CHAPTER 5

LINEAR ARRAY SYNTHESIS

The art of finding out the amplitude and phase distribution at the aperture to yield a specified radiation pattern is known as array synthesis. The pattern synthesis techniques are widely known. The discussion will be restricted to linear array synthesis in this chapter so that a correspondence can be established between the admittance of the slotted array as described in earlier chapters and the aperture distribution to be dealt with in this chapter.

S.A. Schelkunoff [1] has converted the antenna pattern synthesis problem for linear arrays to a functional problem. It has become the basis for many researchers who have worked on the antenna synthesis. C.L. Dolph [2] in his classic paper derived an expression for the calculation of the root positions for the sum pattern by using the Chebyshev functions which has the feature that of, a number of unitary oscillations, followed by a hyperbolic rise, in designing linear arrays that would produce antenna patterns replicating the Chebyshev characteristics. Dolph-Chebyshev patterns provide a minimum beamwidth for a given sidelobe level. But later T.T. Taylor [3] has proved that Dolph's method is unrealizable in practice and demonstrated a new technique for the pattern synthesis. However both Dolph and Taylor pattern synthesis are applicable only for symmetric sidelobes on both sides of the main beam and the main beam position is fixed at an angle of $\theta = 90^\circ$. R.S. Elliott [4], overcome this difficulty by taking Taylor's as the starting pattern and doing a little modification. He [5] has extended the technique for the realization of difference pattern also. Later he [6] has modified the technique to control the individual sidelobe heights by doing the perturbation on the nulls. Starting with a suitably modified Taylor pattern R.F. Hynemann[7] developed perturbation procedure based on the log derivative of the envelop function. This led to the computable shifts in the pattern zeros which caused an
approximation to the requisite pattern. W.L. Stutzman [8] used an iterative sampling method to make the sidelobe peaks conform to a specified shape within a given tolerance. However, the perturbation procedure used by Elliott is totally different from that of Hynemann and Stutzman. W.D. White [9] has derived a flexible technique by which he not only puts the constraint on maxima, but imposes the constraint on the minima of the pattern also, by considering discrete distribution instead of continuous distribution taken by Elliott. A.T. Villenuve [10] in his classic paper has derived an expression to realize the pattern's derived by Taylor, by using discrete aperture distribution rather than continuous distribution assumed by Taylor. His method is superior to Taylor and Elliott, since he has considered the discrete distribution rather than continuous distribution. But this method is applicable only for the case of symmetrical sidelobes and it is not possible to synthesize asymmetric pattern. R.S. Elliott [11] derived a new technique which is very efficient and can be used to realize both sum and difference pattern. Later H.J. Orchard and Elliott [12] have modified the technique to realize the shaped pattern using the optimization technique. K.S. Rao and B.N. Das [13] has improved the Elliott's technique without using the optimization technique to realize the shaped beam pattern. This method suffers from the limitation of fast convergence. Later R.S. Elliott [14] has further improved his old technique by using the minimax method to synthesis desirable pattern. The present pattern synthesis technique enlarges upon the previously known technique of Schelkunoff for arriving at the clear insight of the null locations and their placement on the unit circle. The accuracy of the procedure has been verified by comparing the results with the already published results of Elliott [15]. The fast convergence gives a distinct improvement over the existing pattern synthesis technique.

5.1 LINEAR ARRAY:

Let us consider an N element array with the equal
interelement spacing \( d \) as shown in fig. (5.1). Where the angle \( \Theta \) is that of an incoming plane wave relative to the axis of the receiving array. The generalized array factor can be expressed as

\[
A(\Theta) = I_0 + I_1 e^{jkdcos\Theta} + I_2 e^{jk2dcos\Theta} + \ldots + I_{N-1} e^{jk(N-1)dcos\Theta}
= \sum_{n=0}^{N-1} I_n e^{jnkdcos\Theta}
\]

............(5.1)

If the current has a linear phase progression \( \chi \), then

\[
I_n = A_n e^{j\chi n}
\]

\[
A(\Theta) = \sum_{n=0}^{N-1} A_n e^{jn(kdcos\Theta + \chi)}
\]

............(5.2)

Where the \((n+1)\)th element leads the \(n\)th element in phase by \( \chi \). The eqn (5.1) becomes,

\[
A(\Theta) = \sum_{n=0}^{N-1} A_n e^{jn(kdcos\Theta + \chi)}
\]

............(5.3)

Let \( \chi = kdcos\Theta + \chi \)
then,

\[
A(\Theta) = \sum_{n=0}^{N-1} A_n e^{jn\chi}
\]

............(5.5)

This array factor is a function of \( \chi \) and may be recognized as Fourier series and periodic with period \( 2\pi \). The structure of array factor is completely determined by its values for \( 0 \leq \Theta \leq \pi \).

