CHAPTER 2

TRANSIENT ANALYSIS FOR STATE-DEPENDENT QUEUES WITH CATASTROPHES

2.1 INTRODUCTION

Transient analytical results are directly useful for studying the finite-time properties of queueing systems. It is often impossible and very difficult to obtain the exact time-dependent analysis of the state-dependent queueing systems. In many potential applications, the behavior of the solution in finite time is also important and the analysis becomes intractable in many cases. Even in Markovian queueing systems, if the underlying birth and death rates are complex, then the difficulty is compounded while obtaining the transient solutions. Only a few stochastic systems have closed-form transient solutions for the distributions associated with the process. Morse (1955) obtained the time-dependent system size probabilities of $M/M/1$ queue. Rothkopf and Oren (1979) studied some generalized $M/M/s$ queue with time-dependent arrival and service rates. A large number of articles have been published on queueing systems with catastrophes (Chao and Zheng 2003 and Brockwell 1985 and 1986). Negative customers
or catastrophes occur naturally in many applications in the areas of manufacturing systems and computer communication systems.

It is well known that the steady-state system size probabilities and busy period distributions for the discouraged arrivals queueing model and infinite server queueing model are identical. In this chapter, we consider the discouraged arrivals queueing system and infinite server queueing system incorporating catastrophes and study their transient behavior. Due to complexity of arrival and service rates, we employ continued fraction methodology to obtain closed-form analytical expressions for the transient analysis of the systems under the influence of catastrophes. Section 2.2 discusses the general model for any state-dependent queueing system. Section 2.3 discusses the discouraged arrivals queueing system with catastrophes. Explicit expressions for system size probabilities along with factorial moments are obtained. The Laplace-Stieltjes transform of the busy period along with mean busy period are obtained in terms of confluent hypergeometric functions. Section 2.4 discusses the infinite server queueing system with catastrophes. Explicit expressions for the (transient) system size probabilities are obtained and associated average measures for system size and busy period are provided. The total number of customers served during the busy period is also obtained for both the models. It is interesting to note that though the steady-state system size probabilities are identical for these two queueing models in the absence of catastrophes, when catastrophes are present, the steady-state probabilities differ. In section 2.5, numerical illustrations are provided for both the models. The effects of varying arrival rate and catastrophes on system size probabilities are displayed in graphs.
2.2 MODEL DESCRIPTION

The system under study can be described in the following way. We consider a queueing system with infinite capacity. The arriving customers form a single waiting line based on the order of their arrivals. Customers are served on a first-come-first-served basis. Let \( \{X(t), t \in R^+\} \) be the number of customers in the system at time \( t \). We assume that the arrival and service rates are \( \lambda_n \) and \( \mu_n \), respectively where the number of customers in the system at time \( t \) is \( n \); in any small interval \( (t, t + h) \), an arrival occurs with probability \( \lambda_n h + o(h) \); a service being completed with probability \( \mu_n h + o(h) \). It is obvious that in this interval, neither an arrival nor service takes place with probability \( 1 - (\lambda_n + \mu_n) h + o(h) \).

Apart from arrival and service processes, the catastrophes also occur at the busy server as a Poisson process with rate \( \alpha \), i.e., the catastrophe occurs at the busy server during \( (t, t + h) \) with probability \( \alpha h + o(h) \). Whenever a catastrophe occurs, all the customers in the system are wiped out immediately, the server gets inactivated momentarily and the server is ready for service whenever a new arrival occurs subsequently.

Let \( P_n(t) = P(X(t) = n), \ n = 0, 1, 2, \ldots \), denote the probability that there are \( n \) customers in the system at time \( t \). \( P(s, t) = \sum_{n=0}^{\infty} P_n(t)s^n \), its probability generating function and \( m(t) \), its mean.

Based on the above assumptions, the state probabilities \( P_n(t), n = 0, 1, 2, \ldots \), can be described by the differential-difference
equations governing the system as follows:

\[
\frac{dP_0(t)}{dt} = -\lambda_0 P_0(t) + \mu_1 P_1(t) + \alpha (1 - P_0(t)) \quad (2.1)
\]

\[
\frac{dP_n(t)}{dt} = -(\lambda_n + \mu_n + \alpha) P_n(t) + \lambda_{n-1} P_{n-1}(t) + \mu_{n+1} P_{n+1}(t), \quad n = 1, 2, 3, \ldots \quad (2.2)
\]

Assume that initially there is no customer in the system, so that

\[
P_0(0) = 1. \quad (2.3)
\]

In the following sequel, we denote by \( g^*(s) \) as the Laplace transform of \( g(.) \). Taking Laplace transforms of (2.1) and (2.2) and using (2.3), we get

\[
(s + \lambda_0 + \alpha) P_0^*(s) = 1 + \frac{\alpha}{s} + \mu_1 P_1^*(s) \quad (2.4)
\]

and

\[
(s + \lambda_n + \mu_n + \alpha) P_n^*(s) = \lambda_{n-1} P_{n-1}^*(s) + \mu_{n+1} P_{n+1}^*(s),
\]

\[
\quad n = 1, 2, 3, \ldots \quad (2.5)
\]

