CHAPTER 5

EXISTENCE OF COMMON FIXED POINT FOR A PAIR OF MAPPINGS WITH LIPSCHITZIAN ITERATES

(51-68)
CHAPTER - 5

EXISTENCE OF COMMON FIXED POINT
FOR A PAIR OF MAPPINGS WITH LIPSCHITZIAN ITERATES

In this chapter, an existence of fixed point in $p$-uniformly convex Banach space is established for a pair of mappings having Lifshitz norm. Subsequently, some existence results on fixed point in a Hilbert space, in $L^p$ spaces, in Hardy spaces $H^p$ and in Sobolev spaces $W^{p,k}$, for $1 < p < +\infty$ and $k \geq 0$, are obtained as application of our main result.

In the remark of the definition 2.3 of chapter-2, we have given the definition of Lipschitzian mapping for single-valued mappings as below:

Let $D$ be nonempty (generally bounded, closed and convex) subset of Banach space $X$ and $T: D \to D$. Thus $T$ is $k$-lipschitzian if

$$\|Tx - Ty\| \leq k\|x - y\|$$

for all $x, y \in D$ and $k > 0$.

Now, we denote by $\|T^n\|$ the Lipschitz norm of $T^n$, $n = 1, 2, \ldots$, i.e.

$$\|T^n\| = \sup \left\{ \frac{\|T^n x - T^n y\|}{\|x - y\|} : x \neq y, x, y \in D \right\}.$$
T is said to be uniformly $k$-Lipschitzian if $\|T^n\| = k$ for all $n \geq 1$ and $T$ is nonexpansive if $k = 1$.

For the class of nonexpansive mappings, following is the well known existence result [70]:

**THEOREM 5.1** (Browder [15a], Gohde [38a], Kirk [60]). Let $D$ be a nonempty bounded closed convex subset of a uniformly convex Banach space $X$. If $T: D \rightarrow D$ is nonexpansive then it has fixed point.

However, it is possible to find examples for which fixed point free nonexpansive self-mappings exists. Even, if $B$ is the closed unit ball $\ell^2$, $\epsilon > 0$, $e_1 = (1,0,0,...)$ and $S$ is the right shift operator, then the mapping $T: B \rightarrow B$ defined by $Tx = \epsilon(1-\|x\|)e_1 + Sx$ is a fixed point free mapping having Lipschitz constant $1 + \epsilon$. This example shows that **Theorem 5.1** may fail to hold for the class of mapping $T$ having Lipschitz constant $k > 1$, no matter how near to 1 we choose $k$.

Later, the concept of uniformly $k$-Lipschitzian was introduced by Goebel and Kirk [36] in 1973 for the first time and existence of fixed point for uniformly $k$-Lipschitzian mapping was given by Goebel and Kirk [36] in uniformly convex Banach space [101]. A different and more general approach was proposed by Lifshitz [65] and recently by Dominguez Benavides et al. (cf. [8], [9], [10]). In particular, Lifshitz [65] established the following:
THEOREM 5.2 [65] (Lifshitz 1975). Let \( D \) be a nonempty bounded closed convex subset of a Hilbert space and suppose \( T:D \rightarrow D \) is uniformly \( k \)-Lipschitzian for \( k < \sqrt{2} \). Then \( T \) has a fixed point in \( D \).

Lifshitz [65] found an example of a fixed point free uniformly \( \frac{\pi}{2} \)-Lipschitzian mapping which leaves invariant bounded closed convex subset of \( l^2 \) (cf. [3], [38]). However the validity of Theorem 5.2 for \( \sqrt{2} \leq k < \frac{\pi}{2} \) remains open.

The existence of a fixed point of a uniformly \( k \)-Lipschitzian mapping has been widely investigated by many authors (cf. [3], [6], [9], [10], [21], [43], [65], [67], [68], [77], [87], [96], [98], [101], [109]).

