CHAPTER V

NUMERICAL METHODS AND THE CONVERGENCE OF EIGENMODES

§ 5.1 Introduction:

A large number of numerical schemes are involved in the analysis of the eigenvalue problems for the global density waves in self-gravitating disks. In this chapter, we discuss the numerical schemes we have used for various calculations that we had to carry out in the course of our work.

The eigen-matrices are, in general, 300 x 300 matrices with matrix-elements, which themselves are certain integrals over the disk boundaries. In practice, however, one has to truncate the size of the matrices involved in order to solve the problem by numerical methods. While truncating, appropriate convergence of the eigenvalues and eigenfunctions has to be ensured. Here we have discussed the numerical calculations involved to form the eigen-matrices in Section 5.2. The eigenroutine is described in Section 5.3. Finally, the problem of convergence has been presented in Section 5.4.

§ 5.2 Construction of Eigen-Matrices:

Before proceeding to study the equilibrium and the stability of the disks, one has to solve the transcendental equation (4.2.2) for prescribed values of the disk boundaries so as to evaluate the roots, \( \lambda_k \). A similar calculation has to be carried out to find the roots for the equation (3.3.20)
We have calculated the roots of the above mentioned transcendental equations by means of an algorithm based on the Mueller's iteration scheme of successive bisection and inverse interpolation. In principle, there is an infinite spectrum for \( \lambda \) with a fixed disk boundary. However, it is sufficient to obtain the first fifteen roots in each case with a given \((a, b)\). The zeros, \( \lambda_k \), accurate up to the sixth place after the decimal, are given in the table (5.1).

The gravitational potential, \( \psi_0(r) \), and the azimuthal velocity, \( v_0(r) \), for Yabushita disks involve certain infinite integrals of the products of modified Bessel functions (cf. equations (3.3.21), (4.2.8) and (4.2.9)). We have used 8-point Gaussian Quadrature for these infinite integrations. We assume a large enough upper limit for these integrals to start with, and then, successively increase until the values of the net integrals converge within a specified accuracy. However, we find that, in most of the cases, the absolute values of these integrals are small as compared to other quantities in the above-mentioned equations; these integrals do not affect the problem significantly. The equilibrium profiles for the Hunter's disk models can be obtained analytically, thereby, the computational time required for the analysis is reduced considerably for these disks. Similar infinite integrals are encountered again while studying the linearized stability problem via equations (4.3.15) and (4.3.16). These integrals converge considerably faster as compared to integrals which appear in equations governing the equilibrium disks.
The Matrix-Elements:

The matrix-elements in equations (3.5.29) and (4.3.20) are certain integrated quantities over the disk, with the integrands which are products of some Bessel functions and the equilibrium quantities like \( \sigma_0(r) \), \( V_0(r) \) etc. The functions, \( F_m(\lambda_j r) \), are oscillatory in nature with increasing number of nodes in the interval \((a,b)\) for larger values of \( \lambda_j \). Thus the oscillatory nature of integrands increases as matrices, involving larger number of zeros are considered. The integration of such highly oscillatory integrands is by no means trivial and very accurate routines have to be employed in order to obtain accurate results. We have used 32-point Gaussian quadrature for the evaluation of these finite integrals, which essentially form the eigen-matrix, \( M \), in Chapters III and IV. To ensure the convergence of these integrals, we divide the interval \((a,b)\) of integration and carry out the integration in the segments, again, by 32-point quadrature method. The integrals over the segments are then summed up to yield the net values of the integrals in the interval \((a,b)\) and compared with the corresponding values obtained earlier. The number of segments in the interval \((a,b)\) is increased, each time carrying out the integrals, until proper convergence of the matrix-elements has been achieved. Since the eigen-matrices involved are non-symmetric, even slight changes in the values of the elements could result in wide variations in the eigenvalues. In fact, the convergence of the matrix elements is reflected in the resulting eigenvalues. Table (5.2) shows the eigenvalues as
obtained in three different cases with $n_d = 1, 2, 3$, where $n_d$ represents the number of segments in the interval $(a,b)$. The relative errors are quite small as is evident by comparing the eigenvalues. However, it may be noticed that some unstable modes may appear due to the inaccuracy in the value of the matrix-elements. As the matrix elements are refined they disappear or else, exhibit large deviations. Such modes do not represent valid modes of oscillations. Here, one may notice such a behaviour for an unstable mode for an annular disk, found with $m = 2$, $\alpha = 0.25$ and $\beta = 0.5$; $\Omega = (1.0723, -0.0161)$ when $n_d = 1$. This mode shows relative errors (as defined in equation (4.3.21)), $\Delta \omega_y = 0.003$, $\Delta \omega_i = 0.54$ with $n_d = 2$ and finally disappears completely (i.e. stabilized) when $n_d = 3$. Thus, we may conclude that the above discussed eigenvalue does not represent a genuine spiral mode of the system. On the other hand, the eigenvalues $(1.5231, -0.2952)$ and $(1.4748, -0.0542)$ are hardly affected as $n_d$ is increased, with small $(\Delta \omega_y, \Delta \omega_i)$ as given by $(1.31 \times 10^{-4}, 7 \times 10^{-5})$ and $(1.4 \times 10^{-4}, 7 \times 10^{-3})$ which would suggest that the corresponding eigenvalues represent valid unstable modes of oscillations.

