Chapter 3

Spin precession of charged particles in local Lorentz frame

3.1 Formalism

In the previous chapter, we have seen that the purely geodetic effects do not appear in the spin precession frequency of a charged particle moving in presence of electromagnetic field in curved spacetime background (except in the case of the Ernst spacetime). Therefore, it is very much desirable to see if the purely geodetic terms appear when the present investigation is extended to local Lorentz frame. This is the subject of discussion in this chapter. We begin with a discussion of the local Lorentz tetrad as discussed by Weinberg (1972).

The principle of equivalence allows one to erect, at every point X, a set of coordinates $\psi_X^{(a)}$ which are locally inertial at that point. In any general non-inertial coordinate system, a metric is
\( g_{ik} = \lambda_{i}^{(a)} \lambda_{k}^{(b)} \eta_{ab} \) \hspace{1cm} (3.1.1)

where

\[
\lambda_{i}^{(a)} (X) \equiv \left\{ \frac{\partial \psi^{(a)}_{X}(x)}{\partial x^{i}} \right\}_{x=X} \hspace{1cm} (3.1.2)
\]

\( \lambda_{i}^{(a)} \) is a set of four linearly independent covariant vector fields which is called a tetrad. The index in the bracket is the tetrad index which runs from 0 to 3. These four-vectors, so long they are linearly independent, can be of any length and can have any angle among themselves.

Under a general non-inertial coordinate transformation, a tetrad components transform as following:

\[
\lambda_{i}^{(a)} (x) \rightarrow \lambda_{i}^{(a)} = \frac{\partial x^{k}}{\partial x^{a}} \lambda_{k}^{(a)} \hspace{1cm} (3.1.3)
\]

Consider a contravariant vector field \( V^{i} \), then the corresponding tetrad components are given by

\[
V^{(a)} = \lambda_{i}^{(a)} V^{i} \hspace{1cm} (3.1.4)
\]

\( V^{i} \) is a single four-vector whereas \( V^{(a)} \) are four scalars in a locally inertial frame. Similarly, the tetrad components of a covariant vector \( V_{i} \) are given
3.2 Calculations

We are interested in calculating the spin precession frequency for a charged particle in presence of electromagnetic field on curved spacetime background. We have considered a test particle to be moving in a circular orbit on the equatorial plane of the compact object. Further without loss
of generality, we have taken the $\theta$ component of the spin vector to be zero (see equations 2.4.28-31 and the discussion following this). As we are interested in finding the spin precession frequency in local Lorentz frame, we obtain local tetrad components of the spin vector through

$$
S^{(R)} = \lambda^{(R)}_R S^R,
$$

$$
S^{(\phi)} = \lambda^{(\phi)}_\phi S^\phi
$$

(3.2.10)

$S^\theta$ component has been considered to be zero. The tetrad components $\lambda^{(a)}_i$ are obtained through the relation

$$
g_{ij} \lambda^{(a)}_i \lambda^{(b)}_j = \eta_{(ab)}
$$

(3.2.11)

As we have already discussed in the last chapter that the polar coordinates have inherent rotation with respect to the Cartesian coordinates and therefore we rewrite the tetrad components in terms of their Cartesian components, which are given below:

$$
S^{(x)} = S^{(R)} \cos \phi - RS^{(\phi)} \sin \phi,
$$

$$
S^{(y)} = S^{(R)} \sin \phi + RS^{(\phi)} \cos \phi
$$

(3.2.12)

Using (2.4.32-33), (3.2.9), and (3.2.11), one rewrites the spin equations
in local Lorentz frame as:

\[
\frac{dS^{(x)}}{d\tau} = \frac{1}{R \lambda_R(R) \lambda^{(\phi)}_R} \left[ - S^{(x)} \cos \phi \sin \phi \Delta + S^{(y)} \{ \cos^2 \phi \Delta \\
- R \lambda_R^{(\phi)} \left( L_2 R \lambda^{(\phi)}_R + U^{(\phi)} \lambda^{(R)}_R \right) \} \right] \quad (3.2.13)
\]

