{y_n\} \) defined by (5.4.3) is a Cauchy sequence in \( S(X) \cap T(X) \) with respect to \(N_1\), (since \(N_1\) is equivalent to \(N_2\) on \(X\)).
Again from Lemma 1, \( (X, N_1) \) is a Banach space, since \( S(X) \cap T(X) \) is a closed subspace of \( X \), \( \{y_n\} \) converges to a point \( w \) in \( S(X) \cap T(X) \). On the other hand, since the subsequences \( \{y_{2n}\} \) and \( \{y_{2n+1}\} \) of \( \{y_n\} \) are also Cauchy sequences in \( S(X) \cap T(X) \) with respect to \( N_1 \), they also converges to the same limit \( w \).

Hence there exists two point \( u, v \) in \( X \) such that \( Su = w \) and \( Tv = w \). By (5.4.2), we have

\[
N_2^2(Au - y_{2n+1}) = N_2^2(Au - Bx_{2n+1})
\]

\[
\leq \phi(N_2^2(Su - Tx_{2n+1}))
\]

\[
N_2(Su - Au) \cdot N_2(Tx_{2n+1} - Bx_{2n+1})
\]

\[
N_2(Su - Tx_{2n+1}) \cdot N_2(Su - Au)
\]

\[
N_2(Su - Tx_{2n+1}) \cdot N_2(Tx_{2n+1} - Bx_{2n+1})
\]

\[
N_2(Su - Tx_{2n+1}) \cdot N_2(Su - Bx_{2n+1})
\]

\[
N_2(Su - Tx_{2n+1}) \cdot N_2(Tx_{2n+1} - Au)
\]

\[
N_2(Su - Bx_{2n+1}) \cdot N_2(Tx_{2n+1} - Au)
\]

\[
N_2(Su - Au) \cdot N_2(Tx_{2n+1} - Au)
\]
\[
N_2(Su-Bx_{2n+1}) \cdot N_2(Tx_{2n+1-Bx_{2n+1}}) = \phi(N_2^2(Su-y_{2n}), N_2(Su-Au) \cdot N_2(y_{2n+y_{2n+1}}),
\]
\[
N_2(Su-y_{2n}) \cdot N_2(Su-Au), N_2(Su-y_{2n}) \cdot N_2(y_{2n+y_{2n+1}}),
\]
\[
N_2(Su-y_{2n}) \cdot N_2(Su-y_{2n+1}), N_2(Su-y_{2n}) \cdot N_2(y_{2n-Au}),
\]
\[
N_2(Su-y_{2n+1}) \cdot N_2(y_{2n-Au}), N_2(Su-Au) \cdot N_2(y_{2n-Au}),
\]
\[
N_2(Su-y_{2n+1}) \cdot N_2(y_{2n+y_{2n+1}})
\]

Since \( \lim_{n \to \infty} C_n = \lim_{n \to \infty} N_2(y_{2n+y_{2n+1}}) = 0 \) in the proof of Lemma 3 letting \( n \to \infty \), we have

\[
N_2^2(Au-w) = \phi(0,0,0,0,0,0,N_2^2(w-Au),0),
\]

which is a contradiction. Hence \( Au = w = Su \), i.e., \( u \) is a coincidence point of \( A \) and \( S \). Similarly, we can show that \( v \) is a coincidence point of \( B \) and \( T \).

This completes the proof.

If we define \( \phi : (\mathbb{R}^+)^9 \to \mathbb{R}^+ \) by

\[
\phi(t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8,t_9) = f(\max\{t_1,t_2,t_3,t_4,t_5,t_6/2,t_7,t_8/2,t_9/2\})
\]

where \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) satisfying

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(5.5.2) \( f(t) < t \) for each \( t > 0 \)

