CHAPTER 1
INTRODUCTION

1.1 INTRODUCTION

Fixed point theory is the cardinal branch of nonlinear analysis which has given great effects on the advancement in the branch of linear and nonlinear analysis. Nonlinear analysis was developed in 1950’s, when Browder worked in the combination of functional analysis and variational analysis. However, the earlier results has already been obtained in the 1920’s (see [4] ). Several problems in Physics, Chemistry, Biology and Economics leads to nonlinear models. Nonlinear differential and integral equations, variational inequalities and more general optimization problems are some of the important subjects in nonlinear analysis.

Let $X$ be a set and $F : X \rightarrow X$ be a mapping. A solution of the equation $F(x) = x$ is called a fixed point of $F$. Theorems dealing with existence and construction of solution to the operator equation $F(x) = x$ form a part of fixed point theorem. Fixed points have been used in analysis to solve various kinds of integral and differential equations. For example let $X = C[a,b]$ denote the set of all continuous real valued functions on $[a,b]$. Consider a function
\[ f : [a, b] \times [a, b] \times R \rightarrow R. \] Assume that for every fixed \( x \in X \) and \( s \in [a, b] \), the function \( f(s, t, x(t)) \) is an integrable function of \( t \) on \([a, b]\) and that for every fixed \( x \in X \) the function \( \int_a^b f(s, t, x(t)) \, dt \) is a continuous function of \( s \) on \([a, b]\). Let \( x_0 \in X \) be given and consider the integral equation,

\[
x(s) = x_0(s) + \int_a^b f(s, t, x(t)) \, dt, \quad x \in X
\]

If we define \( F : X \rightarrow X \) by

\[
F(x)(s) = x_0(s) + \int_a^b f(s, t, x(t)) \, dt, \quad x \in X, \quad s \in [a, b]
\]

then \( a \in X \) is a fixed point of \( F \) if and only if \( a \in X \) is a solution of the above integral equation. The solution of such integral equation can often be used to solve various differential equations.

The origin of fixed point theory rests in the use of successive approximations to the existence and uniqueness of solutions particular to differential equations. For more details one can refer [12], [64], [65], [74], [75] and [76].

However, S. Banach ([4],[5]), has given an abstract framework for broad applications well beyond the scope of elementary differential and integral equations, in which he recognized the fundamental role of metric completeness. In the mid 1960’s, as a result of pioneering work of Browder ([8], [9]) gained a new motivation with the development of nonlinear functional analysis which is an active and vital branch of Mathematics. The main work in this de-
development were the existence theorem of Browder ([6], [7]), Gohde [39] and Kirk ([56], [57]) and the early metric results of Edelstein ([24], [25]). Some applications of fixed point theorems in Theoretical Economics has been obtained by Kakutani.

The contributions of fixed point theory are mainly that a function will have i) a fixed point or not (existence) ii) a unique fixed point, under some condition on the function that can be stated in general terms. The existence of a fixed point will depend on the nature of the space and the type of the operator. There may be more than one mapping from a set into itself and the question is whether these mappings have a common fixed point or not.

The amount of research and investigations of fixed point theory has greatly increased in 1970’s. Fixed point theorems using more generalized contractive mappings were done by several authors namely Caristi [11], Chatterjea [13], Ciric [19], Reich [80] and Sehgal [84].

In 1980’s Sessa [85] and Jungck ([50], [51]) used the notions of weak commutativity and compatible maps to obtain common fixed points. This was turning point in the theory and a number of interesting results have been found by various researchers (see [52], [78]). In this connection Park and Bae, Fisher, Kang and Cho, Murthy and Pathak used the improved notions of compatibility of mappings and obtained common fixed point theorems. Later there had been progress in various directions such as set valued maps,
partially ordered metric space, Menger spaces etc.