After some algebra, (2.4) and (2.5) simplify to

\[
P_0^*(s) = \frac{1 + \frac{\alpha}{s}}{(s + \lambda_0 + \alpha) - \mu_1 P_1^*(s)} \quad (2.6)
\]

and

\[
\frac{P_n^*(s)}{P_{n-1}^*(s)} = \frac{\lambda_{n-1}}{(s + \lambda_n + \mu_n + \alpha) - \mu_{n+1} P_{n+1}^*(s)}, \quad n = 1, 2, 3, \ldots \quad (2.7)
\]

By iteration, from (2.6) and (2.7), we obtain continued fraction (Wall 1948) expressions as

\[
P_0^*(s) = \frac{1 + \frac{\alpha}{s}}{(s + \lambda_0 + \alpha) - \frac{\lambda_0}{s + \lambda_1 + \mu_1 + \alpha} - \frac{\lambda_1}{s + \lambda_2 + \mu_2 + \alpha} - \cdots} \quad (2.8)
\]

\[
\]
and
\[
\frac{P_n^*(s)}{P_{n-1}^*(s)} = \frac{\lambda_{n-1}}{(s + \lambda_n + \mu_n + \alpha)} - \frac{\lambda_n\mu_{n+1}}{(s + \lambda_{n+1} + \mu_{n+1} + \alpha)} - \frac{\lambda_{n+1}\mu_{n+2}}{(s + \lambda_{n+2} + \mu_{n+2} + \alpha)} \ldots, \quad n = 1, 2, 3, \ldots \tag{2.9}
\]

To obtain explicit expressions for the system size probabilities \(P_n(t), n = 0, 1, 2, \ldots\), we consider in the following sections the two queueing systems with appropriate arrival and service rates.

### 2.3 QUEUES WITH DISCOURAGED ARRIVALS

A queuing model in which potential customers are discouraged by the queue length is an example for state-dependent queues (Natvig 1974). This discouraged arrival queueing systems are useful to model a computing facility that is solely dedicated to batch job processing (Ng chee Hock 1996). Job submissions are discouraged when the facility is heavily occupied and are modelled as a Poisson process with state-dependent arrivals. The time taken to process each job is exponentially distributed with a constant service rate, regardless of the number of jobs in the system. For the system under consideration, the arrival and service rates, when the system size is \(n\), are given by

\[
\lambda_n = \frac{\lambda}{n+1}, \quad n = 0, 1, 2, \ldots \tag{2.10a}
\]

\[
\mu_n = \mu, \quad n = 1, 2, 3, \ldots \tag{2.10b}
\]

Using (2.10) in (2.8), one obtains

\[
P_0^*(s) = \frac{1 + \frac{\mu}{\lambda}}{(s + \lambda + \alpha)} - \frac{\lambda\mu}{(s + \frac{\lambda}{2} + \mu + \alpha)} - \frac{\lambda\mu}{(s + \frac{\lambda}{3} + \mu + \alpha)} \ldots \tag{2.11}
\]
Now, using the identity (A.6) of Appendix, the above equation can be expressed as a ratio of hypergeometric functions as follows:

$$P^*_0(s) = \left(1 + \frac{\alpha}{s}\right) \left\{ s + \lambda + \alpha + (s + \alpha + \mu) \right. \right. \\
\times \left. \left. \left[ \frac{s\lambda + \lambda\alpha}{(s + \mu + \alpha)^2} + 1 \right] \frac{1}{\Gamma(1; \frac{s\lambda + \lambda\alpha}{(s + \mu + \alpha)^2} + 1; \frac{-\lambda\mu}{(s + \mu + \alpha)^2})} \right. \right. \\
\left. \left. - \left( \frac{\lambda}{s + \mu + \alpha} + 1 \right) \right]^{-1} \right\}^{-1}$$

so that

$$P^*_0(s) = \left(1 + \frac{\alpha}{s}\right) \left\{ (s + \mu + \alpha) \left( \frac{s\lambda + \lambda\alpha}{(s + \mu + \alpha)^2} + 1 \right) \right. \\
\times \left. \frac{1}{\Gamma(1; \frac{s\lambda + \lambda\alpha}{(s + \mu + \alpha)^2} + 1; \frac{-\lambda\mu}{(s + \mu + \alpha)^2}) - \mu} \right\}. \quad (2.12)$$

Further, using the identity (A.4) and the property $\Gamma(0; c; z) = 1$ in (2.12) and after some algebra, we have

$$P^*_0(s) = \left(1 + \frac{\alpha}{s}\right) \frac{1}{\Gamma(2; \frac{s\lambda + \lambda\alpha}{(s + \mu + \alpha)^2} + 2; \frac{-\lambda\mu}{(s + \mu + \alpha)^2})} \left( s + \alpha \right) \left( \frac{s\lambda + \lambda\alpha}{(s + \mu + \alpha)^2} + 1 \right). \quad (2.13)$$

Substituting (2.10) in (2.9), we get

$$\frac{P^*_n(s)}{P^*_{n-1}(s)} = \frac{\frac{\lambda}{n}}{(s + \frac{\lambda}{n+1} + \mu + \alpha)} - \frac{\frac{\lambda\mu}{n+1}}{(s + \frac{\lambda}{n+2} + \mu + \alpha)} - \frac{\frac{\lambda\mu}{n+2}}{(s + \frac{\lambda}{n+3} + \mu + \alpha)} \ldots n = 1, 2, 3, \ldots. \quad (2.14)$$