\(-1.0 \leq \cos\Theta \leq 1.0 \) or \(-kd \leq kdcos\Theta \leq kd \)

............(5.6)

5.2 UNIT CIRCLE REPRESENTATION:

By Schelkunoff's methodology we can represent any array to a polynomial

The array factor given by equation (5.5) can be expressed as the product of \((N-1)\) virtual couplets with their null points at the zeros of \( A(\Theta) \). Since there are \((N-1)\) roots for a polynomial of \( N \)th order, these roots correspond to the zeros of the pattern and thus corresponds to the zeros of the unit circle as shown in Fig (5.2). Let \( Z = e^{j(kdcos\Theta + \chi)} \). When the variable \( Z = Z_n \) (where \( Z_n \) is a root of the polynomial), \( A(Z) \) will become zero & results a null in the
radiation pattern.

The degree of the polynomial which represents an array is always one less than the apparent number of elements. The actual number of elements is at most equal to the apparent number. The total length of the array is the product of the apparent separation and the degree of the polynomial. For \( N \)-element uniform array the array factor is given by

\[
|A(Z)| = |1 + Z + Z^2 + \cdots + Z^{N-1}| \quad \ldots \ldots (5.7)
\]

The ratio between the principal maximum and the secondary lobe is approximately equal to 13.5 dB and is independent of the number of elements.

Since \( \nu = k d \cos \theta + \kappa \) is always pure real, \( e^{j\nu} \) is always complex and imaginary, and the absolute value of \( e^{j\nu} = Z \) is always unity. If \( Z \) is plotted in the complex plane, it will always lie on the circumference of the unit circle as shown in fig. (2).

As \( \theta \) moves from 0 degree to 180 degrees \( \nu \) traverses from \( k d + \kappa \) to \( -k d + \kappa \) and \( Z \) moves in a clockwise direction. Because of symmetry, the range of \( \theta \) is considered from 0 to 180 degrees, thus the range of \( \nu \) described by \( Z \) is \( \nu = 2k d \) radians.

By the fundamental theorem of algebra a polynomial of \( (N-1) \)th degree has \( (N-1) \) zeros (some of which may be multiple zeros) and can be factored into \( (N-1) \) binomials. Thus

\[
|A(Z)| = |(Z-Z_1)(Z-Z_2)\ldots\ldots\ldots(Z-Z_{n-1})| \quad \ldots \ldots (5.8)
\]

Since \( Z \) is always on the unit circle, the pattern will have zero only when \( Z_1 \) also lies on the unit circle, and \( Z_1 \) is within the range. This is true for all other roots. So, in the visible region the pattern will have \( (N-1) \) nulls. It is evident that the relative radiation field in any direction is given by the products of the distances from \( Z \) to the null
points of the array.

Hence the visible regions of $\Theta$ and $\lambda$ are given by (5.6). Suppose that exactly one period appears in the visible region & since the period is $2\pi$ we have $2\pi = 2kd = 2 \times (2\pi / \lambda) \times d = d/\lambda = 1/2$. Thus exactly one period of the array factor appears in the visible region when the element spacing is one half wavelength. For spacing less than $\lambda/2$ the visible region is less than one period (i.e.) $2kd < 2\pi$. For spacings greater than $\lambda/2$ more than one period occurs in the visible region and results in the occurrence of grating lobes in the visible region. In most situations, it is not desirable to have grating lobes. i.e. the arrays have to be designed with the spacing less than one wavelength, usually close to a half-wavelength, to avoid the grating lobes. R.S. Elliott [15] has derived a relation between the interelement space and the number of elements as

$$d_{\text{max}} = (N/(N+1)) \times \lambda \quad \text{(5.9)}$$

If the overall length of the array is maintained constant, but the number of elements is increased, it is possible to improve the directivity still further if the nulls are properly spaced in the range of operation for uniform array. It was found that the maximum directivity and gain obtainable were directly related to the length of the array.