Let \( p > 1 \) and denote by \( \lambda \) the number in \([0,1]\) and by \( W_p(\lambda) \) the function \( \lambda \cdot (1 - \lambda)^p + \lambda^p \cdot (1 - \lambda) \).

The functional \( \| \cdot \|_p \) is said to be uniformly convex (c.f. Zalinescu [115]) on the Banach space \( X \) if

\[
\| \lambda x + (1 - \lambda) y \|_p^p \leq \lambda \| x \|_p^p + (1 - \lambda) \| y \|_p^p - W_p(\lambda) \cdot c_p \cdot \| x - y \|_p^p.
\]

Xu [109] proved that the functional \( \| \cdot \|_p \) is uniformly convex on the whole Banach space \( X \) if and only if \( X \) is \( p \)-uniformly convex, i.e. there exists a constant
such that the moduli of convexity (see [38]),
\[ \delta_X(\varepsilon) \geq c_p \varepsilon^p \]
for all \(0 \leq \varepsilon \leq 2\).

Further, Gornicki has [40] proved following theorem in \(p\)-uniformly convex Banach space via Banach space inequalities:

**THEOREM 5.3.** Let \(p > 1\) and let \(X\) be a \(p\)-uniformly convex Banach space, \(D\) a nonempty bounded closed convex subset of \(X\), and \(A = [a_{n,k}]_{n,k=1}^{\infty}\) a strongly ergodic matrix. If \(T : D \to D\) is a mapping such that

\[
g = \lim \inf_{l \to \infty} \inf_{m=0,1,2,\ldots} \sum_{k=1}^{\infty} a_{n,k} \| T^{k+m} x \| < 1 + c_p,
\]

then \(T\) has a fixed point in \(D\).

Here \(A = [a_{n,k}]_{n,k=1}^{\infty}\) is a strongly ergodic matrix [17]:

\[(a_1) \quad \Lambda_{n,k} a_{n,k} = 0,\]

\[(a_2) \quad \Lambda_k \lim_{n \to \infty} a_{n,k} = 0,\]

\[(a_3) \quad \Lambda_k \sum_{k=1}^{\infty} a_{n,k} = 1,\]

\[(a_4) \quad \lim_{n \to \infty} \sum_{k=1}^{\infty} |a_{n,k+1} - a_{n,k}| = 0.\]

Now, in this chapter, we shall prove existence of fixed point for a pair of mappings with Lipschitz iterate by means of techniques of asymptotic center and inequalities in Banach spaces. We extend the results of [40, 44] and others.
5.1 COMMON FIXED POINTS FOR PAIRS OF LIPSCHITZIAN MAPPINGS

In this section, we establish existence of common fixed point for a pair of mappings with Lipschitzian iterates.

Before presenting our main result, we need the following:

We denote by $\|L^n\|$ the Lipschitz norm of a pair $\{s^n, t^n\}$, $n = 1, 2, \ldots$, i.e.

$$\|L^n\| = \sup \left\{ \frac{\|s^n_x - t^n_y\|}{\|x - y\|} : x \neq y, x, y \in X \right\}.$$  

**Lemma 5.1.1 [40].** Let $p > 1$ and let $X$ be a $p$-uniformly convex Banach space, $D$ a nonempty closed convex subset of $X$ and let $\{x_n\} \subset X$ be a bounded sequence. Then there exists a unique point $z$ in $D$ such that

$$\lim_{n \to \infty} \sup \sum_{k=1}^{\infty} a_{n,k} \|x_k - z\|^p \leq \lim_{n \to \infty} \sup \sum_{k=1}^{\infty} a_{n,k} \|x_k - x\|^p - c_p \|x - z\|^p \quad (5.1.1)$$

for every $x$ in $D$, where $c_p > 0$ is the constant given in (5.1.1) and $A = [a_{n,k}]_{n,k=1}$ is a strongly ergodic matrix.
Now, we prove our main result as below:

**THEOREM 5.1.1.** Let $p > 1$, $X$ be a $p$-uniformly convex Banach space and let $D$ be a nonempty bounded closed convex subset of $X$ and $A = [a_n,k]_{n,k} = 1$ a strongly ergodic matrix. If $S,T:D \to D$ be two mappings such that

$$\lim \sup_{n \to \infty} \|T^n x - S^n x\| = 0$$

and

$$g = \lim \inf_{i \to \infty} \inf_{m=0,1,2,\ldots} \sum_{k=1}^{\infty} a_{n,k} \|L^{k+m}p\| < 1 + C_p,$$

Then $S$ and $T$ have a common fixed point in $D$.

**PROOF.** Let $\{n_i\}$ and $\{m_i\}$ be the sequences of natural numbers such that

$$g = \lim \inf_{i \to \infty} \inf_{m=0,1,2,\ldots} \sum_{k=1}^{\infty} a_{n_i,k} \cdot \|L^{k+m}_i p\|.$$

For any $z_0 \in D$, we can inductively define a sequence $\{z_j\}$ in the following manner:

$z_j$ and $z_{j+1}$ are unique asymptotic centers of the sequences $\{T^n z_{j-1}\}_{n=1}$ and $\{S^n z_j\}_{n=1}$ respectively, i.e., $z_j$ and $z_{j+1}$ are the unique points in $D$ that minimizes the functionals respectively:

$$r_{j-1}(x) = \lim \sup_{i \to \infty} \sum_{k=1}^{\infty} a_{n_i,k} \cdot \|x - T^{k+m_i} z_{j-1}\|$$

and

$$D_j(x) = \lim \sup_{i \to \infty} \sum_{k=1}^{\infty} a_{n_i,k} \cdot \|x - S^{k+m_i} z_j\|,$$

over $x$ in $D$. From the construction of $\{z_j\}$, it follows that

$$r_{j-1}(z_j) \leq r_{j-1}(z_{j-1})$$
and

\[ r_{j-1}(z_j) \leq D_{j-1}(z_j) \leq D_j(z_j) \leq D_j(z_{j-1}) \].

In view of inequality (5.1.1) for any fixed \( N, k, m_j \in \mathbb{N} \), and \( 0 \leq \lambda \leq 1 \), we have

\[
\|\lambda z_j + (1 - \lambda) \cdot s^N z_j - T^{k+m_i} z_{j-1}\|_p
\]
\[
= \|\lambda (z_j - T^{k+m_i} z_{j-1}) + (1 - \lambda) (s^N z_j - T^{k+m_i} z_{j-1})\|_p
\]
\[
\leq \lambda \cdot \| z_j - T^{k+m_i} z_{j-1}\|_p + (1 - \lambda) \cdot \| s^N z_j - T^{k+m_i} z_{j-1}\|_p
\]
\[
- c_p \cdot \| \lambda z_j - s^N z_j \|_p.
\]

Multiply both sides of this inequality by suitable elements of the matrix \( A \) and summing, we have

\[
\sum_{k=1}^{\infty} a_{n_i,k} \cdot \|\lambda z_j + (1 - \lambda) \cdot s^N z_j - T^{k+m_i} z_{j-1}\|_p
\]
\[
\leq \lambda \cdot \sum_{k=1}^{\infty} a_{n_i,k} \cdot \| z_j - T^{k+m_i} z_{j-1}\|_p
\]
\[
+ (1 - \lambda) \cdot \sum_{k=1}^{\infty} a_{n_i,k} \cdot \| s^N z_j - T^{k+m_i} z_{j-1}\|_p
\]
\[
- c_p \cdot \| \lambda z_j - s^N z_j \|_p \quad \text{for } i = 1, 2, \ldots.
\]