Meanwhile, the notion of 2-metric space was investigated by Gahler in 1963. A number of authors have studied the aspects of fixed point theory based on 2-metric space. They have been motivated by various concepts already known for ordinary metric spaces and have thus introduced analogues of various concepts in the framework of the 2-metric spaces. In fact most of the fixed point theorems are extended from metric space to 2-metric space. [16], [20], [46], [47], [87], [88], [93] are a few references in this area. Banach contraction principle was generalized by A.K.Sharma [86] in 1979. The concept of weakly commuting pair of self maps in 2-metric space was introduced by Naidu et. al. [71]. Y.J.Cho extended the compatible mappings and compatible mappings of type (A) in 2-metric space. He obtained common fixed point theorems in this setting by assuming that the 2-metric \( d \) is continuous. Naidu et.al. [71] introduced weak commutativity in 2-metric space while Murthy et.al.[15] defined compatible maps in 2-metric space. In 2004, B.Singh et. al. [89] introduced semi compatible maps in 2-metric space, which was more general than compatibility of maps.

Later researchers observed that there is no relation between metric and 2-metric functions. Ha et.al. [42] showed that a 2-metric need not be a continuous function in its variables, where as an ordinary metric is. Further we can easily verify that there is no easy relationship between the result obtained in the two settings.
In particular the contraction mapping theorem in metric space and 2-metric space are unrelated.

These considerations led Bapure Dhage [21], in his Ph.D Thesis, to introduce a new class of generalized metric called D-metrics. In his work he attempted to develop topological structures in such spaces. But in 2003, Zead Mustafa and B.Sims [70] proved that most of the claims concerning the fundamental topological properties of D-metric space are incorrect such as the convergence of sequences. Thus they introduced an appropriate definition of generalized metric space or G-metric space. They also studied the properties, G-metric topology, convergence and continuity, completeness, compactness and even the product of G-metric spaces also. Recently M.Abbas et.al. [1] proved common fixed point theorem for non commuting maps without continuity in generalized metric space.

1.2 DEFINITION AND PRELIMINARIES

Here we recall the basic definitions, examples and results.

**Definition: 1.2.1** Let $X$ be any set. Let $d(x, y)$ be a function defined on the set $X \times X$ satisfying the following conditions:

1. $d(x, y) \geq 0$
2. $d(x, y) = 0$, if and only if $x = y$
3. $d(x, y) \leq d(x, z) + d(z, y)$ triangle inequality.
Such a function $d(x,y)$ is called a metric on $X$, it is a mapping of $X \times X \to \mathbb{R}$. A set $X$ with a metric $d$ is called a metric space. The members of $X$ are called points and the function $d(x,y)$ is the distance from the point $x$ to the point $y$.

**Example: 1.2.1** The set $\mathbb{R}$ of all real numbers with metric $d(x,y) = |x - y|$ is a metric space.

**Definition: 1.2.2** The Sequence of points $\{x_n\}$ is said to converge to a point $x$ of $X$ if $d(x_n, x) \to 0$ as $n \to \infty$ or for every positive value of $\epsilon$ there exists an integer $n_0$ depends on $\epsilon$, such that $0 \leq d(x_n, x) < \epsilon$ for each $n \geq n_0$. The point $x$ is called the limit of the sequence.

**Definition: 1.2.3** A sequence $\{x_n\}$ in a metric space $(X,d)$ is called a Cauchy sequence if for every $\epsilon > 0$, there exists an integer $n_0$ such that $d(x_n, x_m) < \epsilon$ for each $m > n \geq n_0$

**Definition: 1.2.4** A metric space $(X,d)$ is said to be complete if every Cauchy sequence of points of $X$ converges to a point of $X$.

**Definition: 1.2.5** Let $f$ and $g$ be self maps of a metric space $(X,d)$. A point $x \in X$ is said to be a coincidence point of $f$ and $g$ if $fx = gx$.

**Definition: 1.2.6** Let $f$ and $g$ be self maps of a metric space $(X,d)$. Then $f$ and $g$ are said to be commuting if and only if $fg = gf$.
**Definition: 1.2.7** Two self maps $S$ and $T$ of a metric space $(X,d)$ are said to be weakly commuting if and only if $d(STx,TSx) \leq d(Sx,Tx)$ for all $x \in X$.

**Note: 1.2.1** Commuting mappings are weakly commuting, but the converse is not true.

In [49], Jungck gave a generalization of the Banach’s contraction theorem by using the concept of commuting mappings as

**Theorem: 1.2.1** A continuous self map $S$ of a complete metric space $(X,d)$ has a fixed point in $X$ if and only if there exists a number $\alpha \in (0,1)$ and a mapping $T : X \to X$ which commutes with $S$ and satisfying the following:

1. $TX \subset SX$
2. $d(Tx,Ty) \leq d(Sx,Sy)$ for all $x,y \in X$. In fact $S$ and $T$ have a unique common fixed point in $X$.