5.3 AN IMPROVED PATTERN SYNTHESIS TECHNIQUE:

The generalized array factor of an equispaced linear array of $N+1$ elements is given by (5.2) which can be factored out as

$$A(w) = \prod_{n=1}^{N} (w-w_n) \quad \text{........(5.10)}$$

by defining $w = e^{j\chi}$ with $\chi = kdcos\Theta \quad \text{........(5.11)}$

were, $w_1, w_2, \ldots, w_n$ are the $N$ roots which are all lying at the Schelkunoff unit circle. Equation (5.10) can be normalized and written as
\[
A(w) = C \frac{\sum_{k=1}^{N} (w-w_n)}{w^{N/2}}
\]

...(5.12)

in which C is a complex constant, whose magnitude governs the level of the array factor, but whose phase is chosen to be -B, which insures that A(w) is always a pure real function and alternates its sign from one lobe to the next.

Let starting pattern can be expressed as

\[
A_0(w) = C_0 \frac{\sum_{k=1}^{N} (w-w'_n)}{w^{N/2}}
\]

...(5.13)

in which the values of \(w_n\) and \(C_0\) are known. Now let us assume that the desired pattern \(A'(w)\) is close to that of \(A_0(w)\). It means

\[
w'_n = w_n \cdot e^{j\gamma_n} = w_n \cdot (1 + j\gamma_n)
\]

...(5.14)

with a small perturbation of . Now \(A(w)\) becomes,

\[
A(w) = (C_0 + \delta C) \frac{\sum_{k=1}^{N} (w-w'_n \cdot e^{j\gamma_n})}{w^{N/2}}
\]

\[
\quad \quad = (C_0 + \delta C) \frac{\sum_{k=1}^{N} (w-w_n(1 + j\gamma_n))}{w^{N/2}}
\]

...(5.15)

The perturbation on C serves two purposes. It gives the linkage so that both \(A_0(w)\) and \(A(w)\) are pure real, also it permits an adjustment of pattern level so that the lobe heights in \(A_0(w)\) and \(A(w)\) can be compared. By expanding (5.15) and keeping only the zeroth, and the first order terms, eqn (5.15) will gives

\[
A(w) - A_0(w) = \delta C \frac{\sum_{n=1}^{N} w_n}{w-w_n^*}
\]

...(5.16)

Including the main beam there exist N lobes which gives N distinct peak positions \(w_m\) in the Schelkunoff unit circle corresponding to the peak of the \(m\)th lobe in the starting pattern. By substituting this \(w_m\) values and the values of \(A'(w)\) \(A_0(w)\) in eqn (5.16) gives N simultaneous equations with N+1 variables. It is not a deterministic system of equations. The reason is that there is not a unique set of roots positions on the unit circle. Corresponding to a specified pattern, for if one were to add a common increment to each ,thus ensuring the roots around the unit circle without changing their relative positions the pattern would be
unchanged. Thus one has to fix the position of one of the roots, that is, set one of the \( \gamma_n \)'s equal to zero. Now it is convenient to fix one of the roots, say the Nth at the position \( \gamma_n = \pi \) and the pattern in \(-\pi < \gamma < \pi\).

Now equation (5.16) can be reduced to N-set of equations as:

\[
\begin{align*}
A(w_m) - A_0(w_m) & = \delta C \sum_{n=1}^{\infty} \frac{w^*_n}{w_m - w_n^*} \delta \gamma_n \quad \text{(5.17)}
\end{align*}
\]

Where \( w_1, w_2, \ldots, w_N \) correspond to the peak positions of the starting pattern which can be calculated by using Peak-Finder subroutine or it can be taken approximately as the mid value between the successive roots. The substitution of this values and the peak values (5.17) results a set of N-linear simultaneous equations. Solution of (5.17) gives the perturbation values and their insertion in (5.14) gives the new root positions in the unit circle. If \( A(w) \) is not close enough to the desired level then take this as the starting pattern and repeat the same procedure till we get the desired level of specification. The final values of \( \gamma_n \) can be used to calculate the new values of \( w_n \) and that of the discrete amplitude distribution of the elements.