Generation of Eigen-Patterns:

The patterns associated with the eigenmodes of oscillation have been constructed by using subroutine PLTDSK. A grid with a large number of mesh points has been superimposed over the disk. Any perturbed quantities, e.g. the surface-density perturbation, may be obtained from
\[ \hat{\sigma}(r, \theta, t) = \text{Re} \left\{ \hat{h}(r) \exp \left( i(\omega_j t + m \theta) \right) \right\} \quad (5.2.1) \]

for a given eigen-frequency, \( \omega_j \), and azimuthal wave-number, \( m \). The radial part of the perturbation, \( \hat{\sigma}(r) \), is calculated from the eigenfunctions \( C_{j}^{(k)} \) corresponding to \( \omega_j \) as

\[ \hat{\sigma}(r) = \sum_{k=0}^{n} \sum_{j} \left\{ C_{j}^{(k)} \hat{F}_m(\lambda_k^m r)^n \right\}, \quad n = \frac{n \pm}{3} \quad (5.2.2) \]

where

\[ \hat{F}_m(\lambda_k^m r) \equiv \begin{cases} J_m(\lambda_k^m r) & \text{for a pure disk} \\ J_m(\lambda_k r) - \frac{J_m(\lambda_k a)}{J_m(\lambda_k a)} \gamma_m(\lambda_k r) & \text{for annular disks} \end{cases} \quad (5.2.3) \]

The eigenfunctions \( C_{j}^{(k)} \) are complex for complex eigen-frequency, \( \omega_j \), hence, introduce a radial phase in equation (5.2.1). At a given instant of time, \( t \) (say \( t = 0 \)), one can evaluate the surface-density perturbation at all the mesh points. The perturbations vanish for \( a < r < b \). From the calculated values of \( \hat{\sigma}(r, \theta) \) at 65x65 grid, the perturbations over the disks have been plotted with an arbitrary scale of amplitude.

\( \S \) 5.3 The Eigen-Routine:

The matrices involved in the eigenvalue problems as discussed in Chapters III and IV are real, infinite dimensional and, in general, non-symmetric. Thus, the analytic solutions are intractable except for some special disk modes, e.g. \( N = 1 \) model in the family of Hunter's disks. The determination of the eigenvalues and eigenfunctions of non-symmetric matrices is much more difficult as compared to that for symmetric
matrices in which case, the eigen-values are always well-determined. A variety of considerations enter into the determination of the most suitable and efficient algorithms, viz., the storage of the machine, the precision or the accuracy required, the density of the matrix under consideration etc. The storage may become very important factor if large matrices are involved. In practice, the matrix may be fairly 'sparse', i.e. either the non-zero elements are concentrated on a narrow band centred on the diagonal, or alternatively, they may be distributed in a less systematic manner. The matrix is 'dense' if the presence of zero elements is such as to make it uneconomical to take advantage of their presence.