Taking the limit superior on each side as \( i \to \infty \), we get

\[
\limsup_{i \to \infty} \sum_{k=1}^{\infty} a_{n_i,k} \cdot \|\lambda z_j + (1 - \lambda) \cdot s^N z_j - T^{k+m_i} z_{j-1}\|_p
\]
\[
\leq \lambda \cdot \limsup_{i \to \infty} \sum_{k=1}^{\infty} a_{n_i,k} \cdot \| z_j - T^{k+m_i} z_{j-1}\|_p
\]
\[
+ (1 - \lambda) \cdot \limsup_{i \to \infty} \sum_{k=1}^{\infty} a_{n_i,k} \cdot \| s^N z_j - T^{k+m_i} z_{j-1}\|_p
\]
\[
- c_p \cdot \| \lambda z_j - s^N z_j \|_p,
\]
and
Taking $\lambda = 1$, we get

$$c_p \cdot \|z_j\|_p = \lim \sup \sum_{k=1}^{\infty} a_{n_i,k} \cdot \|S^N z_j\|_p - \|k^m z_j - T^k z_j\|_p.$$
\[
\begin{align*}
&\limsup_{i \to \infty} \left\{ \sum_{k=1}^{N} a_{n_i, k} \cdot \|S^{N} z_j - T^{k+m_j} z_{j-1}\|^p \right. \\
&\quad + \|L\|^p \left. \cdot \sum_{k=1}^{\infty} a_{n_i, k+1} \cdot \|z_j - T^{k+m_j} z_{j-1}\|^p \right\} \\
&\quad - \limsup_{i \to \infty} \sum_{k=1}^{\infty} a_{n_i, k} \cdot \|z_j - T^{k+m_j} z_{j-1}\|^p \\
&\quad + \|L\|^p \cdot \left\{ \sum_{k=1}^{\infty} a_{n_i, k} \cdot \|z_j - T^{k+m_j} z_{j-1}\|^p \\
&\quad - \sum_{k=1}^{\infty} (a_{n_i, k} - a_{n_i, k+1}) \cdot \|z_j - T^{k+m_j} z_{j-1}\|^p \right\} \right. \\
&\quad - \limsup_{i \to \infty} \sum_{k=1}^{\infty} a_{n_i, k} \cdot \|z_j - T^{k+m_j} z_{j-1}\|^p \\
&\quad \leq (\|L\|^p - 1) \limsup_{i \to \infty} \sum_{k=1}^{\infty} a_{n_i, k} \cdot \|z_j - T^{k+m_j} z_{j-1}\|^p \\
&\quad = (\|L\|^p - 1) \cdot r_{j-1}(z_j) \\
&\quad \leq (\|L\|^p - 1) \cdot D_{j-1}(z_{j-1}),
\end{align*}
\]

since

\[\begin{align*}
(b_1) & \quad \sum_{k=1}^{N} a_{n_i, k} \cdot \|S^{N} z_j - T^{k+m_j} z_{j-1}\|^p \to 0 \text{ as } i \to +\infty, \\
(b_2) & \quad \sum_{k=1}^{\infty} (a_{n_i, k} - a_{n_i, k+1}) \cdot \|z_j - T^{k+m_j} z_{k-1}\|^p \to 0, \text{ as } i \to +\infty, \\
(b_3) & \quad r_{j-1}(z_j) \leq D_{j-1}(z_{j-1}).
\end{align*}\]
Therefore, for any fixed $N \in \mathbb{N}$, we have
\[
\|z_j - S^N z_j\|_P \leq (\|L\|_P^N - 1) \cdot D_{j-1}(z_{j-1}).
\]
This inequality for $N = k + m_i$, we multiply by suitable element $a_{n_i,k}$ for $k = 1, 2 \cdots$. Summing up these inequalities and taking the limit superior on each side as $i \to +\infty$, we obtain
\[
c_p \limsup_{i \to \infty} \sum_{k=1}^{\infty} a_{n_i,k} \cdot \|z_j - S^{k+m_i} z_j\|_P
\]
\[
\leq \left[ \lim_{i \to \infty} \sum_{k=1}^{\infty} a_{n_i,k} \cdot \|L^{k+m_i}\|_P - 1 \right] D_{j-1}(z_{j-1})
\]
and
\[
D_j(z_j) \leq B \cdot D_{j-1}(z_{j-1})
\]
where $B = \frac{1}{c_p} \left[ \lim_{i \to \infty} \sum_{k=1}^{\infty} a_{n_i,k} \cdot \|L^{k+m_i}\|_P - 1 \right] < 1$.