Several cases has been considered to emphasis the efficiency of this technique and compared with the results obtained by others.

5.4 REALIZATION OF DOLPH CHEBYSHEV PATTERN:

If the (N+1) roots are equispaced on the unit circle, after which the root at \( w = 1+j0 \) is removed, we know from Demovier's theorem that the excitation is uniform amplitude equispace; a sum pattern results with symmetric sidelobes on both the sides which is shown in Fig(5.3). The null locations derived by C.L.Dolph is given as:

\[
\gamma_n = 2\cos^{-1} \left[ \cos \frac{\xi_n}{U_0} \right] \quad \text{(5.18)}
\]

where \( \xi_n = \frac{2n-1}{2m} \pi \) \( n = 1, 2, 3, \ldots \) \( m \) with \( m \), the order of the Chebyshev differential equation, \( U_0 \).
can be calculated from the desired sidelobe level.

When the values of $\psi_n$ calculated from (5.18) are introduced in (5.11) we will get the null location in the complex plane and the insertion of this nulls in equation (5.12) will give the desired pattern.

A typical Dolph pattern for 19 elements is shown in Fig(5.4) their corresponding amplitude distribution can be obtained by expanding the $\overline{w_n}$ and are given in table - I. It is observed that the the excitation is tapered, with the central elements most excited, with a falling off on each side, and then with an upswing at the ends of the array. This upswing will result more mutual coupling which will cause a great difficulty for the antenna designer.

5.5 REALIZATION OF TAYLOR'S PATTERN:

Taylor chose to start his analysis by considering the general array factor

$$A(\theta) = \int_{-a}^{a} g(f) e^{j k f \cos \theta} df$$

.............. (5.19)

Eqn (5.19) shows that the synthesis problem is one of finding aperture distribution of $g(f)$, given the desired pattern $A(\theta)$. When $g(f)$ has uniform amplitude and uniform progressive phase $g(f) = K e^{-j B f}$ with $K$ and $B$ constants and $e^{-j B f}$ corresponds to progressive phase difference. (5.19) gives

$$A(\theta) = \int_{-\frac{a}{2}}^{\frac{a}{2}} h(f) e^{j k (\cos \theta - B/f)} df$$

.............(5.20)

substituting $u = \frac{\theta}{A} (\cos \theta - B/f)$

$$A(u) = \int_{-\frac{\pi}{A}}^{\frac{\pi}{A}} h(u) e^{j \bar{n} u/a} f df$$

.............(5.21)

The variable $f$ along the length can be replaced by a normalized variable $p$, such that, $p = f/a$ and $dp = (\bar{n}/a) df$

Then $A(u) = \int_{-\frac{\pi}{A}}^{\frac{\pi}{A}} h(p) e^{j p u} dp$ 

.............(5.22)

The aperture distribution is determined by using Woodward's Sampling theorem. $h(p)$ can be expressed as the expansion of Fourier Series as
\[ h(p) = \sum_{m=0}^{\infty} B_m \cos mp \] ............(5.22)

so, \[ A(u) = \frac{a}{\pi} \sum_{m=0}^{\infty} B_m \int_{-\pi}^{\pi} \cos mp e^{ju} du \] ............(5.23)

Using orthogonality properties of the function in the integrand the integral becomes zero except at \( m = u \). So

\[ A(0) = 2aB_0, \quad A(m) = aB_m, \quad m = 1, 2, \ldots \] ............(5.24)

and \[ h(p) = \frac{1}{2a} \left[ A(0) + 2 \sum_{m=0}^{\infty} A(m) \cos mp \right] \] ............(5.25)

The array factor derived by Taylor can be written as

\[ A(u) = \frac{\sin \pi u}{\pi u} \prod_{n=1}^{\bar{n}-1} \left(1 - \frac{u^2}{U_n^2}\right) \] ............(5.26)