We use algorithms based on similarity transformation methods to compute all the eigenvalues and eigenfunctions of a given $N_t \times N_t$ matrix, $M$, with real elements. Thus algorithms produce, from $M$, a sequence of similar matrices ($M_k$) which, in nearly all cases, tend to a form from which the eigenvalues can be recovered with relatively greater ease. If $M$ is real, so are all the consecutive matrices ($M_k$), and we may expect the limiting form to have, along the diagonal, blocks of 1x1 and 2x2 principal submatrices. The 2x2 blocks correspond to complex conjugate pairs of eigenvalues.

It is impossible to design procedures for solving the general unsymmetric eigenvalue problem which are as satisfactory as those for symmetric (or Hermitian) matrices, since
eigenvalues themselves may be very sensitive to small changes in the matrix elements. The determination of an eigen system may present practical difficulties if the matrix is badly balanced, i.e., if corresponding rows and columns have very different norms. Thus, before using an algorithm, one must balance the matrix using the procedure BALANCE which makes the sums of the magnitudes of elements in corresponding rows and columns nearly equal by similarity transformation. This usually gives substantial improvement in the accuracy of the computed eigenvalues, when the original matrix is badly balanced. We use BALANCE even in the case when the original matrix is well-balanced, since the time on the machine is negligible. In fact, BALANCE also recognizes 'isolated' eigenvalues, i.e., eigenvalues which are available by inspection without any computation and its use ensures that such eigenvalues are determined exactly however ill-conditional they may be.

The eigenvalues and eigenvectors can be obtained by using a generalization of the Jacobi's methods for real, symmetric matrices. However, the elegance with which this method works in the case of symmetric case is usually not achieved for real, non-symmetric matrices.

The alternative approach is based on a combination of a reduction to 'Hessenberg' form using subroutines EIMHES and ELTRAN (i.e., elementary Hessenberg and elementary transformation routines) or ORTHES and ORTRAN (corresponding routines for orthogonal transformation method), by means of a sequence of
similarity transformations, followed by the QR algorithm HQR2. ELMHES (or ORTHES) reduces a real general matrix to upper Hessenberg form using stabilized elementary (or orthogonal) similarity transformations, the presence of zero elements in the matrix M can be utilized considerably in ELMHES and if M is very sparse, ELMHES is much more effective. If the procedure BALANCE has been used, the eigenvectors of the original matrix are recovered from those of the balanced matrix using subroutine BALBAK. In fact, BALBAK performs the back-transformation of a set of right-handed eigenvectors.

The essence of QR method is a process whereby a sequence of upper Hessenberg matrices, unitarily similar to the original Hessenberg matrix, is formed which converges to a triangular matrix, that is, an upper Hessenberg matrix whose eigenvalues are the eigenvalues of 1x1 or 2x2 principal submatrices. The rate of convergence of this sequence is improved by shifting the origin at each iteration. The arithmetic throughout the process is kept real by combining two iterations into one, using two origin shifts or a pair of complex conjugate origin shifts. Before each iteration, the last Hessenberg form is checked for a possible splitting into submatrices. If a splitting occurs, only the lower submatrix participates in the next iteration. The similarity transformations used in each iteration are accumulated in the Z-array (for the eigenvectors). The origin shifts at each iteration are the eigenvalues of the lowest 2x2 principal minor. Whenever a lowest 1x1 or 2x2 principal submatrix finally splits from the rest o
the matrix, the eigenvalues of this submatrix are taken to be
eigenvalues of the original matrix and the algorithm proceeds
with the remaining submatrix. This procedure is continued until
the matrix has split completely into submatrices of order 1 or
2. The tolerance in the splitting tests are proportional to the
relative machine precision. The eigenvectors of the quasi-
triangular matrix are determined by a back substitution process
and then transformed to the eigenvectors of the original matrix,
using the information in Z.