(5.5.3) \( f \) is non decreasing

then from inequality (5.4.2) we have the following inequality

(5.5.4) \[ N_2^2 (Ax, By) \leq f \left( \max \left( N_2^2 (Sx_2 -Tx_1) , \right. \right. \]
\[ \left. N_2 (Sx_2 -Ax_2) \cdot N_2 (Tx_1 -Bx_1) \right. \]
\[ \left. N_2 (Sx_2 -Tx_1) \cdot N_2 (Sx_2 -Ax_2) , \right. \]
\[ \left. N_2 (Sx_2 -Tx_1) \cdot N_2 (Tx_1 -Bx_1) , \right. \]
\[ \left. N_2 (Sx_2 -Tx_1) \cdot N_2 (Sx_2 -Bx_1) , \right. \]
\[ 1/2 \left[ N_2 (Sx_2 -Tx_1) \cdot N_2 (Tx_1 -Ax_2) \right] , \]
\[ N_2 (Sx_2 -Bx_1) \cdot N_2 (Tx_1 -Ax_2) , \]
\[ 1/2 \left[ N_2 (Sx_2 -Ax_2) \cdot N_2 (Tx_1 -Ax_2) \right] , \]
\[ 1/2 \left( N_2 (Sx_2 -Bx_1) \cdot N_2 (Tx_1 -Bx_1) \right) \]

for all \( x, y \) in \( X \)

If we define \( \phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) by

\[
\phi (t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9) = \lambda \cdot \max (t_1, t_2, t_3, t_4, t_5, t_6/2, t_7, t_8/2, t_9/2)
\]

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where $k \in (0,1)$, then $\phi \in \Phi$, hence from inequality (5.4.2) we have the following

(5.5.5) \[ N_2^2 (Ax, By) \leq k \cdot \max \{ N_2^2 (Sx_2 - Tx_1), \]

\[ N_2 (Sx_2 - Ax_2) \cdot N_2 (Tx_1 - Bx_1), \]

\[ N_2 (Sx_2 - Tx_1) \cdot N_2 (Sx_2 - Ax_2), \]

\[ N_2 (Sx_2 - Tx_1) \cdot N_2 (Tx_1 - Bx_1), \]

\[ N_2 (Sx_2 - Tx_1) \cdot N_2 (Sx_2 - Bx_1), \]

\[ 1/2 [N_2 (Sx_2 - Tx_1) \cdot N_2 (Tx_1 - Ax_2)], \]

\[ N_2 (Sx_2 - Bx_1) \cdot N_2 (Tx_1 - Ax_2), \]

\[ 1/2 [N_2 (Sx_2 - Ax_2) \cdot N_2 (Tx_1 - Ax_2)], \]

\[ 1/2 [N_2 (Sx_2 - Bx_1) \cdot N_2 (Tx_1 - Bx_1)] \}

If we define $\phi : (\mathbb{R}^+)^9 \to \mathbb{R}^+$ by

$\phi (t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9) = \sum_{i=1}^{9} \alpha_i \cdot t_i$

where each $\alpha_i$ ($i = 1, 2, 3 \cdots$) denotes a non-negative number such that \[ 0 < \sum_{i=1}^{9} \alpha_i + \alpha_6 + \alpha_8 + \alpha_9 < 1 \]

then from inequality (5.4.2) we have the following
\[
N_2^2 (Ax, By) \leq \alpha_1 N_2^2 (Sx_2 -Tx_1) \\
+ \alpha_2 N_2 (Sx_2 -Ax) \cdot N_2 (Tx_1 -Bx_1) \\
+ \alpha_3 N_2 (Sx_2 -Tx_1) \cdot N_2 (Sx_2 -Ax_2) \\
+ \alpha_4 N_2 (Sx_2 -Tx_1) \cdot N_2 (Tx_1 -Bx_1) \\
+ \alpha_5 N_2 (Sx_2 -Tx_1) \cdot N_2 (Sx_2 -Bx_1) \\
+ \alpha_6 N_2 (Sx_2 -Tx_1) \cdot N_2 (Tx_1 -Ax_2) \\
+ \alpha_7 N_2 (Sx_2 -Bx_1) \cdot N_2 (Tx_1 -Ax_2) \\
+ \alpha_8 N_2 (Sx_2 -Ax_2) \cdot N_2 (Tx_1 -Ax_2) \\
+ \alpha_9 N_2 (Sx_2 -Bx_1) \cdot N_2 (Tx_1 -Bx_1)
\]

for all \( x, y \) in \( X \).

