Introduction

Kuttykrishnan A.P “Laplace autoregressive time series models”, Department of Statistics, University of Calicut, 2006
1.1. Introduction

A time series is a series of observations made sequentially in time. The primary objective of time series analysis is to reveal the probability law that governs the observed time series. Thus, if we wish to explain a particular pattern of fluctuations in a series, we need to construct a mathematical model that explains the random features of the series of observations. The construction of such mathematical model is the main objective in time series analysis. If we obtain a satisfactory model for our time series, it may provide knowledge of the physical mechanism generating the data, and it can be used for forecast and thereby control future values of the series. A model is selected in such a way that the selected model is a simple one and reasonably reflects the physical law that governs the data. In selecting a model, one first identifies the salient features from the observed series and then chooses an appropriate model that possesses such features. After estimating the parameters of the model, one verifies whether the model fits the data reasonably well and looks for further improvement whenever possible.

When we analyze a time series using formal statistical methods, we view the collection of observations \( \{x_n, n = 1, 2, \ldots\} \) as a particular realization of the stochastic process \( \{X_t\} \). Hence a complete description of a time series, observed as a collection of \( n \) random variables at arbitrary time points \( t_1, t_2, \ldots, t_n \), is provided by
the joint distribution function $F(x_1, x_2, ..., x_n) = P(X_{t_1} \leq x_1, X_{t_2} \leq x_2, ..., X_{t_n} \leq x_n)$.

This multidimensional distribution function can be written easily provided the random variables are jointly normal. Generally such an assumption is unlikely to be appropriate for many time series, and so the multidimensional distribution cannot be written in a simple form. A simple and more useful way of describing a time series is to give one dimensional distribution function $F(x_i) = P(X_{t_i} \leq x_i)$ and mean value function $\mu_t = E(X_t)$, if it exists. The dependence between two adjacent values $X_{t+k}$ and $X_t$ can be assessed numerically by the autocovariance function, which is defined as $\gamma(k) = E((X_t - \mu_t)(X_{t+k} - \mu_{t+k}))$. It may be noted that $\gamma(k)$ of a time series measures the linear dependence between two points on the same series at different times. As in the classical set up, it is more convenient to define autocorrelation function $\rho(k) = \frac{\gamma(k)}{\sqrt{\text{Var}(X_t) \text{Var}(X_{t+k})}}$ to describe the association between two values on the same series at different times.

A special class of time series, which encountered in most of the practical case, is stationary time series. By stationarity, we mean that the series looks much the same over all time periods and so the statistical properties of $\{X_t\}$ are the same if we change the time origin. If the joint probability distribution of $X_t$ at any set of times $t_1, t_2, ..., t_n$ is the same as the joint probability distribution at times $t_1 + k, t_2 + k, ..., t_n + k$, where $k$ is any arbitrary shift in time, then $\{X_t\}$ is called a strictly stationary time series. These conditions are too strong, hence a milder
version of stationarity, namely weak stationarity, is introduced to apply in practical problems. A time series \( \{X_t\} \) is said to be weakly stationary if (i) \( E(X_t^2) < \infty \), (ii) \( E(X_t) = \mu \) for all \( t \) and (iii) \( \gamma(k) = \gamma(k+h) \) for all \( k \) and \( h \). It may be noted that \( \{X_t\} \) is weakly stationary if it is strictly stationary with finite second moments.

In 1927, Yule developed linear time series models, defined in terms of linear difference equation, with the intention of explaining the dynamic relationship of the sunspot series (see Chatfield (1989)). These linear time series models have now become an integral part of time series literature. The most popular class of linear time series models consists of autoregressive moving average (ARMA) models, including purely autoregressive (AR) and purely moving average (MA) models as special cases. Adding non-stationary models to the mixed ARMA models leads to the autoregressive integrated moving average (ARIMA) model that are popularized and studied by Box and Jenkins (1970).

An autoregressive model is a time series model that can be extremely useful in the representation of certain practically occurring series. An Autoregressive model of order \( p \geq 1 \), abbreviated as AR (p), is defined as

\[
X_n = \rho_1 X_{n-1} + \rho_2 X_{n-2} + \ldots + \rho_p X_{n-p} + \varepsilon_n
\]  

where \( \{\varepsilon_n\} \) is a sequence of independent and identically distributed random variables, and \( \rho_1, \rho_2, \ldots, \rho_p \) are constants. The autoregressive model (1.1.1)
represents the current value $X_n$ of the process through its immediate $p$ past values $X_{n-1}, X_{n-2}, \ldots, X_{n-p}$ and a random shock $\varepsilon_n$ in a linear regression form.

If we define an autoregressive operator of order $p$ by

$$\rho(B) = 1 - \rho_1 B - \rho_2 B^2 - \cdots - \rho_p B^p$$

where $B$ is the back shift operator defined as $B^k X_n = X_{n-k}$, $k = 0, 1, \ldots$, then the AR($p$) process (1.1.1) may be written as $\rho(B) X_n = \varepsilon_n$. If we define

$$\psi(B) = \rho^{-1}(B) = 1 + \psi_1 B + \psi_2 B^2 + \cdots$$

it is equivalent to $X_n = \psi(B) \varepsilon_n$.

An autoregressive process may be stationary or non-stationary. For the process to be stationary, the $\rho$'s must be chosen so that the $\psi(B)$ form a convergent series.