As before, using the identity (A.6) in the above equation, we have, for $n = 1, 2, 3, \ldots,$

$$\frac{P^*_n(s)}{P^*_{n-1}(s)} = \frac{\frac{\lambda}{n} + \frac{\lambda(n+1)}{n(n+1)} + \lambda(n+1)}{(s + \frac{\lambda}{n+1} + \mu + \alpha)} \frac{1}{\Gamma(1; \frac{s\lambda + \lambda\alpha}{(s + \mu + \alpha)^2} + n + 2; \frac{-\lambda\mu}{(s + \mu + \alpha)^2})} \left( n + 2 \right) \frac{1}{\Gamma(1; \frac{s\lambda + \lambda\alpha}{(s + \mu + \alpha)^2} + n + 1; \frac{-\lambda\mu}{(s + \mu + \alpha)^2})} \left( n + 1 \right). \quad (2.15)$$
Iterating the above equation, we get

\[
P^*_n(s) = P^*_0(s) \prod_{j=1}^{n} \frac{P^*_j(s)}{P^*_j(s)}
= \left( \frac{s-\lambda}{s-\mu+\alpha} \right)^n \left( \frac{s-\mu+\alpha}{s+\mu+\alpha} \right) + 1 \cdots \left( \frac{s-\lambda+\lambda}{s+\mu+\alpha} \right) + n + 1 \right. 
\times \left( n+2 \right) \gamma(\lambda, s+\mu+\alpha) + n + 2, \quad n = 0, 1, 2, \ldots
\]  

(2.16)

Thus,

\[
P^*_n(s) = \left( 1 + \frac{\alpha}{s} \right) (n+1) \gamma(s+\mu+\alpha) + (s+\mu+\alpha)^2(n+1) 
\times \sum_{k=0}^{\infty} \frac{(n+2)k}{(s+\mu+\alpha)^{n+2k}}
\]  

(2.17)

where \((.)_k\) is the Pochhammer symbol as defined in (A.2).

After some algebra, (2.17) simplifies to

\[
P^*_n(s) = \left( 1 + \frac{\alpha}{s} \right) \sum_{k=0}^{\infty} \frac{(-1)^k s^{n+k+1} \mu^k}{k!} + \frac{(s+\mu+\alpha)^2(n+k+1)(n+k+1)!}{(s+\alpha)^{n+k+1}} \prod_{l=0}^{n+k+1} \left[ (s+\alpha)^2 + l(s+\mu+\alpha)^2 \right].
\]  

(2.18)

Employing the partial fraction expansion in (2.18), we get

\[
P^*_n(s) = \left( 1 + \frac{\alpha}{s} \right) \sum_{k=0}^{\infty} \frac{(-1)^k s^{n+k+1} \mu^k}{k!} \sum_{l=0}^{n+k+1} \frac{(n+k+1)(-1)^l}{l} 
\times \frac{1}{(s+\mu+\alpha)^{n+2k}[(s+\alpha)^2 + l(s+\mu+\alpha)^2]}, \quad n = 0, 1, 2, \ldots
\]  

(2.19)
On inversion, (2.19) yields, for \( n = 0, 1, 2, \ldots \),

\[
P_n(t) = \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{n+k+1}}{k!n!} \mu^k \sum_{l=0}^{n+k+1} \binom{n+k+1}{l} (-1)^l \\
\times \left[ g_{n+2k,l}(t) + \alpha \int_0^t g_{n+2k,l}(u)du \right] \quad (2.20)
\]

where

\[
g_{0,0}(t) = \frac{e^{-\alpha t}}{\lambda} \quad (2.21)
\]

\[
g_{0,1}(t) = \frac{e^{-(\frac{\lambda}{4}+\mu+\alpha)t}}{\sqrt{\frac{\lambda^2}{4} + \lambda \mu}} \sinh \left( \sqrt{\frac{\lambda^2}{4} + \lambda \mu} \right) t \quad (2.22)
\]

\[
g_{n+2k,0}(t) = \frac{e^{-\alpha t}}{\lambda(2k+n-1)!} \int_0^t e^{-\mu y} y^{n+2k-1} dy, \quad \text{for } n+2k > 0 \quad (2.23)
\]

and for \( n+2k, l \geq 1 \),

\[
g_{n+2k,l}(t) = \frac{e^{-(\frac{\lambda}{4}+\mu+\alpha)t}}{l(n+2k-1)!\sqrt{\frac{\lambda^2}{4} + \lambda \mu}} \\
\times \int_0^t e^{\frac{\lambda}{4} y} y^{n+2k-1} \sinh \left( \sqrt{\frac{\lambda^2}{4} + \lambda \mu} \right) (t-y) dy. \quad (2.24)
\]

Thus, from (2.20), we obtain explicit expressions for the transient probabilities of the system size for discouraged arrival queueing systems with catastrophes.