Jungck [50] also proposed a generalization of the concepts of commuting mappings and weakly commuting mappings, which is called compatible mappings. He also introduced the concept of compatible maps of type (A) and proved common fixed point theorems.

**Definition: 1.2.8** Two self maps $S$ and $T$ of a metric space $(X,d)$ are said to be compatible if $\lim_{n \to \infty} d(STx_n,TSx_n) = 0$ whenever $\{x_n\}$
is a sequence in $X$ such that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t$ for some $t$ in $X$.

**Definition: 1.2.9** Let $S, T : (X, d) \to (X, d)$ be maps. $S$ and $T$ are said to be compatible of type $A$ if $\lim_{n \to \infty} d(STx_n, SSx_n) = 0$ and $\lim_{n \to \infty} d(STx_n, TTx_n) = 0$ Whenever $\{x_n\}$ is a sequence such that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t$ for some $t$ in $X$.

**Remark: 1.2.1** Let $S, T : (X, d) \to (X, d)$ be continuous mappings. If $S$ and $T$ are compatible then they are compatible of type $(A)$.

**Remark: 1.2.2** Let $S, T : (X, d) \to (X, d)$ be compatible mappings of type $(A)$. If one of $S$ and $T$ is continuous, then $S$ and $T$ are compatible.

Recently (see [45], [62]), some fixed point theorems have been proved in a non metric setting wherein the distance function used need not satisfy the triangle inequality. Pathak [72], introduced the notions of $W-$compatible maps of type $(P)$ and $W^*-$compatible maps of type $(P)$ in non metric setting. Here we recall the definitions and preliminaries from [72].

**Definition: 1.2.10** A mapping $T : X \to X$ is $W-$continuous at $x$ if $x_n \to x$ implies $Tx_n \to Tx$.

**Definition: 1.2.11** Mappings $A, S : X \to X$ are called jointly $W-$continuous if there exists a point $x \in X$ such that if $\lim_{n \to \infty} x_n =$
\[ x \text{ and } Ax = Sx, \text{ whenever } \{x_n\} \text{ is a sequence in } X \text{ such that } \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = p \text{ for some } p \in X. \]

**Definition: 1.2.12** Let \( A \) and \( S \) be self maps of a topological space \((X, \tau)\). Then \( A \) and \( S \) are said to be \( W \)-compatible maps of type \((P)\) if the \( W \)-continuity of \( S \) (resp. \( W \)-continuity of \( A \)) implies
\[
\lim_{n \to \infty} AAx_n = Sp \quad (\text{resp. } \lim_{n \to \infty} SSx_n = Ap), \text{ whenever } \{x_n\} \text{ is a sequence in } X \text{ such that } \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = p \text{ for some } p \in X.
\]

**Proposition: 1.2.1** Let \( A \) and \( S \) be self maps of a topological space \((X, \tau)\) which are \( W \)-compatible of type \((P)\). If \( A \) is \( W \)-continuous, then they are compatible.

**Proposition: 1.2.2** Let \( A \) and \( S \) be \( W \)-continuous self maps of a topological space \((X, \tau)\). If the pairs \( \{A, S\} \) is \( W \)-compatible of type \((P)\) and \( Ap = Sp \) then \( AAp = SAp = ASp = SSP \).

**Definition: 1.2.13** Let \( A \) and \( S \) be self maps of a topological space \((X, \tau)\). The pairs \( \{A, S\} \) is said to be \( W^* \)-compatible of type \((P)\) if it is \( W \)-compatible of type \((P)\) and \( Ap = Sp \) implies \( ASp = SAp \).