In a similar way, we obtain
\[
D_j(z_j) \leq B^j D_0(z_0), \quad j = 1, 2, \ldots.
\]
further, we set
\[
r_{j-1}^*(x) = \limsup_{i \to \infty} \sum_{k=1}^{\infty} a_{n_i,k} \cdot \|x - S^{k+m_i} z_{j-1}\|_P
\]
and
\[
D_j^*(x) = \limsup_{i \to \infty} \sum_{k=1}^{\infty} a_{n_i,k} \cdot \|x - T^j z_j\|_P.
\]
Repeating the above arguments, we obtain
\[
D_j^*(z_j) \leq B^j D_0(z_0).
\]
Now, we show the convergence of the sequence $\{z_j\}$. For
a fixed \( N \in \mathbb{N} \), we have \([64, \text{p.} 183]\)

\[
\|z_{j+1} - z_j\| \leq 3^{p-1} (\|z_{j+1} - T^N z_j\| + \|T^N z_j - S^N z_j\| + \|S^N z_j - z_j\|)
\]

\[
\leq 3^{p-1} \left( (1+2^p) \|z_{j+1} - T^N z_j\| + 2^{p-1} \|z_{j+1} - T^N z_j\| + \|S^N z_j - z_j\| \right).
\]

We multiply this inequality for \( N = k + m_i \) by elements \( a_{n_i,k} \) for \( k = 1, 2, \ldots \). Summing up these inequalities and taking the limit superior on each side as \( i \to \infty \), we obtain

\[
\|z_{j+1} - z_j\| \leq 3^{p-1} \left( (1+2^p) \limsup_{i \to \infty} \sum_{k=1}^{\infty} a_{n_i,k} \|z_{j+1} - T^{k+m_i} z_j\| + 2^{p-1} \limsup_{i \to \infty} \sum_{k=1}^{\infty} a_{n_i,k} \|z_{j+1} - S^{k+m_i} z_j\| + \limsup_{i \to \infty} \sum_{k=1}^{\infty} a_{n_i,k} \|z_j - S^{k+m_i} z_j\| \right)
\]

\[
= 3^{p-1} \left( (1+2^p-1) r_j(z_{j+1}) + (1+2^{p-1}) D_j(z_{j+1}) + D_j(z_j) \right)
\]

\[
\leq 3^p (1+2^{p-1}) D_j(z_j)
\]

\[
\leq 3^p (1+2^{p-1}) B^j D_0 (z_0).
\]

Thus
\[ \|z_{j+1} - z_j\| \leq 3 (1 + 2^{p-1}) \left[ B^j \cdot D_0(z_0) \right]^{\frac{1}{P}} \rightarrow 0 \text{ as } j \rightarrow +\infty, \]

which show that \( \{z_j\} \) is a Cauchy sequence. Let 
\[ z = \lim_{j \to \infty} z_j. \]

For fixed \( N \in \mathbb{N} \), we have 
\[ \|z - T^N z\|^p \leq (\|z - z_j\|^p + \|z_j - S^N z_j\|^p + \|S^N z_j - T^N z\|^p) \]
\[ \leq 3^{p-1} \cdot (\|z - z_j\|^p + \|z_j - S^N z_j\|^p + \|S^N z_j - T^N z\|^p). \]