The most simple form of an autoregressive model is an AR (1) and given by the linear difference equation

$$X_n = \rho X_{n-1} + \varepsilon_n \quad (1.1.2)$$

and it may be verified that the parameter $\rho$ must satisfy the condition $|\rho| < 1$ to ensure stationarity of the process. The autocovariance function of the process (1.1.2) satisfied the equation $\gamma(k) = \rho \gamma(k-1)$, for all $k > 0$ and consequently $\rho(k) = \rho^k$.

Another kind of model that has very great practical importance in the representation of the series is the moving average model. The standard form of a moving average model of order $q \geq 1$, denoted by MA ($q$) is given by
\[ X_n = \theta_1 \varepsilon_{n-1} + \theta_2 \varepsilon_{n-2} + \ldots + \theta_q \varepsilon_{n-q} + \varepsilon_n \]  

(1.1.3)

where \( \{\varepsilon_n\} \) is a sequence of independent and identically distributed random variables, and \( \theta_1, \theta_2, \ldots, \theta_q \) are constants. Here we make the current value \( X_n \) linearly dependent on previous \( q \) values of \( \varepsilon_n \) 's.

If we define a moving average operator of order \( q \) by

\[ \theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \ldots + \theta_q B^q \]

then the MA(q) process (1.1.3) may be written as \( X_n = \theta(B) \varepsilon_n \). The simplest form of moving average process, first order moving average process, is given by the difference equation

\[ X_n = \theta \varepsilon_{n-1} + \varepsilon_n. \]

(1.1.4)

The general linear time series model is obtained by including both autoregressive and moving average terms in the model. Such models are known as autoregressive moving average models, denoted by ARMA \((p,q)\) and has the form

\[ X_n - \rho_1 X_{n-1} - \ldots - \rho_p X_{n-p} = \varepsilon_n + \theta_1 \varepsilon_{n-1} + \ldots + \theta_q \varepsilon_{n-q} \]

(1.1.5)

where \( \{\rho_i\} \) and \( \{\theta_j\} \), \( i = 1, 2, \ldots, p; \ j = 1, 2, \ldots, q \) are sequences of constants and \( \{\varepsilon_n\} \) is a sequence of independent and identically distributed random variables. One can write ARMA \((p,q)\) process (1.1.5) as \( \rho(B)X_n = \theta(B)\varepsilon_n \) using autoregressive
(ρ(B)) and moving average (θ(B)) operators. It can be seen that AR (p) model is same as ARMA (p, 0) model and a MA (q) model is same as an ARMA (0,q) model. The mathematical details of these models are available in Box and Jenkins (1970), Box et al. (1994), Chatfield (1975) and Brockwell and Davis (1991).

Until quite recently much of time series modeling has been limited to ARMA models with the assumption that \( \{ ε_n \} \) is a sequence of independent and identically distributed Gaussian random variables. Then the process \( \{ X_n \} \) is known as Gaussian process. If \( \{ X_n \} \) is Gaussian then the conditional mean of \( X_n \) given the past values will be linear, the conditional variance of \( X_n \) given the past values will be a constant and the process will be time reversible. But, there are many naturally occurring time series data sets that cannot be modeled using the assumption that the variables are distributed as Gaussian.

1.2. Non–Gaussian autoregressive time series models

It is well known that the modeling and interpretation of time series data plays a significant role in almost every field of modern research. Many time series models are developed and studied by several researchers using the assumption that the sequence of observations follows Gaussian distribution. However, time series in which observations are of non-Gaussian nature are very common in many areas. In modeling such non-Gaussian time series, the usual practice is to make suitable transformation to remove skewness of the data and fit a Gaussian model. Such technique for a class of non-Gaussian time series has been discussed in Granger and
Newbold (1976) and Janacek and Swift (1990). But in most of the cases, the assumption that transformed data follows Gaussian is unlikely to be true in practice (see Sim (1994)).

In the last two decades, several time series models with non-Gaussian marginal distribution have been introduced and studied by various authors. The need for such models arises from the fact that many naturally occurring time series are clearly non-Gaussian. Some researchers have been trying to find model such that the variables have a given marginal distribution and a given correlation structure corresponding to an ARMA process. One class of non-Gaussian linear time series models that appears to be particularly useful is the first order autoregressive process. Cox (1981) and Bondesson (1981), discussed the problem of existence of solution of an autoregressive model (1.1.2) and established that the linear autoregressive model is properly defined if and only if the marginal distribution belongs to class L. Similar problems were also discussed in Hart (1984), Jayakumar and Pillai (1992) and Pillai and Jayakumar (1994). Pillai and Jose (1994) considered an autoregressive process with structure

\[ X_n = I_n X_{n-1} + \varepsilon_n, \]

where \( P(I_n = 0) = p = 1 - P(I_n = 1), 0 < p < 1 \) and obtained a necessary and sufficient condition on the marginal distribution for the process to be properly defined. They showed that the process is properly defined if and only if the marginal distribution is geometrically infinitely divisible.
The first order autoregressive process (1.1.2) for positive variables is properly defined only if \( \phi_\varepsilon(t) = \frac{\phi_X(t)}{\phi_X(\rho t)} \) is a proper characteristic function for \( 0 \leq \rho < 1 \), where \( \phi_X(t) \) is the characteristic function of stationary process \( \{X_n\} \).