and

\[
\frac{dS^{(y)}}{d\tau} = \frac{1}{R \lambda_R(R) \lambda^{(\phi)}_R} \left[ S^{(x)} \{ \cos^2 \phi \Delta + \lambda_R^{(R)} \left( R U^{(\phi)} \lambda^{(\phi)}_R - L_1 \lambda^{(R)}_R \right) \} \right. \\
\left. + S^{(y)} \cos \phi \sin \phi \Delta \right] \quad (3.2.14)
\]

with

\[
\Delta = L_1 \left( \lambda^{(R)}_R \right)^2 + L_2 R^2 \left( \lambda^{(\phi)}_R \right)^2 \quad (3.2.15)
\]

Solving them one gets the spin precession frequency in local Lorentz frame, which is given by

\[
\omega^2 = \left( U^{(\phi)} - \chi_1 \right) \left( U^{(\phi)} - \chi_2 \right) \quad (3.2.16)
\]

with

\[
\chi_1 = \frac{\lambda^{(R)}_R}{R \lambda^{(\phi)}_R} L_1,
\]

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where \( L_1 \) and \( L_2 \) are given by (2.4.34) and (2.4.35) respectively. Using the above, the spin precession frequency in local Lorentz frame for the physical situations discussed in section (2.3) have been evaluated (Virbhadra and Prasanna 1991) which are given in the following.

(a) Schwarzschild background with superposed magnetic field dipolar at infinity:

\[
\begin{align*}
\omega_s^2_{\text{local}} &= \left(\omega_s^2\right)_{\text{global}} + \left(\omega_s^2\right)_{\text{sch}} + \left\{\omega_s^2\right\}_{\text{geod}} + \frac{3e\mu F_R}{4m_0 M^2 R} \left(1 - \frac{2M}{R}\right)^{-1} U^\psi \\
&= 2g \left\{ \left(1 - \frac{2M}{R}\right)^{1/2} - \left(1 - \frac{M}{R}\right) \right\} \\
&\quad + (g - 2) (RU^\psi)^2 \left\{ \left(1 - \frac{2M}{R}\right)^{1/2} - 1 \right\}
\end{align*}
\]

(3.2.18)

where \( (\omega_s)_{\text{global}} \) and \( F_R \) are respectively given by (2.4.43) and (2.4.44). \( \left(\omega_s\right)_{\text{sch}} \) stands for the purely geodetic spin precession frequency in Schwarzschild
field, which is given as following:

\[
\{\omega_s^2\}^{\text{geod}}_{\text{Sch}} = \left[ \left(2 - \frac{3M}{R}\right) - \left(1 - \frac{2M}{R}\right)^{-1/2} \left(2 - \frac{5M}{R}\right) \right] U^\phi \quad (3.2.19)
\]

(b) Schwarzschild background with superposed uniform magnetic field:

The spin precession frequency of a charged particle in local Lorentz frame for the Wald case is given by

\[
[\omega_s^2]_{\text{local}} = \left(\omega_s^2\right)_{\text{global}} + \{\omega_s^2\}^{\text{geod}}_{\text{Sch}} + \frac{eB_0}{m_0} U^\phi \times \\
\left[ \left\{1 - \left(1 - \frac{2M}{R}\right)^{1/2}\right\} \left\{1 + \left(1 - \frac{2}{g}\right) \frac{R^2U^\phi}{2} \right\} - \frac{M}{R} \right]
\]

\[
(3.2.20)
\]

\( (\omega_s)_\text{global} \) for the Wald case is given by (2.4.46) and \( \{\omega_s\}^{\text{geod}}_{\text{Sch}} \) is given by (3.2.19).