This inequality for \( N = k + m_i \), we multiply by suitable element \( a_{n_i, k} \) for \( k = 1, 2, \ldots \). Summing up this inequality we have 
\[ \sum_{k=1}^{\infty} a_{n_i, k} \cdot \|z - T^{k+m_i} z\|^p \leq 3^{p-1} \left\{ \|z - z_j\|^p \right. \]
\[ + \sum_{k=1}^{\infty} a_{n_i, k} \cdot \|z_j - S^{k+m_i} z_j\|^p \]
\[ + \|z - z_j\|^p \cdot \sum_{k=1}^{\infty} a_{n_i, k} \cdot \|L^{k+m_i}\|^p \left\} \right. \]
for \( i = 1, 2, \ldots \).

Taking the limit superior on each side as \( i \rightarrow +\infty \), we get 
\[ \limsup_{i \to \infty} \sum_{k=1}^{\infty} a_{n_i, k} \cdot \|z - T^{k+m_i} z\|^p \]
\[ \leq 3^{p-1} \left\{ \|z - z_j\|^p + \limsup_{i \to \infty} \sum_{k=1}^{\infty} a_{n_i, k} \cdot \|z_j - S^{k+m_i} z_j\|^p \right. \]
\[ + \|z - z_j\|^p \cdot \lim_{i \to \infty} \sum_{k=1}^{\infty} a_{n_i, k} \cdot \|L^{k+m_i}\|^p \left\} \right. \]
\[ \leq 3^{p-1} \left[ (1 + g) \cdot \|z - z_j\|^p + B^j \cdot D_0(z_0) \right] \rightarrow 0 \]
as \( j \rightarrow +\infty \). Therefore, 
\[ \limsup_{i \to \infty} \sum_{k=1}^{\infty} a_{n_i, k} \cdot \|z - T^{k+m_i} z\|^p = 0. \]
This implies that $Tz = z$. By symmetry, we have $Sz = z$.

Hence $Tz = z = Sz$. This completes the proof.

In **Theorem 5.1.1**, if we put $S = T$, then we will have the following:

**COROLLARY 5.1.1** [40, Theorem 2]. Let $p > 1$ and let $X, D$ be as in **Theorem 5.1.1** and $T : D \to D$ such that

$$g = \lim \inf \inf_{n \to \infty} m = 0, 1, 2, \ldots \sum_{k=1}^{m} a_{n,k} \| T^{k+m} \| p < 1 + c_p,$$

Then $T$ has a fixed point in $D$.

### 5.2 APPLICATIONS

In a Hilbert space $H$, following inequality holds:

$$\| \lambda x + (1-\lambda) y \|^2 = \lambda \| x \|^2 + (1-\lambda) \| y \|^2 - \lambda (1-\lambda) \| x - y \|^2$$

(5.2.1)

for all $x, y$ in $H$ and $\lambda \in [0,1]$.

Thus by (5.2.1), we immediately obtain from **Theorem 5.1.1**, the following:

**THEOREM 5.2.1.** Let $D$ be a nonempty bounded closed convex subset of a Hilbert space $H$, and $A = \{a_{n,k}\}_{n,k \geq 1}$ a strongly ergodic matrix.

If $S, T : D \to D$ are mappings such that

$$\lim \sup_{n \to \infty} \| T^n x - S^n x \| = 0 \quad \text{for all } x \in D$$

and

63
\[ g = \lim_{n \to \infty} \inf_{m=0,1,2,\ldots} \inf_{k=1} \sum a_{n,k} \| T^{k+m} \| < \sqrt{2}. \]

Then \( S \) and \( T \) have a common fixed point in \( D \).

If we put \( S = T \) in Theorem 5.2.1, we have the following corollary:

**COROLLARY 5.2.1 [40, corollary 1]**. Let \( D \) and \( A \) be as in Theorem 5.2.1. If \( T : D \to D \) such that

\[ g = \lim_{n \to \infty} \inf_{m=0,1,2,\ldots} \inf_{k=1} \sum a_{n,k} \| T^{k+m} \| < \sqrt{2}, \]

then \( T \) has a fixed point in \( D \).