Gaver and Lewis (1980) introduced and studied first order autoregressive process of the form (1.1.2) using exponential marginal distributions and obtained

\[
X_n = \begin{cases} 
\rho X_{n-1} & \text{w.p. } \rho \\
\rho X_{n-1} + E_n & \text{w.p. } 1 - \rho
\end{cases}
\]

(1.2.1)

where \( 0 \leq \rho < 1 \) and \( \{E_n\} \) is a sequence of independent and identically distributed exponential random variables and \( X_0 \sim E_1 \). This model, known as EAR (1), has autocorrelation \( \rho(h) = \rho^h \) and generates sample paths in which large values are followed by runs of falling values. However the degeneracy at \( \rho \), which is known as "zero defect", limit the broad applicability of the model. Corresponding to EAR (1) model Lawrance and Lewis (1977) introduced a first order moving average model EMA (1). Jacobs and Lewis (1977) linked the two models into a first order autoregressive-moving average model EARMA (1,1). Lawrance and Lewis (1980) generalized this model and developed an exponential autoregressive-moving average model of order \( (p,q) \).

Lawrance and Lewis (1981) proposed a new autoregressive process with structure
If \( \{X_n\} \) is stationary with exponential distribution, they identified that the innovation sequence \( \{e_n\} \) as a sequence of independent and identically distributed random variables given by

\[
X_n = \begin{cases} \rho X_{n-1} + e_n & \text{w.p. } \gamma \\ e_n & \text{w.p. } 1-\gamma \end{cases}
\]  
(1.2.2)

where \( \{E_n\} \) is a sequence of independent and identically distributed exponential random variables. The stationary autoregressive process \( \{X_n\} \) with structure (1.2.2) and \( \{e_n\} \) is a sequence of independent and identically distributed random variables given by (1.2.3) with \( X_0 \equiv E_1 \) is known as NEAR (1) model. If \( \gamma = 0 \) or \( \rho = 0 \), \( \{X_n\} \) is a sequence of independent and identically distributed exponential random variables, whereas \( \gamma = 1 \), the model reduced to structure given by (1.2.1) where \( \rho \) is replaced by \( 1-\rho \). The autocorrelation function of the NEAR (1) process is obtained as

\[
\rho(h) = (\rho(1-\gamma))^h.
\]

Jayakumar and Pillai (1993) developed and studied a general form of process discussed in Gaver and Lewis (1980) using semi-Mittag-Leffler distribution as marginal distribution. Subsequently number of time series models for positive variables have been studied and constructed using different marginal distributions

Recently number of time series models have been studied and constructed for real valued observations using different marginal distributions by several researchers. Lawrance (1978), Andel (1983) and Dewald and Lewis (1985) developed and studied time series models for real valued variables using Laplace marginal distribution. The applications of such models in the fields of environmental studies, communication theory etc. is given in Gibson (1986) and Damsleth and El-Shaarawi (1989). Anderson and Arnold (1993) and Jayakumar et al. (1995) studied a generalization of the Laplace process using Linnik / $\alpha$-Laplace marginal distribution and discussed the applications in the fields of financial mathematics. Autoregressive models using marginal distributions such as logistic, semi $\alpha$-Laplace, Cauchy and Pakes are studied in Sim (1993), Jayakumar (1997), Balakrishna and Nampoothiri (2003) and Seetha Lekshmi and Jose (2006) respectively.

Tavares (1980) introduced and studied an autoregressive process with structure $X_n = k \min(X_{n-1}, \epsilon_n)$ where $n \geq 1$ and $k > 1$ is a constant. The innovation sequence $\{\epsilon_n\}$ is a sequence of independent and identically distributed random
variables chosen to ensure that \( \{X_n\} \) is a stationary process with a given marginal distribution. The process of this type is known as minification process. Tavares (1980) discussed the minification process using exponential marginal distribution. Yeh et al. (1988), Arnold and Robertson (1989), Pillai (1991), Arnold (1993), Pillai et al. (1995), Kuttykrishnan and Jayakumar (2001) and Ristić (2005) studied minification processes using different marginal distributions and different structures.

1.3. Autoregressive process using Laplace variables

One of the most well-known and widely used symmetric distributions for modeling data with heavier tails than normal is the classical Laplace distribution. The probability density function of Laplace random variable with mean zero and variance \( 2\sigma^2 \) is given by

\[
f(x) = \frac{1}{2\sigma} e^{-\frac{|x|}{\sigma}}, -\infty < x < \infty, \sigma > 0
\] (1.3.1)

and the characteristic function is

\[
\phi_{X}(t) = \frac{1}{1 + \sigma^2 t^2}.
\] (1.3.2)

In this case we write \( X \overset{d}{=} L(\sigma) \).

It is only in recent years that Laplace distribution, together with its various generalizations, has been revived and is now being used in a variety of fields, including biology, economics, environmental science and financial mathematics. Several properties and applications of the Laplace distribution have been reported.
demonstrating that it is a natural and sometimes superior alternative to the Gaussian distribution (for more details see Kotz et al. (2001)).

Although the theory and applications of Laplace distribution are well developed and have appeared in literature in recent years, their applications in time series modeling have not developed that much. Lawrance (1978) discussed autoregressive process using Laplace variables. Andel (1983) studied a first order autoregressive time series model of the form (1.1.2) where $|\rho|<1$ and with Laplace distribution as marginal. The stationary process using Laplace random variables has the structure

$$X_n = \begin{cases} \rho X_{n-1} & \text{w.p. } \rho^2 \\ \rho X_{n-1} + L_n & \text{w.p. } 1-\rho^2 \end{cases}$$

(1.3.3)

where $\{L_n\}$ is a sequence of independent and identically distributed Laplace random variables with characteristic function (1.3.2). This model, known as LAR (1), has autocorrelation $\rho(h) = \rho^h$ and hence exhibits positive and negative autocorrelation. However, as in the case of EAR (1) process LAR (1) process also has the problem of "zero defect" that limit the broad applicability of the model. Dewald and Lewis (1985) discussed the process $\{X_n\}$ with structure (1.2.2) and Laplace distribution as the marginal distribution. The resulting Laplace model is free from the drawback "zero defect" of the Laplace model (1.3.3). They have shown that the innovation sequence $\{e_n\}$ is a convex mixture of Laplace random variables.
The main goal of the research presented here is to develop and study different types of autoregressive time series models for the set of real valued observations using Laplace random variables.