(c) Reissner-Nordström field:

\[
[\omega_s^2]_{\text{local}} = \left(\omega_s^2\right)_{\text{global}} + \{\omega_s^2\}^{\text{geod}}_{R-N} + \frac{eQU^1}{2m_0R} U^\phi \times \\
\]

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where \((\omega_s)^{R-N}_{\text{global}}\) is given by (2.4.49) and \(\{\omega_s^2\}^{\text{geod}}_{R-N}\) is the purely geodetic spin precession frequency in the R-N field, which is given by

\[
\{\omega_s^2\}^{\text{geod}}_{R-N} = \left[ \left( 2 - \frac{3M}{R} + \frac{2Q^2}{R^2} \right) - \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right)^{-1/2} \left( 2 - \frac{5M}{R} + \frac{3Q^2}{R^2} \right) \right] U^2
\]

(d) Ernst field:

The spin precession frequency in the local Lorentz frame is given by

\[
[\omega_s^2]_{\text{local}} = \left( \frac{geB}{m_0 \Lambda^4} \right) \left[ \frac{geB}{m_0} \left( 1 - \frac{2M}{R} \right) + \frac{U^{(\phi)}}{\Lambda} \left( 1 - \frac{2M}{R} \right)^{1/2} \right]
\]

\[
+ \left( g - 2 \right) \frac{eB}{m_0 \Lambda^2} R^2 U^{(\phi)^2} \left( 1 - \frac{2M}{R} \right) + \left( 1 - \frac{2}{g} \right) \frac{R^2 U^{(\phi)^3}}{\Lambda^3}
\]

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\[
\times \left[ (4 - 3\Lambda) \left( 1 - \frac{2M}{R} \right) - \Lambda \left\{ \Lambda^2 \left( 1 - \frac{2M}{R} \right)^{1/2} + \frac{M}{R} \right\} \right] \\
+ \left\{ \omega^2 \right\}_{\text{Ernst local}}^{\text{geod}} \tag{3.2.23}
\]

where the last term \( \left\{ \omega^2 \right\}_{\text{Ernst local}}^{\text{geod}} \) stands for the purely geodetic precession frequency in the local Lorentz frame which is given by

\[
\left\{ \omega^2 \right\}_{\text{Ernst local}}^{\text{geod}} = \frac{U(\phi)^2}{\Lambda^3} \left[ \frac{(\Lambda - 2)(3\Lambda - 4)}{\Lambda^3} \left( 1 - \frac{2M}{R} \right) + \frac{M(\Lambda - 2)}{R\Lambda^5} + \Lambda^3 \right] \\
+ \left( 1 - \frac{2M}{R} \right)^{-1/2} \left\{ 2(2\Lambda - 3) \left( 1 - \frac{2M}{R} \right) + \frac{M\Lambda}{R} \right\} \\
\tag{3.2.24}
\]

with

\[
U(\phi) = \frac{U^\phi}{\Lambda} \tag{3.2.25}
\]

### 3.3 Discussion

Expressed through equations (3.2.18) to (3.2.25) are spin precession frequencies in local Lorentz frames for different cases. Switching off the charge parameter of the test particle, one had noted in the previous chapter that the expressions for the spin precession frequencies in the global frame for
the first three cases vanish whereas for the case of the Ernst spacetime, one gets non-vanishing terms given by (2.4.52). However, in local Lorentz frames one gets purely geodetic terms (along with other terms) in all these cases given by (3.2.19), (3.2.22)(3.2.24). It is clear that $Q = 0$ in (3.2.22) as well as $\Lambda = 1$ in (3.2.24) give (3.2.19) as expected. Both in Schwarzschild and R-N geometries, which are asymptotically flat, the geodetic terms explicitly manifest only in local frames whereas in the Ernst geometry (which is not asymptotically flat), the purely geodetic term appears even in global frame. In the Reissner-Nordström and the Ernst spacetimes, the electrostatic and the magnetostatic fields respectively contribute to the curvature of the spacetime which are clearly reflected in these results. It is interesting to note that the purely geodetic spin precession frequencies are found to be proportional to the respective orbital frequencies of the test particles for all the cases we have investigated.

It would indeed be interesting to extend the calculations to the case of non-circular orbits as well as for orbits off the equatorial plane. As we have neglected the non-linear spin-orbit coupling terms, it could be interesting to see the effects on the spin precession when these are included. Moreover, it is not clear to us why the purely geodetic terms do not appear in the expressions for the spin precession frequency in global frame except in the case of the Ernst field. This requires further serious investigations.