In the light of the above inequalities we have following results from Theorem 1:

**Theorem 2.** Let \((X_S, d)\) be as in Theorem 1 and let \(A, B, S\) and \(T\) be self mappings of \((X_S, d)\) satisfying the conditions (5.4.1), (5.5.1) and (5.5.4). Then

(i) \(A\) and \(S\) have a coincidence point,

(ii) \(B\) and \(T\) have a coincidence point.
THEOREM 3. Let \((X_s,d)\) be as in Theorem 1 and let \(A,B,S\) and \(T\) be self mappings of \((X_s,d)\) satisfying the conditions (5.4.1), (5.5.1) and (5.5.5). Then

(i) \(A\) and \(S\) have a coincidence point.

(ii) \(B\) and \(T\) have a coincidence point.

THEOREM 4. Let \((X_s,d)\) be as in Theorem 1 and let \(A,B,S\) and \(T\) be self mappings of \((X_s,d)\) satisfying the conditions (5.4.1), (5.5.1) and (5.5.6). Then

(i) \(A\) and \(S\) have a coincidence point.

(ii) \(B\) and \(T\) have a coincidence point.

As an immediate consequence of Theorem 1 we have the following theorem:

THEOREM 5. Let \((X_s,d)\) be as in Theorem 1 and \(A = B, S\) and \(T\) be self mappings of \((X_s,d)\) in which \(N_1\) is equivalent to \(N_2\) on \(X\), satisfying (5.4.1), (5.4.2) and (5.5.1), then

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(i) A and S have a coincidence point.

(ii) A and T have a coincidence point.

Similarly from Theorem 2 we have the following results:

THEOREM 6. Let \((X, d)\) be as in Theorem 1 and \(A = B, S\) and \(T\) be self mappings of \((X, d)\) in which \(N_1\) is equivalent to \(N_2\) on \(X\), satisfying \((5.4.1), (5.4.2)\) and \((5.5.4)\), then

(i) A and S have a coincidence point,

(ii) A and T have a coincidence point.

Similarly from Theorem 3, we have the following theorem:

THEOREM 7. Let \((X, d)\) be as in Theorem 1 and \(A = B, S\) and \(T\) be self mappings of \((X, d)\) in which \(N_1\) is equivalent to \(N_2\) on \(X\), satisfying \((5.4.1), (5.5.1)\) and \((5.5.5)\), then

(i) A and S have a coincidence point,

(ii) A and T have a coincidence point.
THEOREM 8. Let \((X_s,d)\) be as in Theorem 1 and \(A = B, S\) and \(T\) be self mappings of \((X_s,d)\) in which \(N_1\) is equivalent to \(N_2\) on \(X\), satisfying (5.4.1), (5.5.1) and (5.5.6), then

(i) \(A\) and \(S\) have a coincidence point,

(ii) \(A\) and \(T\) have a coincidence point.

Indeed in above theorems (i.e. Theorem 5, Theorem 6, Theorem 7, and Theorem 8) \(A, S\) and \(T\) have a coincidence point if and only if \(A\) is one to one.

THEOREM 9. Let \((X_s,d)\) be as in Theorem 1 and let \(A, B, S = T\) be self mappings of \((X_s,d)\), satisfying (5.4.1), (5.4.2) and (5.5.1), then

(i) \(A\) and \(S\) have a coincidence point,

(ii) \(B\) and \(S\) have a coincidence point.

Similarly from Theorem 2 we have the following result:

THEOREM 10. Let \((X_s,d)\) be as in Theorem 1 and let \(A, B, S = T\) be self mappings of \((X_s,d)\), satisfying
\[(5.4.1), \ (5.5.1) \text{ and } (5.5.4), \text{ then} \]

(i) \ A \text{ and} \ S \text{ have a coincidence point},

(ii) \ B \text{ and} \ S \text{ have a coincidence point}.