1.4. Basic concepts

In this Section, we give some preliminary concepts needed for the discussion in the subsequent Chapters.

1.4.1. Self-decomposability

A random variable $X$ is said to be self-decomposable if for every $\alpha \in (0, 1)$, it can be written (in distribution) as

\[ X \overset{d}{=} \alpha X + X_\alpha, \]  

(1.4.1)

where $X$ and $X_\alpha$ are independent random variables.

It may be noted that a distribution is self-decomposable if and only if for every $\alpha \in (0, 1)$, there exists a characteristic function $\phi_{X_\alpha}(t)$ such that

\[ \phi_{X_\alpha}(t) = \frac{\phi_X(t)}{\phi_X(\alpha t)} \]  

where $\phi_X(t)$ is the characteristic function of the random variable $X$.

It will be very simple to show that any non-degenerate self-decomposable distribution is absolutely continuous. Some important self-decomposable distributions are normal, exponential, gamma, Cauchy, Laplace etc. The class of self-decomposable distributions is closed under scale transformation, convolution
and weak convergence. Also it can be proved that a self-decomposable distribution on the real line is infinitely divisible. For more details and discussions on self-decomposability, see Steutel and van Harn (2004). Application of self-decomposability of distributions in time series modeling is discussed in Gaver and Lewis (1980) and Bondesson (1981).

By replacing the ordinary product $\alpha X$ in (1.4.1) by the product $\alpha \oplus X$, Steutel and van Harn (1979) defined a meaningful concept of self-decomposability for discrete random variables. The product $\alpha \oplus X$ is defined as $\alpha \oplus X = \sum_{i=1}^{X} Y_i$ where $P(Y_i = 1) = \alpha = 1 - P(Y_i = 0)$ and $\{Y_i\}$ is a sequence of independent and identically distributed random variables independent of $X$. The self-decomposability property of a discrete random variable can be explained using the probability generating function $P_x(z)$ of $X$. A discrete random variable $X$ is self-decomposable if and only if for every $\alpha \in (0, 1)$, there exists a probability generating function $P_{X_{\alpha}}(z)$ such that $P_{X_{\alpha}}(z) = \frac{P_X(z)}{P_X(1 - \alpha + \alpha z)}, |z| \leq 1$.

### 1.4.2. Stable and semi-stable distributions

Stable distributions are very important class of distributions that allow skewness and heavy tails and have many interesting mathematical properties. The lack of closed formulas for probability density and distribution functions for all but a few stable distributions (normal, Cauchy and Levy distributions) has been a major drawback to the use of stable distributions by practitioners.
A random variable \( X \) with distribution function \( F(x) \) is said to be stable if for each \( n \in \{1, 2, \ldots\} \) there exists \( d_n \in (-\infty, \infty) \) and \( c_n > 0 \) such that

\[
X \overset{d}{=} c_n (X_1 + X_2 + \ldots + X_n) + d_n
\]

(1.4.2)

where \( X_1, X_2, \ldots, X_n \) are independently and identically distributed random variables with distribution function \( F(x) \). The random variable \( X \) is called strictly stable if (1.4.2) holds with \( d_n = 0 \). Using characteristic function \( \phi_X(t) \) of a strictly stable random variable we can write (1.4.2) as

\[
\phi_X(t) = (\phi_X(c_n t))^n.
\]

Now it is simple to obtain following results related to stable distribution.

(1) A distribution with characteristic function \( \phi_X(t) \) is stable if and only if

\[
\phi_X(t) = (\phi_X(\frac{n}{\alpha} t))^n, \quad n > 0.
\]

The positive constant \( \alpha \) is called the exponent of stability of the random variable \( X \) and it can be verified that \( 0 < \alpha \leq 2 \).

It will be clear that for \( \alpha > 0 \), the characteristic function \( \phi_X(t) \) is stable with exponent \( \alpha \) if and only if the constants \( c_n = n^{-1/\alpha} \).

(2) A random variable \( X \) is stable with exponent \( \alpha \) if and only if for every \( n \in \{1, 2, \ldots\} \) it can be written as \( X \overset{d}{=} n^{-1/\alpha} (X_1 + X_2 + \ldots + X_n) \) where \( X_1, X_2, \ldots, X_n \) are independent with \( X_i \overset{d}{=} X \) for all \( i \).

(3) A stable random variable is infinitely divisible and self-decomposable.
The most concrete way to describe all possible stable distribution is through the following representation of characteristic function.