If \( 1 < p \leq 2 \), then we have for all \( x,y \) in \( L^p \) and \( \lambda \in [0,1] \),

\[ \| \lambda x + (1-\lambda)y \|^p \leq \lambda \| x \|^p + (1-\lambda) \| y \|^p \]

\[ - \lambda (1-\lambda) (p-2) \| x-y \|^p. \tag{5.2.2} \]

(The inequality (5.2.2) is contained in Lim, Xu and Xu [68] and Smarzewski [98]).

Assume that \( 2 < p < +\infty \) and \( t_p \) is the unique zero of the function \( g(x) = -x^{p-1} + (p-1)x + p - 2 \) in the interval \( (1, +\infty) \). Let

\[ c_p = (p-1)(1+t_p)^{p-2} = \frac{1 + t_p}{(1 + t_p)^{p-1}} \]

and we have the following inequality:

\[ \| \lambda x + (1-\lambda)y \|^p \leq \lambda \| x \|^p + (1-\lambda) \| y \|^p \]

\[ - \| \lambda x + (1-\lambda)y \|^p \leq \lambda(1-\lambda)(p-1) \| x-y \|^p \tag{5.2.3} \]

for all \( x,y \) in \( L^p \) and \( \lambda \in [0,1] \).

(The inequality (5.2.3) is essentially due to Lim [67]).
From (5.2.2) and (5.2.3), we obtain from Theorem 5.1.1, the following:

**Theorem 5.2.2.** Let \( D \) be a nonempty bounded closed convex subset of \( L^p (1 < p < +\infty) \) and \( A = [a_{n,k}]_{n,k \geq 1} \) a strongly ergodic matrix. If \( S, T : D \rightarrow D \) such that

\[
\lim_{n \to \infty} \sup_{x \in D} \| T^n x - S^n x \| = 0
\]

and

\[
g = \liminf_{n \to \infty} \inf_{m=0,1,2,\ldots} \sum_{k=1}^{\infty} a_{n,k} \| T^k \|^{k+m} < p, \text{ if } 1 < p \leq 2
\]

and

\[
g = \liminf_{n \to \infty} \inf_{m=0,1,2,\ldots} \sum_{k=1}^{\infty} a_{n,k} \| T^k \|^{k+m} < 1 + \epsilon p, \text{ if } p > 2,
\]

then \( S \) and \( T \) have a common fixed point in \( D \).

If we put \( S = T \) in Theorem 5.2.2, then we will have the following result:

**Corollary 5.2.2** [40, Corollary 2.3]. Let \( D \) and \( A \) be as in Theorem 5.2.2 and if \( T : D \rightarrow D \) with

\[
g = \liminf_{n \to \infty} \inf_{m=0,1,2,\ldots} \sum_{k=1}^{\infty} a_{n,k} \| T^k \|^{k+m} < p, \text{ if } 1 < p \leq 2
\]

and

\[
g = \liminf_{n \to \infty} \inf_{m=0,1,2,\ldots} \sum_{k=1}^{\infty} a_{n,k} \| T^k \|^{k+m} < 1 + \epsilon p, \text{ if } p > 2,
\]

then \( T \) has a fixed point in \( D \).
5.3 COROLLARIES IN OTHER BANACH SPACES

Let $H^p$, $1 < p < +\infty$, denote the Hardy space [33] of all functions $x$ analytic in the unit disc $|z| < 1$ of the complex plane and such that

$$
\|x\| = \lim_{r \to 1^-} \left( \frac{1}{2\pi} \int_0^{2\pi} |x(re^{i\theta})|^p \, d\theta \right)^{1/p} < +\infty.
$$