Similarly from Theorem 3, we have the following theorem:

**Theorem 11.** Let \((X_s, d)\) be as in Theorem 1 and let \(A, B, S, T\) be self mappings of \((X_s, d)\), satisfying \(\text{(5.4.1), (5.5.1) and (5.5.5), then} \]

(i) \ A \text{ and} \ S \text{ have a coincidence point},

(ii) \ B \text{ and} \ S \text{ have a coincidence point}.

**Theorem 12.** Let \((X_s, d)\) be as in Theorem 1 and let \(A, B, S, T\) be self mappings of \((X_s, d)\), satisfying \(\text{(5.4.1), (5.5.1) and (5.5.6), then} \]

(i) \ A \text{ and} \ S \text{ have a coincidence point},

(ii) \ B \text{ and} \ S \text{ have a coincidence point}.

**Theorem 13.** Let \((X_s, d)\) be as in Theorem 1 and let \(A = B, S = T\) be self mappings of \((X_s, d)\), satisfying
(5.4.1), (5.4.2) and (5.5.1), then

(i) A and S have a coincidence point.

THEOREM 14. Let \((X_s, d)\) be as in Theorem 1 and let \(A = B\) and \(S = T\) be self mappings of \((X_s, d)\), satisfying

(5.4.1), (5.5.1) and (5.5.4), then

(i) A and S have a coincidence point.

THEOREM 15. Let \((X_s, d)\) be as in Theorem 1 and let \(A = B\) and \(S = T\) be self mappings of \((X_s, d)\), satisfying

(5.4.1), (5.5.1) and (5.5.5), then

(i) A and S have a coincidence point.

THEOREM 16. Let \((X_s, d)\) be as in Theorem 1 and let \(A = B\) and \(S = T\) be self mappings of \((X_s, d)\), satisfying

(5.4.1), (5.5.1) and (5.5.6), then

(i) A and S have a coincidence point.

Remark 1. Theorem 1 includes Murthy, Sharma and Cho [79, Theorem 3.1] if we replace (5.4.2) by the
following condition:

\[(5.5.7) \quad N_2(Ax-By) = \alpha N_2(Sx,Ty) \]
\[
\quad + \beta \cdot \max \{ N_2(Ax-Sy), N_2(By-Ty), \]
\[
\quad 1/2 \cdot [ N_2(Ax-Ty) + N_2(By-Sx) ] \}
\]

for all \( x, y \in X \) where \( \alpha, \beta > 0 \) and \( \alpha + \beta < 1 \).

Remark 2. Theorem 1 includes the result of Cho and Singh [24, Theorem 2] for \( S = T \), if the condition \[(5.4.2) \] is replaced by the following condition

\[(5.5.8) \quad N_2(Ax-By) \leq f(N_2(Sx-Ax), N_2(Sy-By), \]
\[
N_2(Sx-Sy), N_2(Sy-Ax), N_2(Sx-By))
\]

for every \( x, y \) in \( X \), where \( f : (\mathbb{R}^+)^5 \rightarrow \mathbb{R}^+ \), is non-decreasing in each coordinate variable and \( f(t,t,t,\alpha_1 t, \alpha_2 t) < t \), for \( t > 0 \) where \( \alpha_i \in \{0,1,2\} \) with \( \alpha_1 + \alpha_2 = 2 \).

Now we give common fixed point theorems for 2-compatible mappings:
5.6 COMMON FIXED POINT THEOREMS

THEOREM 17. Let \((X_s,d) = (X,N_1,N_2)\) be a Saks space and \(N_1\) be equivalent to \(N_2\) on \(X\). Let \(A,B,S\) and \(T\) be self mappings of \(X\) satisfying the conditions (5.4.1), (5.4.2) , (5.5.1) and (5.6.1) The pairs \(\{A,S\}\) and \(\{B,T\}\) are 2-compatible mappings.

Then \(A,B,S\) and \(T\) have a unique common fixed point in \(X\).

PROOF. By Theorem 1, there exists two points \(u,v\) in \(X\) such that \(Au = Su = w\) and \(Bv = Tv = w\), respectively.