A random variable $X$ is said to have stable distribution if there are parameters $0 < \alpha \leq 2, \sigma \geq 0, -1 \leq \beta \leq 1$ and $\mu \in (-\infty, \infty)$ such that its characteristic function has the following form

$$
\phi_X(t) = \begin{cases} 
\exp \left\{ -\sigma^\alpha |t|^\alpha (1 - i\beta \tan \frac{\pi \alpha}{2} \text{sign} t) + i\mu t \right\} & \text{if } \alpha \neq 1 \\
\exp \left\{ -\sigma |t| (1 + i\beta \frac{2}{\pi} \ln |t| \text{sign} t) + i\mu t \right\} & \text{if } \alpha = 1 
\end{cases} 
$$

(1.4.3)

where $\text{sign}(t) = \begin{cases} 
1 & \text{if } t > 0 \\
0 & \text{if } t = 0 \\
-1 & \text{if } t < 0
\end{cases}$

The representation (1.4.3) shows that a general stable distribution requires four parameters $\alpha \in (0, 2]$, known as index of stability or characteristic exponent, skewness parameter $\beta \in [-1, 1]$, scale parameter $\sigma \geq 0$ and location parameter $\mu \in (-\infty, \infty)$. If $X$ is a random variable with characteristic function (1.4.3), then we denote it by $X \sim S_{\alpha}(\sigma, \beta, \mu)$. These distributions are symmetric around zero if $\beta = \mu = 0$ and so the characteristic function of symmetric stable is

$$
\phi_X(t) = \exp(-\sigma^\alpha |t|^\alpha).
$$

The various properties and applications of stable distributions are discussed in Samorodnitsky and Taqqu (1994).

Note that stable distribution becomes

(i) normal distribution if $\alpha = 2, \beta = 0$
(ii) Cauchy distribution when $\alpha = 1, \beta = 0$

(iii) Levy distribution when $\alpha = \frac{1}{2}, \beta = 1$.

A class of distributions containing stable distributions, namely semi-stable distributions is studied in Kagan et al. (1973).

A class of distributions is called semi-stable if

$$\phi(t) = (\phi(bt))^a, \ a > 1, 0 < |b| < 1,$$

(1.4.4)

where $\phi(t)$ is the characteristic function of the distribution.

Pillai (1968) proved that a semi-stable distribution is infinitely divisible. For more properties and discussions, see Pillai (1971).

1.4.3. Geometric infinite divisibility

A random variable $X$ is infinitely divisible if for each $n \geq 1$, there exist independent and identically distributed random variables $X_1, X_2, ..., X_n$ such that $X \overset{d}{=} X_1 + X_2 + ... + X_n$. Klebanov et al. (1984) introduced the concept of geometric infinite divisibility.

Let $N_p$ be a geometric random variable independent of $X_i$'s and with probability mass function

$$P(N_p = k) = p(1-p)^{k-1}; k = 1, 2, ...$$

(1.4.5)
Then a real valued random variable $X$ is said to have a geometrically infinitely divisible distribution if for any $p \in (0,1)$, there exists a sequence of independent and identically distributed real valued random variables $\{X_i^{(p)}\}$ such that

$$X \overset{d}{=} \sum_{i=1}^{N_p} X_i^{(p)},$$

where $N_p$ and $\{X_i^{(p)}\}$ are independent.

In terms of characteristic function, the relation (1.4.6) can be written as

$$\phi_X(t) = \frac{p\phi_{X_i^{(p)}}(t)}{1-(1-p)\phi_{X_i^{(p)}}(t)},$$

where $\phi_X(t)$ and $\phi_{X_i^{(p)}}(t)$ are the characteristic functions of $X$ and $X_i^{(p)}$ respectively.

Klebanov et al. (1984), Pillai (1990a), Mohan et al. (1993) and Fujita (1993) obtained several characterizations and properties of geometrically infinitely divisible distributions.

Some important results related to geometrically infinitely divisible distributions are

1. A random variable $X$ with characteristic function $\phi(t)$ is geometrically infinitely divisible if and only if $\phi(t) = \lim_{n \to \infty} \frac{1}{1+\theta_n(1-\phi_n(t))}$, where $\{\theta_n\}$ is a sequence of positive numbers and $\{\phi_n(t)\}$ is a sequence of characteristic functions.
(2) A random variable $X$ with characteristic function $\phi(t)$ is geometrically

infinitely divisible if and only if $\psi(t) = \exp\left\{1 - \frac{1}{\phi(t)}\right\}$ represents

characteristic function of an infinitely divisible distribution.

(3) The class of geometrically infinitely divisible distributions forms a subclass

of infinitely divisible distributions.

The applications of geometrically infinitely divisible distributions in the field

of time series modeling are discussed in Pillai and Jose (1994).

1.4.4. Geometric stable distribution

Geometric stable distributions are the weak limits of appropriately

normalized geometric random sums of independent and identically distributed

random variables $X_1, X_2, \ldots$. Let $N_p$ be a geometric random variable independent

of $X_i$'s and with probability mass function (1.4.5). If there exists a weak limit of

\begin{equation}
S_p = a(p) \sum_{i=1}^{N_p} (X_i + b(p)), \tag{1.4.8}
\end{equation}

when $p \to 0$, where $a(p) > 0$ and $b(p) \in (-\infty, \infty)$, then the class of limiting

distributions is called geometric stable distributions. If $b(p) = 0$, then the limiting
distribution is called strictly geometric stable.

Since the sums such as (1.4.8) frequently appear in many applied problems

in various areas (see Gnedenko and Korolev (1996)), these classes of distributions
have a wide variety of applications especially in the field of reliability, biology,
economics, financial mathematics etc.