Now, let $\Omega$ be an open subset of $\mathbb{R}^n$. Denote by $H^{r,p}(\Omega)$, $r \geq 0$, $1 < p < +\infty$, the Sobolev space [6, p.149] of distributions $x$ such that $D^\alpha x \in L^p(\Omega)$ for all $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq k$ equipped with the norm

$$
\|x\| = \left( \sum_{|\alpha| \leq k} \int_\Omega |D^\alpha x(\omega)|^p \, d\omega \right)^{1/p}.
$$

Let $(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$, $\alpha \in \Lambda$, be a sequence of positive measure spaces, where index set $\Lambda$ is finite or countable. Given a sequence of linear sub-spaces $X_\alpha$ in $L^p(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$, we denote by $L^{q,p}, 1 < p < +\infty$ and $q = \max(2, p)$ [70], the linear space of all sequences $x = \{x_\alpha \in X_\alpha : \alpha \in \Lambda\}$ equipped with the norm

$$
\|x\| = \left( \sum_{\alpha \in \Lambda} (\|x_\alpha\|_{L^p(\mu_\alpha)})^q \right)^{1/q},
$$

where $\|\cdot\|_{L^p(\mu_\alpha)}$ denotes the norm in $L^p(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$.

Finally, let $L^p = L^p(S_1, \Sigma_1, \mu_1)$ and $L^q = L^q(S_2, \Sigma_2, \mu_2)$. 

66
where $1 < p < +\infty$, $q = \max(2, p)$ and $(S_1, \Sigma_1, \mu_1)$ are positive measure spaces. Denote by $L_q(L_p)$ the Banach spaces [31, III.2.10] of all measurable $L_p$-value function $x$ on $S_2$ such that

$$
\|x\| = \left( \int_{S_2} \left( \|x(s)\|_p \right)^q \mu_2(ds) \right)^{1/q}.
$$

These spaces are $q$-uniformly convex with $q = \max(2, p)$, [37, 97] and the norm in these spaces satisfies,

$$
\|\lambda x + (1 - \lambda)y\|_q \leq \lambda\|x\|_q + (1 - \lambda)\|y\|_q - d \cdot \mathcal{W}_q(\lambda) \cdot \|x - y\|_q
$$

with a constant

$$
d = d_p = \begin{cases} 
\frac{p-1}{8} & \text{if } 1 < p \leq 2, \\
\frac{1}{p \cdot 2^p} & \text{if } 2 < p < +\infty.
\end{cases}
$$

Hence, from Theorem 5.1.1, we have the following results:

**THEOREM 5.3.1.** Let $D$ be a nonempty bounded closed convex subset of the space $X$, where $X = H^P$, or $X = H^{r, p}(\Omega)$, or $X = L_q, p$, or $X = L_q, (L_p)$, and $1 < p < +\infty$, $q = \max(2, p)$, $r \geq 0$, and let $A = \{a_{n,k}\}_{n,k \geq 1}$ a strongly ergodic matrix. If $S, T : D \rightarrow D$ such that

$$
\lim_{n \rightarrow \infty} \sup_{x \in D} \|T^nx - S^nx\| = 0
$$

and

67
\[ g = \lim \inf_{n \to \infty} \inf_{m=0,1,2, \ldots} \sum_{k=1}^{\infty} a_n, k \cdot \|L_k^{m+q} \| < 1 + d_p, \]

then \( S \) and \( T \) have a common fixed point in \( D \).

If we put \( S = T \) in Theorem 5.3.1, then we will have the following:

**COROLLARY 5.3.1 [40, Corollary 4].** Let \( D \) and \( A \) be as in Theorem 5.3.1 and if \( T : D \to D \) such that

\[ g = \lim \inf_{n \to \infty} \inf_{m=0,1,2, \ldots} \sum_{k=1}^{\infty} a_n, k \cdot \|T_k^{m+q} \| < 1 + d_p, \]

then \( T \) has a fixed point in \( D \).