Since \(A\) and \(S\) are 2-compatible , by Proposition 2,

\[ASu = SSu = SAu = AAu,\]

which implies that \(Aw = Sw\).

Also, since \(B\) and \(T\) are 2-compatible , we have \(Bw = Tw\).

Now, we have to prove that \(Aw = w\). Let if possible \(Aw \neq w\), then by (5.4.2) we have,
\[ N_2^2(Aw \cdot Y_{2n+1}) = N_2^2(Aw \cdot X_{2n+1}) \leq \phi(N_2^2(Sw \cdot X_{2n+1})) \]

\[ N_2(Sw \cdot Aw) \cdot N_2(Tx_{2n+1} \cdot Bx_{2n+1}) \]

\[ N_2(Sw \cdot Tx_{2n+1}) \cdot N_2(Sw \cdot Aw) \]

\[ N_2(Sw \cdot Tx_{2n+1}) \cdot N_2(Tx_{2n+1} \cdot Bx_{2n+1}) \]

\[ N_2(Sw \cdot Rx_{2n+1}) \cdot N_2(Sw \cdot Bx_{2n+1}) \]

\[ N_2(Sw \cdot Tx_{2n+1}) \cdot N_2(Tx_{2n+1} \cdot Aw) \]

\[ N_2(Sw \cdot Bx_{2n+1}) \cdot N_2(Tx_{2n+1} \cdot Aw) \]

\[ N_2(Sw \cdot Aw) \cdot N_2(Tx_{2n+1} \cdot Aw) \]

\[ N_2(Sw \cdot Bx_{2n+1}) \cdot N_2(Tx_{2n+1} \cdot Bx_{2n+1}) \]

\[ N_2(Sw \cdot Y_{2n}) \cdot N_2(Y_{2n} \cdot Y_{2n+1}) \]

\[ N_2(Sw \cdot Y_{2n}) \cdot N_2(Sw \cdot Aw) \]

\[ N_2(Sw \cdot Y_{2n}) \cdot N_2(Y_{2n} \cdot Y_{2n+1}) \]

\[ N_2(Sw \cdot Y_{2n}) \cdot N_2(Sw \cdot Y_{2n+1}) \]

\[ N_2(Sw \cdot Y_{2n}) \cdot N_2(Y_{2n} \cdot Aw) \]

\[ N_2(Sw \cdot Y_{2n+1}) \cdot N_2(Y_{2n} \cdot Aw) \]

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\[ N_2(Sw-Aw) \cdot N_2(Y_{2n} - Aw), \]

\[ N_2(Sw - Y_{2n+1}) \cdot N_2(Y_{2n} - Y_{2n+1}). \]

Letting \( n \to \infty \), we have

\[ N_2^2(Aw - w) \leq \phi(N_2^2(Aw - w), 0, 0, 0, N_2^2(Aw - w), N_2^2(Aw - w)), \]

\[ N_2^2(Aw - w), 0, 0) \]

\[ < N_2^2(Aw - w) \]

which is a contradiction. Hence, we have \( Aw = w = Sw \).

Similarly, we have \( Bw = w = Tw \). Which implies that

\( Aw = Bw = w = Sw = Tw \). Therefore, \( w \) is a common fixed point of \( A, B, S \) and \( T \). The uniqueness of the fixed point is obvious.

This completes the proof.

From Theorem 17 we have the following results:

Theorem 18. Let \((X_s, d)\) be as in Theorem 17 and let \(A, B, S\) and \(T\) be self mappings of \((X_s, d)\) satisfying the conditions (5.4.1), (5.5.1), (5.5.4) and (5.6.1), then \(A, B, S\) and \(T\) have a unique common fixed point in \(X\).
THEOREM 19. Let \((X_s, d)\) be as in Theorem 17 and let \(A, B, S\) and \(T\) be self mappings of \((X_s, d)\) satisfying the conditions \((5.4.1), (5.5.1), (5.5.5)\) and \((5.6.1)\), then \(A, B, S\) and \(T\) have a unique common fixed point in \(X\).