The class of geometric stable distributions is a four-parameter family
denoted by $GS_\alpha(\alpha, \beta, \mu)$ and conveniently described in terms of characteristic
function

$$\phi_X(t) = \frac{1}{1 + \sigma^\alpha |t|^\alpha \varpi_{\alpha, \beta}(t) - i \mu t},$$

where

$$\varpi_{\alpha, \beta}(t) = \begin{cases} 1 - i \beta \text{sign}(t) \tan(\pi \alpha/2) & \text{if } \alpha \neq 1 \\ 1 + i \beta \frac{2}{\pi} \text{sign}(t) \log|t| & \text{if } \alpha = 1. \end{cases}$$

(1.4.10)

The parameter $\alpha \in (0, 2]$ is the index of stability and determines the tail of the
distribution, $\beta \in [-1, 1]$ determines the skewness of the distribution. The parameters
$\mu \in (-\infty, \infty)$ and $\sigma \geq 0$ correspond to location and scale. As the tail $P(X > x)$ of a
geometric stable random variable $X$ is regularly varying at infinity, this class of
distributions seems to be appropriate for modeling heavy tailed asymmetric data.
One such area of application is mathematical finance, where price change over a
certain period of time can be regarded as the sum of changes over shorter periods of
time.

Special cases of geometric stable distributions include exponential, Mittag-
Leffler, Laplace, asymmetric Laplace and Linnik distributions.

Note that geometric stable distribution becomes
(i) exponential distribution if $\sigma = 0$

(ii) symmetric Laplace distribution if $\alpha = 2, \mu = 0$

(iii) asymmetric Laplace distribution if $\alpha = 2, \mu \neq 0$

(iv) symmetric Linnik distribution if $\alpha \in (0, 2), \beta = \mu = 0$

(v) Mittag-Leffler distribution with Laplace transform $\frac{1}{1 + \sigma^\alpha s^\alpha}$ if $0 < \alpha \leq 1$

$\beta = \mu = 0$ and $X$ is defined on $(0, \infty)$.

Various properties and representations of geometric stable distribution are discussed in Mittinik and Rachev (1991), Ramachandran (1997), and Kozubowski and Rachev (1999). Aly and Bouzar (2000) studied geometric infinity divisibility and geometric stability properties of distributions and developed first order stationary autoregressive process with geometric stable marginal distributions. Some important properties of geometric stable distribution are as follows:

(1) A random variable $X$ is geometric stable if and only if its characteristic function $\phi(t)$ has the form

$$\phi(t) = \frac{1}{1 - \ln \psi(t)}, \quad (1.4.11)$$

where $\psi(t)$ is the characteristic function of a stable distribution.

(2) If $X$ is a $\text{GS}_\alpha(\sigma, \beta, \mu)$ random variable then

$$X \overset{d}{=} \begin{cases} \mu Z + Z^{1/\alpha} \sigma Y & \text{if } \alpha \neq 1 \\ \mu Z + Z \sigma Y + \sigma Z \beta(2/\pi) \ln(\sigma Z) & \text{if } \alpha = 1, \end{cases} \quad (1.4.12)$$
where $Y \sim S_\alpha(1,\beta,0)$, $Z$ has standard exponential distribution, and $Y$ and $Z$ are independent.

(3) If $X$ is a $GS_\alpha(\sigma,\beta,\mu)$ random variable, then for any $\tau > 0$, absolute moments $E|X|^\tau$ exists if and only if $\tau < \alpha$, where $0 < \alpha < 2$.

This result does not apply to the exponential ($\sigma = 0$) and to the Laplace ($\alpha = 2$) distributions, which have moments of all orders.

(4) The geometric stable distribution is self-decomposable
   
   (i) if $\alpha = 2$ without restriction on $\mu$ and $\beta$
   
   (ii) if $0 < \alpha < 1$ or $1 < \alpha < 2$ for $\mu = 0$ and
   
   (iii) if $\alpha = 1$ for $\mu = \beta = 0$.


1.4.5. Asymmetric Laplace distribution

The Laplace distribution is symmetric, and there were several asymmetric extensions in generalizing the Laplace distribution. In the last several decades, different forms of skewed Laplace distributions have been introduced and studied by various authors (see Holla and Bhattacharya (1968) and Yu and Zhang (2004)). Kozubowski and Podgórski (2000) studied asymmetric Laplace distribution having probability density function
where $\sigma > 0$, $-\infty < \mu < \infty$.

The additional parameter $\kappa > 0$ is related to $\mu$ and $\sigma$ as

$$\frac{1}{\kappa} - \kappa = \frac{\mu}{\sigma}$$

and hence

$$\kappa = \frac{2}{\frac{\mu}{\sigma} + \sqrt{4 + (\frac{\mu}{\sigma})^2}}.$$ 

If $X$ is a random variable with probability density function (1.4.13), then we represent it as $X \equiv \text{AL} (\mu, \sigma)$.

The $n$th arbitrary moment of $X \equiv \text{AL} (\mu, \sigma)$ is given by

$$E(X^n) = n! \left( \frac{\sigma}{\kappa} \right)^n \frac{1 + (-1)^n \kappa 2(n+1)}{1 + \kappa^2}.$$ 

The characteristic function of $\text{AL} (\mu, \sigma)$ random variable is given by

$$\phi_X(t) = \frac{1}{1 + \sigma^2 t^2 - i \mu t}, \sigma > 0, -\infty < \mu < \infty. \quad (1.4.14)$$

The class of asymmetric Laplace distributions with characteristic function (1.4.14) arises as limiting distribution of a random (geometric) sum of independent and identically distributed random variables with finite second moments. Hence the class of asymmetric Laplace distributions forms a subclass of geometric stable.
distributions where the geometric stable distributions, similar to stable laws, have tail behavior governed by the index of stability $\alpha \in (0,2]$. The class of distributions with characteristic function (1.4.14) corresponds to the geometric stable subclass with $\alpha = 2$ (see Kozubowski and Rachev (1999)). By specifying $\sigma = 0$ and $\mu > 0$, we have an exponential distribution with mean $\mu$ and obtain symmetric Laplace distribution if $\mu = 0$ and $\sigma \neq 0$. The asymmetric Laplace distribution plays an analogous role among geometric stable laws as Gaussian distributions do among the stable laws. The applications of asymmetric Laplace distribution in the fields of biology, financial mathematics, environmental science etc are well established by different authors (see Kotz et al. (2001) and Julia and Vives-Rego (2005)).