**Theorem: 1.2.2** Let \( A, B, S \) and \( T \) be self maps of a Hausdorff \( d \)-complete topological space \((X, \tau)\) satisfying the following conditions:
1. \( AX \subseteq TX, BX \subseteq SX \)
2. \( d(Ax, By) \leq G(\max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By)\}) \) for all \( x, y \in X \), where \( G \) satisfies :

10

a. $0 < G(y) < y$ for each $y > 0$, $G(0) = 0$

b. $g(y) = \frac{y}{y - G(y)}$ is a non increasing function on $(0, \infty)$.

c. $\int_0^{y_1} g(y) < \infty$ for each $y_1 > 0$.

d. $G(y)$ is non decreasing.

and also the pairs \{A, S\} and \{B, T\} are $W^*-$compatible of type (P) and jointly $W-$continuous. Then $A, B, S$ and $T$ have a unique common fixed point in $X$.

1.3 2-METRIC SPACES

Definition: 1.3.1 A 2-metric space is a set $X$ with a real valued $d$ on $X \times X \times X$ satisfying the following conditions:

1. for distinct points $x, y \in X$ there exists a point $z \in X$ such that $d(x, y, z) \neq 0$

2. $d(x, y, z) = 0$ if at least two of $x, y, z$ are equal

3. $d(x, y, z) = d(P\{x, y, z\})$ (Symmetry in all three variables, where $P$ is the permutation function)

4. $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$ for all $x, y, z, u \in X$.

The function $d$ is called a 2-metric for the space $X$ and $(X, d)$ is called a 2-metric space.

Note: 1.3.1 It has been shown by Gahler that the 2-metric $d$ is non negative although $d$ is a continuous function of any one of its three arguments. A 2-metric $d$ which is continuous in all of its arguments will be called continuous.
**Definition: 1.3.2** A sequence \( \{x_n\} \) in a 2-metric space \((X, d)\) is said to be convergent to a point \(x\) in \(X\) denoted by \(\lim_{n \to \infty} x_n = x\) or \(x_n \to x\) as \(n \to \infty\), if \(\lim_{n \to \infty} d(x_n, x, z) = 0\) for all \(z \in X\). The point \(x\) is called the limit of the sequence \(\{x_n\}\) in \(X\).

**Definition: 1.3.3** A sequence \(\{x_n\}\) in a 2-metric space \((X, d)\) is called a Cauchy sequence if \(\lim_{n,m \to \infty} d(x_n, x_m, z) = 0\) for all \(z \in X\).

**Definition: 1.3.4** A 2-metric space \((X, d)\) is said to be complete if every Cauchy sequence in \(X\) is convergent.

Note that in a 2-metric space \((X, d)\) a convergent sequence need not be a Cauchy sequence, but every convergent sequence is a Cauchy sequence when the 2-metric \(d\) is continuous on \(X\).

Naidu et.al. [71] introduced the concept of weakly commuting pairs of self maps on a 2-metric space and weak continuity of a 2-metric. Y.J.Cho [15] extended the concepts of compatible mappings and compatible mappings of type (A) in metric space to the setting of 2-metric space and derived some relations between these mappings.

**Definition: 1.3.5** Two self maps \(S\) and \(T\) of a 2-metric space \((X, d)\) is said to be weakly commuting if and only if \(d(STx, TSx, z) \leq d(Sx, Tx, z)\) for all \(x, z\) in \(X\).
**Definition: 1.3.6** Two self maps $S$ and $T$ of a 2-metric space $(X, d)$ are said to be compatible if $\lim_{n,m \to \infty} d(STx_n, TSx_n, z) = 0$ whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t$ for some $t$ in $X$.

**Definition: 1.3.7** Two self maps $S$ and $T$ of a 2-metric space $(X, d)$ are said to be compatible of type (A) if $\lim_{n,m \to \infty} d(STx_n, SSx_n, z) = 0$ and $\lim_{n,m \to \infty} d(STx_n, TTx_n, z) = 0$ whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t$ for some $t$ in $X$.

**Note: 1.3.2** Commuting maps are weakly commuting but the converse need not be true.

**Note: 1.3.3** Any weakly commuting mappings are compatible but the converse need not hold.

### 1.4 GENERALIZED METRIC SPACES

Dhage introduced a generalized form of metric space which is defined as follows:

**Definition: 1.4.1** Let $X$ be any set and let $D: X \times X \times X \to R$ be a function that satisfies

a). $D(x, y, z) = 0$ if and only if $x = y = z$

b). $D(x, y, z) \geq 0$
c). $D(x, y, z) = D(P\{x, y, z\})$ (Symmetry in three variables, where $P$ is the permutation function)

d). $D(x, y, z) \leq D(x, y, u) + D(x, u, z) + D(u, y, z)$

e). $D(x, y, z) \leq D(x, z, z) + D(z, y, y)$ for all $x, y, z, u$ in $X$.