Some of the eigenvalues may have been isolated on the
diagonal by the subroutine BALANCE. This information is trans-
mitted to HQR2 through some parameters, LØW and IGH. As a
result, HQR2 immediately extracts the eigenvalues in rows 1 to
LØW-1 and IGH+1 to N, and so applies the QR procedure to the
submatrix situated in rows and columns LØW through IGH.

Each eigenvalue \( \lambda_i \) and its corresponding eigen-
vector \( X_i \) is always exact for some matrix, \( M + E_i \) where
\( \| E_i \| / \| M \| \) is of the order of magnitude of the machine
precision. The residual vector, \( MX_i - \lambda_i X_i \) is, therefore,
always small relative to \( \| M \| \), independent of its 'accuracy'
regarding \( M \) as exact. Thus the eigen-routines are based on the
algorithms which are numerically stable.

A measure of performance of the routine, \( \mu \), based
on the backward error analysis, is defined as

\[
\mu = \max_{1 \leq i \leq N_t} \frac{\| M Z_i - \lambda_i Z_i \|}{10. n_t. E. \| M \|. \| Z_i \|}
\]  

(5.3.1)
where each pair \( \lambda_i \) and \( Z_i \) is an eigenvalue and corresponding eigenvector of the matrix \( M \) of order \( n_t \) and \( C \) is the precision of arithmetic on the test machine.

In equation (5.3.1), \( ||M|| \) and \( ||Z_i|| \) represent the Euclidean norms of the given matrix, \( M \), and the eigenvector \( Z_i \) respectively. The Euclidean norms are defined as follows:

\[
||X||_2 = \left( \sum_i |X_i|^2 \right)^{1/2}; \quad \text{for vectors } X
\]

and

\[
||M||_2 = \left( \sum_i \sum_j |M_{ij}|^2 \right)^{1/2}; \quad \text{for the matrix, } M,
\]

where summation is over \( 1 \leq i \leq n_t \).

If \( \mu < 1 \), the routine has performed satisfactorily and if \( \mu > 100 \), the performance is poor. With \( 1 < \mu < 100 \), the performance is progressively marginal. In each computer run, we have examined the parameter \( \mu \) and found the performance satisfactory in most of the cases.

For more details on the eigen-routines, one may refer to Parlett and Reinsch (1969), Martin and Wilkinson (1971), Peters and Wilkinson (1971), Francis (1961a,b) and also Smith et al (1974).

§ 5.4 Convergence of Eigenvalues and Eigenfunctions:

We have carried out the solution of the eigenvalue-eigenfunction problem using matrices of successively increasing dimension in order to study the numerical accuracy of obtained results. Eigenvalues generally depend on the value of the dimension \( n_t = 3 \times n \) (where \( N \) is the number of terms retained in
expansions for perturbed quantities), of the eigenmatrix and they do not always converge rapidly to certain values when \( n_t \) is increased. We have considered Hunter's \( N = 2 \) pressureless disk model and Yabushita's disk I to study the behaviour of eigen-frequencies for \( m = 2 \) modes. For \( n = 1 \) (i.e. only one term retained in the expansions for perturbed quantities) we obtain a pair of complex conjugate modes alongwith a real (or stable) mode. The maximum number of nodes in the interval, is \( n - 1 \) (that is, no node for \( n = 1 \)). As \( n \) is increased, the number of complex conjugate pairs and the stable modes increases. Thus new modes of oscillations appear, while the values of already existing modes are refined. In principle, the self-gravitating disks would admit infinite unstable (complex) mode as well as infinite number of stable (or real) modes. The truncation of the eigenmatrix to a size \( n_t \) thus, contains modes up to \( n-1 \) nodes only. The eigen modes of perturbations with much shorter wavelengths would be omitted by the truncation of \( M \). From figures (5.1) and (5.2) we note that with progressive increase in the dimension of the eigenmatrix, the variations in the unstable eigen-frequencies in complex \( \omega \)-plane diminishes. However, the convergence is rather slow. The eigenroutine tends to show numerical errors for very large matrices, hence the convergence of the eigen-values cannot correctly, be pursued indefinitely. We have used a maximum dimension of 45x45 in most of our calculations.
The convergence of the lower modes with small growth rates in the eigen-branch (where we call the sequence of unstable modes in complex ( \( \zeta \)-plane as the branch) is quite satisfactory in the case of both the disk models considered. These lower modes exhibit open pattern (i.e. relatively smaller number of nodes in the disk, and grow comparatively slower. We can discuss the results obtained for these open modes with confidence, as valid modes of oscillations. In fact, the open modes are essentially the outcome of a more appropriate global analysis. The local density wave theory could not, properly, deal with such open modes, since it depend on the assumption of \( \kappa \sqrt{r} \gg 1 \). We remark here that the convergence of the eigenvalues in 'hot' disks is extremely poor in some cases.