**Remark 1:** If \( \mu = 0 \), then (2.19) becomes

\[
P^*_n(s) = \frac{\lambda^n}{n!} \frac{1}{s(s+\alpha)^n} + \frac{\lambda^{n+1}}{n!} \sum_{l=1}^{n+1} \binom{n+1}{l} (-1)^l \\
\times \frac{1}{(s+\alpha)^n \left\{ (s + \frac{\lambda/l+\alpha}{2})^2 - \left( \frac{\lambda/l+\alpha}{2} \right)^2 \right\}}, \quad n = 0, 1, 2, \ldots \quad (2.25)
\]
On inversion, we get

\[
P_n(t) = \frac{1}{n!} \int_0^t e^{-\alpha u} u^{n-1} \lambda u \, du + \frac{1}{n!} \sum_{l=1}^{n+1} \binom{n+1}{l} (-1)^l \frac{\lambda/l}{(\alpha + \lambda/l)} \left\{ \int_0^t e^{-\alpha u} u^{n-1} \lambda^n u \, du - e^{-\alpha t} \int_0^t u^{n-1} e^{-(\lambda/l)(t-u)} \lambda^n u \, du \right\}.
\]

(2.26)

Thus, for \( \mu = 0 \), the transient probabilities of the system size for the discouraged arrival process with catastrophes are obtained as a special case. Such models occur in immigration-catastrophe process dealt with by Swift (2001) wherein the immigration of insect population and their destruction by insecticides were considered.

### 2.3.1 Factorial Moments

To obtain explicit expressions for the factorial moments of the system size, we define

\[
P^*_A(z, s) = \sum_{n=0}^\infty P^*_n(s) z^n.
\]

(2.27)

Using the identity (A.7), from (2.19) and (2.27), we have

\[
P^*_A(z, s) = \left(1 + \frac{\alpha}{s}\right) \frac{1}{s} F_1 \left(2; \frac{s \lambda + \lambda \alpha}{(s + \mu + \alpha)^2} + 2; \frac{\lambda \alpha}{s + \mu + \alpha} - \frac{\lambda \mu}{(s + \mu + \alpha)^2} \right) \left(\frac{s \lambda + \lambda \alpha}{(s + \mu + \alpha)^2} + 1\right). \tag{2.28}
\]

Let \( M_n(t) \) denote the \( n^{\text{th}} \) factorial moment of \( X(t) \). Differentiating (2.28) \( n \)-times with respect to \( z \) and setting \( z = 1 \), it is easy to see that

\[
M^*_n(s) = \frac{(1 + \frac{\alpha}{s})(n + 1)! \left(\frac{\lambda}{s + \mu + \alpha}\right)^n}{(s + \alpha)(\frac{s \lambda + \lambda \alpha}{(s + \mu + \alpha)^2} + 1) \ldots (\frac{s \lambda + \lambda \alpha}{(s + \mu + \alpha)^2} + n + 1)} \times \frac{1}{s} F_1 \left(n + 2; \frac{s \lambda + \lambda \alpha}{(s + \mu + \alpha)^2} + n + 2; \frac{s \lambda + \lambda \alpha}{(s + \mu + \alpha)^2}\right),
\]

\( n = 1, 2, 3, \ldots \) \( \tag{2.29} \)
The above equation can be expressed as

\[
M_n^*(s) = (1 + \frac{\alpha}{s}) \sum_{k=0}^{\infty} \frac{\lambda^{n+k+1}}{k!} \left[ \frac{s + \alpha}{(s + \mu + \alpha)^2} \right]^k \sum_{l=0}^{n+k+1} \binom{n+k+1}{l} \frac{1}{(s + \mu + \alpha)^n[(s + \alpha)\lambda + l(s + \mu + \alpha)^2]}
\]

which on inversion yields

\[
M_n(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{n+k+1} \binom{n+k+1}{l} (-1)^l \frac{\lambda^{n+k+1}}{k!} \int_0^t h^{(k)}(t-u) \left[ g_{n,l}(u) + \alpha \int_0^u g_{n,l}(v)dv \right] du, \quad n=1,2,3,\ldots
\]

where \( h(t) = e^{-(\alpha+\mu)t} (1-\mu t) \) and \( h^{(k)}(t) \) denotes the \( k \)-fold convolution of the function \( h(t) \) with itself, and the function \( g_{n,l}(t) \) are defined as in (2.21) - (2.24).

In particular, the mean system size \( m(t) \) is obtained as

\[
m(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{k+2} \binom{k+2}{l} (-1)^l \frac{\lambda^{k+2}}{k!} \int_0^t h^{(k)}(t-u) \left[ g_{1,l}(u) + \int_0^u g_{1,l}(v)dv \right] du.
\]

2.3.2 Steady-State Probabilities

The steady-state system size probabilities for the discouraged arrival queue with catastrophes are obtained as follows:

Multiplying equation (2.19) by \( s \) on both sides and taking
limit as $s \to 0$, one has

$$
\lim_{s \to 0} s P_n^s(s) = \lim_{s \to 0} s \left( 1 + \frac{\alpha}{s} \right) \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{n+k+1}}{k!n!} \mu^k \sum_{l=0}^{n+k+1} \binom{n+k+1}{l} \frac{1}{(s + \mu + \alpha)^{n+2k}[\lambda + l(s + \mu + \alpha)^2]},
$$

$n = 0, 1, 2, \ldots$.  

By the application of Tauberian theorem, we get

$$
P_n = \alpha \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{n+k+1}}{k!n!} \mu^k \sum_{l=0}^{n+k+1} \binom{n+k+1}{l} (-1)^l \frac{1}{(\alpha + \mu)^{n+2k}[\alpha \lambda + l(\alpha + \mu)^2]},
$$

$n = 0, 1, 2, \ldots \quad (2.33)$

For $\alpha = 0$, (2.33) reduces to

$$
P_n = e^{-\frac{\lambda}{\mu} \frac{n}{n!}}, \quad n = 0, 1, 2, \ldots, \quad (2.34)
$$

a well known result (Gross and Harris 1998).