Then the set equipped with the metric $D$ is called the generalized metric space and the function $D$ is called the generalized metric.

In order to overcome the limitations of the definition of $D$-metric space, Mustafa et al. [70] defined an appropriate form of the generalized metric space which is given as:

**Definition: 1.4.2** Let $X$ be any set and let $G : X \times X \times X \rightarrow R^+$ be a function that satisfying

a). $G(x, y, z) = 0$ if and only if $x = y = z$

b). $0 < G(x, y, z)$ for all $x, y \in X$ with $x \neq y$.

c). $G(x, y, z) = G(P\{x, y, z\})$ (Symmetry in three variables, where $P$ is the permutation function)

d). $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$

e). $G(x, y, z) \leq G(x, z, z) + G(z, y, y)$ for all $x, y, z$ in $X$ (rectangle inequality).

Then the set equipped with the metric $G$ is called the generalized metric space and the function $G$ is called the generalized metric.

When $G(x, y, z)$ is the perimeter of the triangle with vertices at $x, y$ and $z$ in $R^2$ then all the above properties are satisfied.

**Definition: 1.4.3** A $G$-metric $(X, G)$ is said to be symmetric if
\(G(x, y, y) = G(x, x, y)\) for all \(x, y \in X\).

**Definition: 1.4.4** A sequence \(\{x_n\}\) in a G-metric space \((X, G)\) converges to a point \(x\) in \(X\) if and only if \(G(x_n, x_n, x) \to 0\) as \(n \to \infty\).

**Definition: 1.4.5** A sequence \(\{x_n\}\) in a G-metric space \((X, G)\) is said to be G-Cauchy if for every \(\epsilon > 0\), there exists \(N \in \mathbb{N}\) such that \(G(x_n, x_m, x_l) < \epsilon\) for all \(n, m, l \geq N\).

**Definition: 1.4.6** A G-metric space \((X, G)\) is said to be complete if every G-Cauchy sequence in \((X, G)\) is G-convergent in \((X, G)\).

Mustafa and Sims defined open ball, Cauchy sequence, convergence, G-bounded open set and also studied many properties of G-metric space. For more details one can refer [70].

“Fuzziness” is a term related to almost all branches of Mathematics. The word fuzziness is related to decision making. So it has important in this era, where one’s life depends upon a good decision. Fuzzy set was introduced by Zadeh [103] in 1965 as a new way to represent vagueness in everyday life. Subsequently it was developed by many authors and used in various fields. In the early 70’s and 80’s much work has been done on probabilistic metric spaces. The motivation of introducing the probabilistic metric space is the fact that in many situations the distance between two points is inexact rather than a single real number. But when the uncertainty is
due to fuzziness rather than randomness, the concept of fuzzy metric space is more suitable. Kramosil and Michalek [59] introduced the fuzzy metric space by generalizing the concept of probabilistic metric space to the fuzzy situation. Motivated by this O.Kaleva and Seikkala [54], in 1984, studied the properties of fuzzy metric space. Heilpern [44] was the first to prove a fixed point theorem in fuzzy metric space. Followed by this O.Kaleva and Seikkala [54], R.Badard [3] and D.Butnariu [10] attempted to study the existence of fixed points using fuzzy mappings. M.Grabiec [27] extended the fixed point theorems of Banach [5] and Edelstein [25] to fuzzy metric space. The fuzzy Banach contraction theorem in [27] is as follows

**Theorem: 1.4.1** Let \((X, M, \ast)\) be a complete fuzzy metric space such that \(\lim_{t \to \infty} M(x, y, t) = 1\) for all \(x, y \in X\). Let \(T : X \to X\) be a mapping satisfying \(M(x, y, kt) \geq M(x, y, t)\) for all \(x, y \in X\) and \(k \in (0, 1)\). Then \(T\) has a unique fixed point.