1.4.6. Semi $\alpha$-Laplace distribution

Linnik (1963) considered a family of symmetric distributions defined on $(-\infty, \infty)$ with the characteristic function

$$\phi(t) = \frac{1}{1 + |t|^\alpha}, 0 < \alpha \leq 2,$$

and several researchers discussed the properties and method of generation of random variables belong to this family of distributions (see Devroye (1990), Lin (1994, 1998) and Kotz et al. (2001)).

When $\alpha = 2$, the function (1.4.15) corresponds to the characteristic function of symmetric Laplace distribution and hence Pillai (1985) termed the distribution with characteristic function (1.4.15) as $\alpha$-Laplace distribution. Note that this
distribution is also known in the literature as Linnik distribution. Lin (1998) proved that Linnik distribution is geometric infinitely divisible and discussed characterization of Linnik distribution using geometric summation. There are no closed-form expressions for distribution and density functions for Linnik random variable except for $\alpha = 2$, which correspond to the Laplace distribution. The probability density function of the Linnik random variable with characteristic function (1.4.15) has the following representation

$$f_\alpha(x) = \frac{\sin \frac{\pi \alpha}{2}}{\pi} \int_0^\infty \frac{v^\alpha \exp(-v|x|)}{1 + v^{2\alpha} + 2v^\alpha \cos \frac{\pi \alpha}{2}} \, dv, \text{ if } x > 0$$

and

$$f_\alpha(x) = f_\alpha(-x), \text{ if } x < 0.$$ 

Sabu George and Pillai (1987) derived the probability density function of $\alpha$-Laplace distribution in terms of the Meijer's G function and studied multivariate extension of $\alpha$-Laplace distributions.

Unlike Laplace distribution, the Linnik distribution has infinite variance and the mean is finite only for $1 < \alpha < 2$. The absolute moments of the distribution are given by

$$E|X|^\delta = \frac{2^\delta \Gamma(1+\delta)\Gamma(\frac{1-\delta}{\alpha})\Gamma((1+\delta)/2)}{\sqrt{\pi} \Gamma(1-\frac{\delta}{2})}, \text{ } 0 < \delta < \alpha, 0 < \alpha \leq 2.$$
Various authors discussed the applications of the Linnik distribution in different fields that ranges from engineering to finance (see Devroye (1986) and Anderson and Arnold (1993)). The Linnik distribution is considered as the best choice for modeling whenever data exhibit both high kurtosis and heavy tails than Gaussian tails. Christoph and Schreiber (1998) studied the discrete version of Linnik distribution, namely discrete Linnik distribution.

Pillai (1985) has introduced a general class of distributions, termed semi $\alpha$-Laplace distributions, of which $\alpha$-Laplace distribution is a special case. Let $\phi(t)$ be the characteristic function, which is never zero, defined by

$$\phi(t) = \frac{1}{1 + \psi(t)}.$$  \hspace{1cm} (1.4.16)

Then the distribution with characteristic function (1.4.16) is called semi $\alpha$-Laplace distribution if $\psi(t)$ has the property

$$\psi(t) = a \ \psi(bt), \ 0 < b < 1,$$ \hspace{1cm} (1.4.17)

where $a$ is the unique solution of the equation

$$a b^\alpha = 1, \ 0 < \alpha \leq 2,$$ \hspace{1cm} (1.4.18)

where $b$ and $\alpha$ are known as order and exponent of the semi $\alpha$-Laplace distribution, respectively. It can be noted that the solution of the functional equation

$$\psi(t) = a \ \psi(bt), \ 0 < b < 1, a > 0$$

is

$$\psi(t) = |t|^\alpha h(t)$$  \hspace{1cm} (1.4.19)
where \( h(t) \) is periodic in \( \ln|t| \) (see Pillai (1985)).

It may be noted that when \( h(t) \) is a constant the semi \( \alpha \)-Laplace distribution reduces to \( \alpha \)-Laplace distribution.

Pillai (1985) and Divanji (1988) discussed properties of the semi \( \alpha \)-Laplace distribution and some of the properties are

1. A random variable \( X \) is a semi \( \alpha \)-Laplace random variable with order \( b \) if and only if the distribution function \( F(x) \) satisfies the equation

\[
F(x) = p F_1(x) + (1 - p) F_2(x)
\]

for some \( p \in (0,1) \), where \( F_1(x) \) is the distribution of \( bX \) and \( F_2 = F \ast F_1 \).