### 2.3.3 Busy Period Analysis

The busy period analysis forms an integral part of queueing system as the distribution of its duration is important from the server’s point of view and is also helpful in the efficient planning of the system and resources (Srivastava and Kashyap 1982, and Wolff 1989).

A busy period normally starts with the arrival of a customer to an empty system and ends when the system becomes empty. Description of the state of the system during the busy period is made here by means of ‘zero-avoiding state probabilities’. Let the length of
the busy period, a random variable, be denoted by $T$. Let the arrival of a customer to an empty system starts the busy period and now the system is assumed to be in state one. We will now find the first passage time from state one to state zero (system with no customer). That is, the busy period analysis concerns the first-passage time from state 1 to state 0.

The zero-avoiding state probabilities $Q_n(t)$, $n = 0, 1, 2, \ldots$, satisfy the system of forward Kolmogorov equations given below:

\[
\frac{dQ_0(t)}{dt} = \mu_1 Q_1(t) + \alpha (1 - Q_0(t)) \quad (2.35)
\]

\[
\frac{dQ_1(t)}{dt} = -(\lambda_1 + \mu_1 + \alpha) Q_1(t) + \mu_2 Q_2(t) \quad (2.36)
\]

\[
\frac{dQ_n(t)}{dt} = -(\lambda_n + \mu_n + \alpha) Q_n(t) + \lambda_{n-1} Q_{n-1}(t)
+ \mu_{n+1} Q_{n+1}(t), \quad n = 2, 3, 4, \ldots \quad (2.37)
\]

with the initial conditions

\[
Q_1(0) = 1. \quad (2.38)
\]

Taking Laplace transform of the equations (2.35)–(2.37) and using the initial condition (2.38), after some algebraic manipulation, we have,

\[
Q_0^*(s) = \frac{\mu_1}{s + \alpha} Q_1^*(s) + \frac{\alpha}{s(s + \alpha)} \quad (2.39)
\]

\[
Q_1^*(s) = \frac{1}{(s + \lambda_1 + \mu_1 + \alpha) - \frac{\mu_2 Q_2^*(s)}{Q_1^*(s)}} \quad (2.40)
\]

and

\[
\frac{Q_n^*(s)}{Q_{n-1}^*(s)} = \frac{\lambda_{n-1}}{(s + \lambda_n + \mu_n + \alpha) - \frac{\mu_{n+1} Q_{n+1}^*(s)}{Q_n^*(s)}}, \quad n = 1, 2, 3, \ldots \quad (2.41)
\]
By iteration, from (2.40) and (2.41), we obtain continued fraction expressions as

\[ Q_1^*(s) = \frac{1}{(s + \lambda_1 + \mu_1 + \alpha) - \frac{\lambda_1\mu_2}{(s + \lambda_2 + \mu_2 + \alpha)} - \frac{\lambda_2\mu_3}{(s + \lambda_3 + \mu_3 + \alpha)} - \ldots} \] (2.42)

Using (2.10) in (2.39)-(2.42) and using the identity (A.6), \( Q_1^*(s) \) can be expressed as the ratio of hypergeometric functions as follows:

\[ Q_1^*(s) = \frac{2}{(s + \alpha + \mu)\left(\frac{\lambda(s+\alpha)}{(s+\alpha+\mu)^2} + 2\right)} \times \left\{ \begin{array}{l} _1F_1(3; \frac{\lambda(s+\alpha)}{(s+\mu+\alpha)^2} + 3; -\frac{\lambda\mu}{(s+\mu+\alpha)^2}) \\ _1F_1(2; \frac{\lambda(s+\alpha)}{(s+\mu+\alpha)^2} + 2; -\frac{\lambda\mu}{(s+\mu+\alpha)^2}) \end{array} \right\}. \] (2.43)

Substituting (2.43) in (2.39), we get

\[ Q_0^*(s) = \frac{\alpha}{s(s + \alpha)} + \frac{2\mu}{(s + \alpha)(s + \alpha + \mu)\left(\frac{\lambda(s+\alpha)}{(s+\alpha+\mu)^2} + 2\right)} \times \left\{ \begin{array}{l} _1F_1(3; \frac{\lambda(s+\alpha)}{(s+\mu+\alpha)^2} + 3; -\frac{\lambda\mu}{(s+\mu+\alpha)^2}) \\ _1F_1(2; \frac{\lambda(s+\alpha)}{(s+\mu+\alpha)^2} + 2; -\frac{\lambda\mu}{(s+\mu+\alpha)^2}) \end{array} \right\}. \] (2.44)

which agrees with Parthasarathy and Selvaraju (2001) for \( \alpha = 0 \). Using the identity (A.8), \( Q_0^*(s) \) can be rewritten as

\[ Q_0^*(s) = \frac{\alpha}{s(s + \alpha)} - \frac{2(s + \alpha + \mu)}{\lambda(s+\alpha)} \times \left\{ \begin{array}{l} _1F_1(3; \frac{\lambda(s+\alpha)}{(s+\mu+\alpha)^2} + 3; -\frac{\lambda\mu}{(s+\mu+\alpha)^2}) \\ _1F_1(2; \frac{\lambda(s+\alpha)}{(s+\mu+\alpha)^2} + 2; -\frac{\lambda\mu}{(s+\mu+\alpha)^2}) \end{array} \right\} - 1 \}