Where \((\bar{n} - 1)\) is number of equal height sidelobes in the innermost region. The above pattern is an even function in \( u \) and therefore provides symmetrical radiation pattern with a main beam in the center and the symmetrical sidelobe topography on either side of it. The denominator serves to erase first (innermost) nulls on each side of the center of the function for uniform distribution, \((\sin \pi u) / (\pi u)\). These nulls are replaced by the zeros of the numerator, whereas all the remaining nulls of the space factor \( A(u) \) occur exactly in similar fashion as that of \( \sin(\pi u) / \pi u \). Writing the numerator of the equation (5.26) again as

\[ \prod_{n=1}^{\bar{n}-1} \left[ 1.0 - \frac{u^2}{U_n^2} \right] \]

Where

\[ U_n^2 = \bar{n}^2 \left[ \frac{A^2 + (n-1/2)^2}{A^2 + (\bar{n}-1/2)^2} \right] \] ............(5.27)

\[ = \frac{\bar{n}^2}{A^2 + (n-1/2)^2} \left[ A^2 + (n-1/2)^2 \right] \] ............(5.28)

substituting

\[ \frac{\bar{n}^2}{A^2 + (n-1/2)^2} = \sigma^2 \] ............(5.29)

Since \( w = e^{j\chi} \) with \( \chi = kd(\cos \theta - \cos \theta_0) \)

\[ w = (2\pi/\lambda) * d(\cos \theta - \cos \theta_0) \]

Hence the nulls in the complex plane is defined as

\[ w_n = (\pi d/a)u_n \] where \( u_n \) is defined by 5.28

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By using the equation (5.25) the required aperture distribution can be calculated.

A typical Taylor pattern for \( \tilde{n} = 5 \) and SLL = 20 dB is shown in Fig(5.5a) and the corresponding aperture distribution is shown in Fig(5.5b).

By substituting the values of \( u_n \) from equation (5.29) in equation (5.30) we will get the null locations in the complex plane which will enable us to calculate the required pattern.

As a specific example, suppose a 19-element equispaced linear array is to be excited so as to produce the pattern as shown in Fig (5.6a) with an interelement spacing of \( 0.7\lambda \). The aperture distribution shown in Fig(5.6b) is sampled at 19 equal intervals and are given in column (1) of table (2). Using this current distribution in eqn (5.2) the corresponding pattern is as shown in Fig (5.7), which is not close enough to the desired one. By introducing the nulls \( w_n \) and factored out \( (w-w_n) \) the resulting current distribution can be obtained and are shown in column 2 of table (2). When this current distributions are used in equation (5.2) we will get the pattern as shown in Fig (5.8). It clearly indicates that for the case of small arrays the taylor's sampling method is not at all giving the desired pattern. By this we are able to overcome this difficulty.

The beamwidth in \( u \) domain is given by

\[
\beta = \frac{\tilde{n} B_0}{2/\lambda^2 + (n-1/2)^2} \quad \ldots \ldots \ldots (5.31)
\]

\[
= B_0 \quad \ldots \ldots \ldots (5.32)
\]

Where

\[
B_0 = (2/\pi) \times [(\text{arccosh} b)^2 - \text{arccosh}(b/\sqrt{2})^2]^{1/2} \quad \ldots \ldots (5.33)
\]

which is the beamwidth of the ideal Taylor pattern. When one goes to realizable pattern of Taylor, the beam broadening takes place. As can be seen from equation (5.33), the beam broadening is equal to the dilatation factor, which in turn depends upon the number of equal sidelobes chosen for the inner most region. Suitable choice of the \( n \) will provide the minimum beam width for the specified sidelobe levels. \( n \) must
be chosen such that a unit increase in \( n \) does not increase. Larger values of \( n \) will sharpen the main beam, but \( n \) should not exceed the visible range otherwise the effect of beam sharpening will be negligible.
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REFERENCES:


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FIG. 5.1 LINEAR ARRAY

FIG. 5.2 SCHEKUNOFF'S UNIT CIRCLE

FIG. 5.3 UNIFORM PATTERN
FIG. 5.4a  DOLPH'S PATTERN FOR 20 dB

FIG. 5.4b
\[ \tilde{n} = 5 \text{dB} \]
\[ \text{SLL} = 20 \text{dB} \]

\( u = \frac{2a}{\Lambda} (\cos \Theta) \)

**FIG. 5.5 a**

**FIG. 5.5 b**
Figure 5.6a

Figure 5.6b

Figure 5.6c
FIG. 5.7  INITIAL PATTERN

FIG. 5.8  FINAL PATTERN