Referring to figure (4.8) which compares the eigen-patterns obtained by using 27x27 and 45x45 matrices, it is evident that for the eigen-values, under consideration, the eigen-patterns do not exhibit any major variations. The convergence of the principal mode (or the dominant mode) can be seen in table (5.3). The relative errors for these calculations are discussed in Chapter IV. Finally, we may remark again that the check on the convergence of various eigenvalues and eigen-functions is very important, before a particular mode can be assumed a genuine mode of oscillation in a global problem.
Table (5.1)

ROOTS OF $F^m_{fr} = 0$; at $r = a$ and $r = b$

<table>
<thead>
<tr>
<th>$\alpha/b$</th>
<th>$m = 0$</th>
<th></th>
<th></th>
<th>$m = 2$</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
<td>0.25</td>
<td>0.50</td>
<td>0.75</td>
<td>0.25</td>
<td>0.5</td>
<td>0.75</td>
</tr>
<tr>
<td>1</td>
<td>4.097686</td>
<td>6.246062</td>
<td>12.553266</td>
<td>5.319868</td>
<td>6.813843</td>
<td>12.761297</td>
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<tr>
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<td>12.528667</td>
<td>18.836415</td>
<td>37.694697</td>
<td>13.121498</td>
<td>19.045705</td>
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</tr>
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<td>75.396014</td>
<td>25.425757</td>
<td>37.798231</td>
<td>75.431362</td>
</tr>
<tr>
<td>7</td>
<td>29.304675</td>
<td>43.976623</td>
<td>87.962700</td>
<td>29.573912</td>
<td>44.067336</td>
<td>87.993003</td>
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<td>50.260516</td>
<td>100.52931</td>
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<td>100.55582</td>
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<td>12</td>
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</tr>
<tr>
<td>13</td>
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<td>81.678350</td>
<td>163.36180</td>
<td>54.591471</td>
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<tr>
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<td>58.634562</td>
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<td>175.92824</td>
<td>58.770530</td>
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<td>175.94340</td>
</tr>
<tr>
<td>15</td>
<td>62.823916</td>
<td>94.245128</td>
<td>188.49468</td>
<td>62.950873</td>
<td>94.287346</td>
<td>188.50882</td>
</tr>
</tbody>
</table>
Table (5.2)
The convergence of various eigen-frequencies with the increase in number of segments, \( n_d \), in the interval \((a, b)\) of integration is shown.

\[
\text{Azimuthal number, } m = 2
\]
\[
The \text{Ratio, } a/b (~\Xi~\alpha~) = 0.25
\]
\[
The \text{Ratio, } M_d^\alpha M_b^\beta (~\Xi~\beta~) = 0.5
\]

Pressure-Coefficient, \( C = 0 \) (cold disk)