Now using the identities (A.9) and (A.10) and after some algebra, the Laplace-Stieltjes transform of the density function of the
busy period is obtained as

$$sQ_0^*(s) = \frac{(s + \lambda)I_1(2; \frac{\lambda(s+\alpha)}{(s+\mu+\alpha)^2}) + 2; \frac{-\lambda\mu}{(s+\mu+\alpha)^2}) - s(\frac{\lambda(s+\alpha)}{(s+\mu+\alpha)^2} + 1)}{\lambda I_1(2; \frac{\lambda(s+\alpha)}{(s+\mu+\alpha)^2}) + 2; \frac{-\lambda\mu}{(s+\mu+\alpha)^2})}$$

(2.45)

and the mean busy period $E(T)$ is obtained as

$$E(T) = -\frac{d}{ds}sQ_0^*(s) \bigg|_{s=0} = \frac{\lambda \alpha}{(\alpha + \mu)^2} + 1 - I_1(2; \frac{\lambda\alpha}{(\mu+\alpha)^2}) + 2; \frac{-\lambda\mu}{(\mu+\alpha)^2}) \frac{\lambda I_1(2; \frac{\lambda\alpha}{(\mu+\alpha)^2}) + 2; \frac{-\lambda\mu}{(\mu+\alpha)^2})}{\lambda I_1(2; \frac{\lambda(s+\alpha)}{(s+\mu+\alpha)^2}) + 2; \frac{-\lambda\mu}{(s+\mu+\alpha)^2})}$$

(2.46)

. In the absence of catastrophes, that is for $\alpha = 0$ we have

$$E(T) = \frac{1 - I_1(2; 2; \frac{-\lambda}{\mu})}{\lambda I_1(2; 2; \frac{-\lambda}{\mu})} = \frac{1 - e^{-\frac{\lambda}{\mu}}}{\lambda e^{-\frac{\lambda}{\mu}}}$$

(2.46a)

Mean number of customers served during the busy period

Now, we discuss the mean number of customers served during a busy period. Let $N_B$ be the number of customers served during the busy period and $E(N_B)$, its mean. It is observed that $N_B$ is also the number of transitions between two successive visits to state 0. Under steady-state conditions, from the theory of regenerative process, the long-run proportion of arrivals finding the system empty can be expressed (Wolff 1989) as

$$P_0 = \frac{1}{E(N_B)}$$

so that

$$E(N_B) = \frac{1}{P_0} = \frac{1}{\alpha \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{k+1} \mu^k}{k!(\alpha + \mu)^{k+2}} \sum_{l=0}^{k+1} \frac{1}{l!} (-1)^{l+1} \frac{1}{\alpha l + l(\alpha + \mu)^2}}$$

(2.47)

where we have used (2.33).


2.4 INFINITE SERVER QUEUEING SYSTEMS

The $M/M/\infty$ queueing systems has been used in the context of communication system design (Newell 1984). Massey and Whitt (1993) studied networks of infinite server queues with non-stationary Poisson arrivals. Keilson and Servi (1993) analyzed the $M/M/\infty$ system, when the servers had Markov-modulated service rates by using a matrix-geometric approach. Further, the results concerning dynamic storage allocation in computers can be found in Coffman et al. (1985).

The arrival and service rates are given as

$$\lambda_n = \lambda, \quad n = 0, 1, 2, \ldots \quad (2.48a)$$

$$\mu_n = n\mu, \quad n = 1, 2, 3, \ldots \quad (2.48b)$$

Use of (2.48) in (2.8) immediately yields

$$P_0^*(s) = \frac{1 + \frac{\alpha}{s}}{(s + \lambda + \alpha)} \frac{\lambda \mu}{s + \lambda + \mu + \alpha} \frac{2\lambda \mu}{s + 2\mu + \alpha} \ldots \quad (2.49)$$

Employing the identity (A.6), the above can be expressed as

$$P_0^*(s) = \left(1 + \frac{\alpha}{s}\right) \frac{1}{s + \alpha} \frac{1}{F_1} \left(1 + \frac{s + \alpha}{\mu}; n + 1; -\frac{\lambda}{\mu}\right). \quad (2.50)$$

Now, substituting (2.48) in (2.9), we have

$$\frac{P_n^*(s)}{P_{n-1}^*(s)} = \frac{\lambda}{(s + \alpha + n\mu)} \frac{1}{F_1} \left(1 + \frac{s + \alpha}{\mu} + n + 1; -\frac{\lambda}{\mu}\right), \quad n = 1, 2, 3, \ldots \quad (2.51)$$

where we have used the identity (A.6). Proceeding in a similar manner as in section 3, we get

$$P_n^*(s) = \frac{(1 + \frac{\alpha}{s}) \lambda^n}{(s + \alpha)(s + \alpha + \mu) \ldots (s + \alpha + n\mu)} \times 1\left(n + 1; \frac{s + \alpha}{\mu} + n + 1; -\frac{\lambda}{\mu}\right), \quad n = 0, 1, 2, \ldots \quad (2.52)$$
It now follows, for $n = 0, 1, 2, \ldots$,

$$P_n^*(s) = \sum_{k=0}^{\infty} \frac{(1 + \frac{a}{s})(-1)^k(n + k)! \lambda^{n+k}}{k!n!} \frac{1}{\prod_{l=0}^{n+k} (s + \alpha + l\mu)} \cdot (2.53)$$