**Theorem: 1.4.2 (Fuzzy Edelstein Contraction Theorem)** Let \((X, M, \ast)\) be a compact space. Let \(T : X \to X\) be a mapping satisfying \(M(Tx, Ty, .) > M(x, y, .)\) for all \(x \neq y\). Then \(T\) has a unique fixed point.

Grabiec’s result were further generalized by Subrahmanyam [96] for a pair of commuting mappings. O.Hadzic [43] obtained fixed point theorem for multivalued mappings in fuzzy metric space. Meanwhile fixed point theorems for contractive type mappings are discussed in
the works of Fang [28].

In 1994, George and Veeramani [35], redefined the notion of metric space introduced by Kramosil and Michalek so as to introduce Hausdorff topology in fuzzy metric space. Thus they studied a stronger form of metric fuzziness. In particular it is well known that every metric induces a fuzzy metric in the sense of George and Veeramani and conversely every fuzzy metric in the sense of George and Veeramani (and also Kramosil and Michalek) generates a metrizable topology ([36], [37], [38], [59], [67], [82]). Moreover, Jung et.al.[48] established minimization theorem in fuzzy metric space and thus obtained fixed point theorems in fuzzy metric space which was an extension of the Downing-Kirk fixed point theorem in metric space (see [23]). In the early 21st century there was a considerable growth of the fixed point theory in fuzzy metric space. Fang [28] proved some fixed point theorems in fuzzy metric space which improves, generalize, unify and extend some main results of Banach [5], Edelstein [25] and many others. Mishra [68], Chugh and Kumar [17], Chugh et.al.[18] introduced compatible maps, compatible maps of type (A) and compatible maps of type (P) respectively in fuzzy metric space and obtained common fixed point theorems. Cho [15] introduced the concept of compatible maps of type $\alpha$ in fuzzy metric space.

The notion of fuzzy 2-metric space was introduced by Sushil Sharma [98] and extended the theorem given by Fisher [29] which
stated as follows

**Theorem: 1.4.3** Let $S$ and $T$ be continuous mappings of a complete metric space $(X, d)$ into itself. Then $S$ and $T$ have a common fixed point in $X$ if and only if there exists a continuous mapping $A$ of $X$ into $S(X) \cap T(X)$ which commutes with $S$ and $T$ and satisfy:

$$d(Ax, Ay) \leq \alpha d(Sx, Ty)$$

for all $x, y \in X$ and $0 < \alpha < 1$. Indeed $S, T$ and $A$ have a unique common fixed point.

The fuzzy version of this theorem given by Sushil Sharma is as follows

**Theorem: 1.4.4** Let $(X, M, \ast)$ be a complete fuzzy metric space with the condition $\lim_{t \to \infty} M(x, y, t) = 1$ and let $S$ and $T$ be continuous self maps in $X$. Then $S$ and $T$ have common fixed point in $X$ if there exists a continuous mapping $A$ of $X$ into $S(X) \cap T(X)$ which commutes with $S$ and $T$ and

$$M(Ax, Ay, qt) \geq \min\{M(Ty, Ay, t), M(Sx, Ax, t), M(Sx, Ty, t)\}$$

for all $x, y \in X$, $t > 0$ and $0 < q < 1$. Then $S, T$ and $A$ have unique common fixed point.

The above theorem in fuzzy 2-metric space is given by

**Theorem: 1.4.5** Let $(X, M, \ast)$ be a complete fuzzy 2-metric space and let $S$ and $T$ be continuous self maps in $X$. Then $S$ and $T$ have common fixed point in $X$ if there exists a continuous mapping $A$ of
$X$ into $S(X) \cap T(X)$ which commutes with $S$ and $T$ and

$$M(Ax, Ay, a, qt) \geq \min\{M(Ty, Ay, a, t), M(Sx, Ax, a, t), M(Sx, Ty, a, t)\}$$

for all $x, y, a \in X$, $t > 0$ and $0 < q < 1$ with the condition

$$\lim_{t \to \infty} M(x, y, z, t) = 1$$

for all $x, y, a \in X$. Then $S, T$ and $A$ have unique common fixed point.

Motivation to define generalized metric space from metric space lead G.Sun et.al. [41] to introduce the new notion called Q-fuzzy metric space which is the generalization of fuzzy metric space. Moreover, a few fixed point theorems are obtained by him in this space.