THEOREM 20. Let \((X_s, d)\) be as in Theorem 17 and let \(A, B, S\) and \(T\) be self mappings of \((X_s, d)\) satisfying the conditions \((5.4.1), (5.5.1), (5.5.6)\) and \((5.6.1)\), then \(A, B, S\) and \(T\) have a unique common fixed point in \(X\).

THEOREM 21. Let \((X_s, d)\) be as in Theorem 17 and let \(A, B, S\) and \(T\) be self mappings of \((X_s, d)\) satisfying the conditions \((5.4.1), (5.4.2), (5.5.1)\) and \((5.6.1)\), then \(A, B\) and \(S\) have a unique common fixed point in \(X\).

Theorem 22. Let \((X_s, d)\) be as in Theorem 17 and let \(A, B, S\) and \(T\) be self mappings of \((X_s, d)\) satisfying the conditions \((5.4.1), (5.5.1), (5.5.4)\) and \((5.6.1)\), then \(A, B\) and \(S\) have a unique common fixed point in \(X\).
THEOREM 23. Let \((X_s,d)\) be as in Theorem 17 and let 

\(A, B, \text{ and } S = T\) be self mappings of \((X_s,d)\) satisfying the 
conditions \((5.4.1), (5.5.1), (5.5.5)\) and \((5.6.1)\), then 

\(A, B, \text{ and } S\) have a unique common fixed point in \(X\).

THEOREM 24. Let \((X_s,d)\) be as in Theorem 17 and let 

\(A, B, \text{ and } S = T\) be self mappings of \((X_s,d)\) satisfying the 
conditions \((5.4.1), (5.5.1), (5.5.6)\) and \((5.6.1)\), then 

\(A, B, \text{ and } S\) have a unique common fixed point in \(X\).

REMARK 3. Theorem 17 includes the result of Murthy, 
Sharma and Cho [79, Theorem 4.1] if we replace condition 
\((5.4.2)\) by \((5.5.7)\) and \((5.6.1)\) by compatibility of 
type \((\text{A})\).

REMARK 4. Theorem 17 includes the result of Sharma and 
Sahu [119, Theorem 3.1] if we replace condition \((5.4.2)\) 
by the following condition:
\begin{align*}
(5.6.2) \quad N_2(Ax-By) & \leq \phi \left(N_2(Sx-Ty), N_2(Ax-Sx), N_2(By-Ty), \\
& \quad N_2(Ax-Ty), N_2(By-Sx) \right)
\end{align*}

for all \( x, y \) in \( X \), where \( \phi : (\mathbb{R}^+)^5 \to \mathbb{R}^+ \) which is non-decreasing and upper semi-continuous and

\[
\psi(t) = \max \{ \phi(t,0,t,t,t), \phi(t,t,t,2t,0), \\
\quad \phi(t,t,t,0,2t) \} < t \quad \text{for each } t > 0
\]

and (5.6.1) by compatibility of type (A).

REMARK 5. Theorem 17 includes Murthy, Sharma and Cho [79] if condition (5.4.2) is replaced by the following condition:

\begin{align*}
(5.6.3) \quad N_2^2(Ax-By) & \leq \phi \left( \max \{ N_2^2(Sx-Ty), \\
& \quad N_2(Sx-Ax) \cdot N_2(Ty-By), \right) \\
& \quad N_2(Sx-By) \cdot N_2(Ty-Ax), \\
& \quad N_2(Sx-Ax) \cdot N_2(Ty-Ax), \\
& \quad N_2(Sx-By) \cdot N_2(Ty-By) \} }
\end{align*}
nondecreasing and \( \phi(t) < t \) for each \( t > 0 \), and (5.6.1) is replaced by compatibility of type (A).

**REMARK 6.** Theorem 17 includes Cho and Singh [24, Theorem 3] for \( S = T \) and condition (5.4.2) is replaced by (5.5.8).

**REMARK 7.** Theorem 17 includes Cho and Singh [23] for \( A = B \) and \( S = T \) and condition (5.5.8) in place of (5.4.2).

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