By partial fraction expansion, (2.53) can be expressed as

$$P_n^*(s) = \sum_{k=0}^{\infty} \frac{(1 + \frac{a}{s})(-1)^k(n + k)!}{n!k!} \left( \frac{\lambda}{\mu} \right)^{n+k} \times \sum_{l=0}^{n+k} \frac{(-1)^l}{l!(n + k - 1)!(s + \alpha + l\mu)} , n = 0, 1, 2, \ldots \quad (2.54)$$

Taking inverse, we get

$$P_n(t) = e^{-\alpha t} \frac{[\frac{\lambda}{\mu} (1 - e^{-\mu t})]^n}{n!} \exp \left( -\frac{\lambda}{\mu} (1 - e^{-\mu t}) \right)$$

$$+ \alpha \int_0^t e^{-\alpha y} \frac{[\frac{\lambda}{\mu} (1 - e^{-\mu y})]^n}{n!} \exp(-\frac{\lambda}{\mu} (1 - e^{-\mu y}))dy, \quad n = 0, 1, 2, \ldots \quad (2.55)$$

Thus (2.55) gives explicit time-dependent system size probabilities for the infinite server queueing systems with catastrophes.

**Remark 2:** If $\mu = 0$, then (2.55) reduces to

$$P_n(t) = e^{-(\alpha + \lambda)t} \frac{(\lambda t)^n}{n!} + \alpha \int_0^t \frac{\lambda u^n}{n!} \exp(-\alpha - \lambda u)du, n = 0, 1, 2, \ldots \quad (2.56)$$

Thus, the transient probabilities of the system size for an immigration and catastrophe process are obtained explicitly. We observe that (2.56) agrees with the result of Swift (2001).
2.4.1 Mean System Size

Proceeding on similar lines as we have done for the discouraged arrivals queue, an explicit expression for mean is obtained in this section. As before, we define

\[ P^*_B(z, s) = \sum_{n=0}^{\infty} P^*_n(s) z^n \]  

(2.57)

where \( P^*_n(s) \) is given as in (2.54). Thus, it follows that

\[ P^*_B(z, s) = \left(1 + \frac{\alpha}{s}\right) \frac{1}{(s + \alpha)} \frac{\Gamma(1 + \frac{s+\alpha}{\mu} + 1; \frac{\lambda z}{\mu} - \frac{\lambda}{\mu})}{(s + \alpha)} \]

so that

\[ P^*_B(z, s) = \left(1 + \frac{\alpha}{s}\right) \sum_{k=0}^{\infty} (\lambda(z-1))^k \sum_{l=0}^{k} \frac{(-1)^l}{l!(k-l)!\mu^l(s+l\mu+\alpha)} \] .

On inversion, the above yields

\[ P_B(z, t) = e^{-\alpha t} \exp\left[-\frac{\lambda}{\mu}(1 - z)(1 - e^{-\mu t})\right] \]

\[ + \alpha \int_0^t e^{-\alpha u} \exp\left[-\frac{\lambda}{\mu}(1 - z)(1 - e^{-\mu u})\right] du . \]  

(2.58)

Now, differentiating (2.58) with respect to \( z \) and evaluating at \( z = 1 \), we obtain the mean

\[ m(t) = \frac{\lambda}{(\alpha + \mu)}(1 - e^{-(\alpha+\mu)t}) . \]  

(2.59)

Thus, the average number of customers in the system at time \( t \) is derived. The factorial moments can be obtained as in section 2.3.1.
2.4.2 Steady-State Probabilities

As before, the application of Tauberian theorem to (2.54) yields the steady-state system size probabilities as

\[
P_n = \lim_{s \to 0} s P_n^*(s) = \lim_{s \to 0} s \sum_{k=0}^{\infty} \frac{(1 + \frac{\alpha}{s})(-1)^k(n + k)!((\frac{\lambda}{\mu})^{n+k}}{n!k!} \times \sum_{l=0}^{n+k} \frac{(-1)^l}{l!(n + k - l)!(s + \alpha + l\mu)}, \quad n = 0, 1, 2, \ldots \]

which simplifies to

\[
P_n = \sum_{k=0}^{\infty} \frac{(-1)^k(\frac{\lambda}{\mu})^{n+k}}{n!k!} \sum_{l=0}^{n+k} \left(\frac{n + k}{l}\right)(-1)^l \frac{\alpha}{\alpha + l\mu}, \quad n = 0, 1, 2, \ldots \] (2.60)

If \(\alpha = 0\), then (2.60) becomes

\[
P_n = e^{-\frac{\lambda}{\mu}} \frac{(\frac{\lambda}{\mu})^n}{n!}, \quad n = 0, 1, 2, \ldots \] (2.61)

which agrees with (2.34) as expected.

**Remark 3:** It is observed that the steady-state solutions are same for both discouraged arrival and infinite server queueing systems in the absence of catastrophes whereas the steady-state solutions (2.33) and (2.60) differ when catastrophes occur in the systems.