2. The semi \( \alpha \)-Laplace distribution is geometrically infinitely divisible.

3. For suitable choice of \( A \) and \( \alpha < 1 \), the characteristic function of a semi \( \alpha \)-Laplace can be written as

\[
\phi(t) = \frac{1}{1 + |t|^{\alpha} (1 - A \cos(k \ln|t|))}
\]

where

\[
k = \frac{2\pi}{\ln b}, \quad 0 < b < 1.
\]

4. The absolute moments \( E|X|^\delta \) for the semi \( \alpha \)-Laplace distribution with exponent \( \alpha \) exist if and only if \( 0 \leq \delta < \alpha \).
1.5. Outline of the thesis

The thesis is concerned with autoregressive time series models using Laplace random variables. In our study we focus on first order autoregressive process and we develop such models using the self-decomposability and geometric infinite divisibility properties of the variables. In the present work, we introduce and study the properties of first order autoregressive process using Laplace, asymmetric Laplace, geometric stable and discrete Laplace variables. Bivariate time series models with first order autoregressive structure are developed corresponding to the bivariate Laplace, marginal Laplace and Linnik distributions and bivariate semi $\alpha$-Laplace distributions. The properties of such models are discussed and obtained autocorrelation structure of the process. The technique of estimating model and distribution parameters of the autoregressive process is given. An application of the process is discussed.

The thesis is structured in seven chapters. In the introductory chapter, Chapter-I, we discussed the general aspect and development of non-Gaussian autoregressive time series models. The applications and relevance of the study of such models are pointed out there. Some basic concepts with regard to distribution theory that is needed for constructing time series models are also given in this Chapter.

In the second Chapter, we consider first order autoregressive model using Laplace, asymmetric Laplace and generalized Laplace variables and the existence of such processes are established. The models are developed using the self-
decomposability property of these distributions. Empirical analysis of some important time series data shows that asymmetric and heavy tailed distributions are more suitable for modeling the data. Hence we have given emphasis to first order asymmetric Laplace autoregressive process. It is identified that the distribution of the innovation sequence of the process is a convex mixture of exponential random variables with different parameters. The random coefficient representation of the process is given. Properties of the model such as autocorrelation, joint distribution and asymptotic behavior are studied. Also the problem of estimation of parameters of the model and distribution is addressed. The sample path behavior of the process for various parameters is given here. But the processes discussed here have the problem of "zero defect" that causes successive values of the process to be fixed multiples of the previous values.

In Chapter-III, we introduce a two-parameter time series model that is free from the drawback "zero defect" using asymmetric Laplace marginal distribution. The innovation sequence of the process is obtained as a sum of two independent random variables $U_n$ and $V_n$ where $U_n$ is a convex mixture of exponential random variables with different parameters and $V_n$ is an asymmetric Laplace random variable. The properties of the process are studied. Also we develop a tractable one-parameter model using geometric infinitely divisible property of the asymmetric Laplace distribution and discuss the properties of the model. An autoregressive process with geometric stable distribution as marginal distribution is developed. Also we considered some important cases by which the geometric stable
autoregressive process is reduced to some well known autoregressive processes. We introduce the concept of tailed asymmetric Laplace distribution and an autoregressive process using tailed asymmetric Laplace marginal distribution is developed. Also we developed a three-parameter autoregressive model using Laplace, asymmetric Laplace and semi $\alpha$-Laplace variables. If the marginal distribution of the stationary three-parameter autoregressive process is Laplace then we obtain the solution of innovation sequence as a convex mixture of Laplace random variables. Further we establish the condition for existence of stationary solution of the three-parameter autoregressive process when the marginal distribution is semi $\alpha$-Laplace distribution. The condition for existence of solution as a mixture of semi $\alpha$-Laplace with negative weights is given. The autocorrelation function, joint characteristic function and sample path behavior of the process are obtained. A second order autoregressive process with Laplace marginal distribution is introduced and the distribution of the innovation sequence in this case is obtained. The first order moving average process with Laplace marginal distribution is developed in this Chapter.

In Chapter-IV, we define a bivariate process with Laplace marginal distribution and establish the existence of the process. We discuss autocorrelation structure of the process and consider the question of obtaining bivariate Laplace autoregressive process with negative correlation. The autocorrelation structures of the marginal processes of the bivariate Laplace process are discussed and the conclusion is that the marginal process has the autocorrelation of $\text{ARMA}(p,q)$ process with $p \leq 2$ and $q \leq 1$ and some particular cases are examined.
The Chapter-V is about a class of bivariate distribution, namely marginal Laplace and Linnik distribution, which is a particular case of operator geometric stable distributions. Here we study some properties and characterizations of marginal Laplace and Linnik distributions and present a first order autoregressive marginal Laplace and Linnik process. The autocorrelation and asymptotic behavior of the process are derived in this Chapter. We consider a generalized class of distributions, namely marginal asymmetric Laplace and asymmetric Linnik distribution by introducing asymmetric component in the marginal distributions and time series model corresponding to this distribution is also developed.

Also we have introduced a new class of bivariate distributions called bivariate semi $\alpha$-Laplace distribution, containing bivariate Laplace distributions. Some characterizations of bivariate semi $\alpha$-Laplace distribution are obtained. Relation of bivariate semi $\alpha$-Laplace distribution with bivariate semi stable distribution is established. An autoregressive model with bivariate semi $\alpha$-Laplace distribution as marginal distributions is developed and its properties are studied.

The Chapter-VI is devoted to discrete Laplace distribution and time series model using this distribution. Since the discrete Laplace distribution is geometrically infinitely divisible it is possible to construct an autoregressive model equivalent to one-parameter model developed in Chapter-III. The distribution of innovation sequence of the stationary process when variables follow discrete Laplace distribution is identified in this chapter and hence we develop a time series model using discrete Laplace distribution. We extended this result to the skewed discrete
Laplace distribution and constructed a time series model corresponding to this skewed variables. The properties of the models are also studied.

In Chapter-VII we discuss an application of the asymmetric Laplace autoregressive process.