2.4.3 Busy Period Analysis

Now we consider the busy period distribution for the infinite server queueing system whose performance measures will be compared with the discouraged arrivals queue discussed earlier.
Substituting (2.48) in (2.39)-(2.42) and using the identity (A.6), \( Q_1^*(s) \) can be expressed as a ratio of hypergeometric functions as follows:

\[
Q_1^*(s) = \frac{1}{s + \alpha + \mu} \frac{1}{F_1(2; \frac{s + \alpha}{\mu} + 2; \frac{-\lambda}{\mu})} \quad (2.62)
\]

Proceeding as in the previous section, we get the Laplace-Stieltjes transform of the density function of the busy period and its mean as

\[
sQ_0^*(s) = \frac{s + \lambda}{\lambda} - \frac{s}{\lambda F_1(1; \frac{s + \alpha}{\mu} + 1; \frac{-\lambda}{\mu})} \quad (2.63)
\]

\[
E(T) = \frac{1 - F_1(1; \frac{\alpha}{\mu} + 1; \frac{-\lambda}{\mu})}{\lambda F_1(1; \frac{\alpha}{\mu} + 1; \frac{-\lambda}{\mu})} \quad (2.64)
\]

For \( \alpha = 0 \), we have

\[
E(T) = \frac{1 - F_1(1; 1; \frac{-\lambda}{\mu})}{\lambda F_1(1; 1; \frac{-\lambda}{\mu})} = 1 - e^{-\frac{\lambda}{\mu}}
\]

which coincides with (2.46a). Note that in the absence of catastrophes, the mean busy period for the discouraged arrivals queueing system and the infinite server queueing system coincide, but not otherwise.

Mean number of customers served during the busy period

Here, we study the mean number of customers served during the busy period. Let \( N_B \) be the number of customers served during the busy period and \( E(N_B) \) its mean. Now proceeding as in the previous section

\[
P_0 = \frac{1}{E(N_B)}
\]
so that

$$E(N_B) = \frac{1}{P_0} = \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k}{\mu^k k!} \sum_{l=0}^{k} \frac{1}{(-1)^l \binom{k}{l} \frac{\alpha}{\alpha + \beta}}$$

(2.65)

where we have used (2.60).

2.5 NUMERICAL ILLUSTRATIONS FOR THE TWO MODELS

In this section, the effect of the system parameters on the system size probabilities are discussed numerically. The graphs for the steady-state system size probabilities for the infinite server queueing model and queues with discouraged arrivals are plotted for varying parametric values.

In figures 2.1a and 2.1b, we study the behavior of the steady-state probabilities for $P_1, P_2$ and $P_3$ using (2.33) and (2.60) for $\mu = 5, \alpha = 8$ and varying $\lambda$. For both discouraged arrival and infinite server queueing systems, the graphs display a steep increase initially (for smaller system size), followed by a downward trend towards stability. They also show that, for the same parametric values, for the infinite server queueing model, stability is reached faster than the discouraged arrival queueing model for increasing values of $\lambda$. Figure 2.1c shows that for both the systems, the probability $P_0$ displays a similar behavior with marginal variation in $P_0$ values for varying values of $\lambda$ and $\mu = 5, \alpha = 8$.

Figures 2.2a - 2.2c display the effect of catastrophes for the models under discussion. It is observed from figures 2.2a and 2.2b that
the steady-state probabilities show a downward trend towards stability for increasing values of $\alpha$ with $\lambda = 4$ and $\mu = 5$. The graph for $P_0$ shows an upward trend towards stability for the same values as in figure 2.2c, as expected. Figures 3.3a and 3.3b show the effect of catastrophes on the average busy period. For both the models under discussion it is observed that the average busy period shows a downward trend towards stability for increasing values of $\alpha$. Though the busy period distribution are different for both the models based on their Laplace transforms, the average busy period displays almost the same trend for varying values of $\alpha$.

Figures 2.4a and 2.4b shows the effect of the average number of customers served during the busy period for varying values of $\alpha$ which also shows a steep downward trend for smaller values of $\alpha$ towards reaching stability for increasing values of $\alpha$. Figure 2.4c compares the two models with regard to the average number of customers served during the busy period for varying values of $\alpha$. The displayed graph shows that both the models display very similar trend for varying values of $\alpha$. 
Figure 2.1a: $P_n$ versus $\lambda$ with $\alpha = 8$ and $\mu = 5$
for discouraged arrival queue

Figure 2.1b: $P_n$ versus $\lambda$ with $\alpha = 8$ and $\mu = 5$
for infinite server queue
Figure 2.1c: Comparison of $P_0$ versus $\lambda$ with $\alpha = 8$ and $\mu = 5$

Figure 2.2a: $P_n$ versus $\alpha$ with $\lambda = 4$ and $\mu = 5$ for discouraged arrival queue
Figure 2.2b: $P_n$ versus $\alpha$ with $\lambda = 4$ and $\mu = 5$ for infinite server queue

Figure 2.2c: Comparison of $P_0$ versus $\alpha$ with $\lambda = 4$ and $\mu = 5$
Figure 2.3a: $E(T)$ versus $\alpha$ with $\mu = 5$
for discouraged arrival queue

Figure 2.3b: $E(T)$ versus $\alpha$ with $\mu = 4$
for infinite server queue
**Figure 2.3c:** Comparison of $E(T)$ versus $\alpha$ with $\lambda = 2.6$ and $\mu = 3$

**Figure 2.4a:** $E(N_B)$ versus $\alpha$ with $\mu = 2$
for discouraged arrival queue
Figure 2.4b: \( E(N_B) \) versus \( \alpha \) with \( \mu = 3 \) for infinite server queue

Figure 2.4c: Comparison of \( E(N_B) \) versus \( \alpha \) with \( \lambda = 3.5 \) and \( \mu = 4 \)