<table>
<thead>
<tr>
<th>No. of segments ( n_d )</th>
<th>Eigen mode</th>
<th>( (\omega_r, \omega_i) )</th>
<th>( (\omega_r, \omega_i) )</th>
<th>( (\omega_r, \omega_i) )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.5231,-,0.2952</td>
<td>1.5229,-,0.2948</td>
<td>1.5231,-,0.2950</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.4748,-,0.0542</td>
<td>1.4743,-,0.0563</td>
<td>1.4745,-,0.0567</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.0723,-,0.0161</td>
<td>1.0753,-,0.0052</td>
<td>1.0700,-,0.0044</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.5820,-,0.0090</td>
<td>0.5801,-,0.0064</td>
<td>0.5800,-,0.0044</td>
<td></td>
</tr>
</tbody>
</table>
Table (5.3)

CONVERGENCE OF EIGENMODES WITH THE INCREASE IN THE DIMENSION OF MATRIX

\( m = 2 \) \quad \text{Pressure-coefficient, } C = 0.0 \quad \text{(cold disk)}

\( a/c = 0.25 \) \quad \beta = 1.0

<table>
<thead>
<tr>
<th>Dimension of the Truncated Eigenmatrix</th>
<th>( \omega_{\text{real}} )</th>
<th>( \omega_{\text{imag.}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>1.2727</td>
<td>-0.2654</td>
</tr>
<tr>
<td>15</td>
<td>1.3481</td>
<td>-0.3508</td>
</tr>
<tr>
<td>21</td>
<td>1.3591</td>
<td>-0.3640</td>
</tr>
<tr>
<td>27</td>
<td>1.3617</td>
<td>-0.3473</td>
</tr>
<tr>
<td>33</td>
<td>1.3625</td>
<td>-0.3229</td>
</tr>
<tr>
<td>39</td>
<td>1.3631</td>
<td>-0.2999</td>
</tr>
<tr>
<td>45</td>
<td>1.3650</td>
<td>-0.2800</td>
</tr>
</tbody>
</table>
CONVERGENCE OF EIGENMODES FOR N=2 COLD MODEL (m = 2)

\[ \omega_{\text{imag}} \]

\[ \omega_{\text{real}} \]

FIG. 5.1
Fig. 5.2
CHAPTER V

EPILOGUE

The grand spiral structure of most disk galaxies is one of their most striking morphological features. The spiral type turns out to have a link with the physical properties of the galaxies, such as the gaseous and stellar content, the radial distribution of various constituents, the disk to bulge ratio etc. This implies that spiral structure is a very sensitive indicator of basic properties of the galaxies; and hence the importance of the theoretical understanding of the phenomenon. The density wave theory has been fairly successful in explaining many observational features related with spiral galaxies. The features such as HII/HII ratios, distribution of progressively older stellar objects in azimuthal direction, the star formation rates etc. have been attempted to be explained on the basis of the density wave concept.

The original theory, as proposed by Lin and his coworkers is not self-consistent and faces severe theoretical criticisms on many counts, as is described already in Chapter I. The persistence of the Lin's density waves need some sources of excitation, which would replenish the spirals in the system continually against any possible dissipative effects which would lead to the damping of the density waves.

As a possible mechanism of excitation, we have considered the process of mass and momentum exchange between the two main subsystems of disk-galaxies, viz., the gas and the
stars. The stellar component is more adequately described by a kinetic approach, using collisionless, Boltzmann (or Vlasov) equation. However, we have adopted fluid equations for both the components, as obtained from the moments of the Vlasov equations for each component. The appropriate source and sink terms, corresponding to the exchange-process, have been properly taken into account. In fact, the cyclic exchange between the two subsystems is reasonably important for disk galaxies, where young, massive stars are continuously forming out of gas and dust concentrated in giant clouds. Most observational evidences indicate the star formation rate $\sim 5 \, M_\odot$ per year for the Milky Way. On the other hand, the massive stars explode (at their end), producing large amounts of gas and dust in the form of supernova-remnants. There are reasons to believe that the supernova events are not frequent, and they occur at a rate of $\sim 3-5$ per century. However, gases are continuously ejected in the form of stellar winds from stars. We have considered the effect of such a process on the density waves. It is found that the transfer of mass from a colder, gaseous component to the warmer, stellar component leads to the amplification of otherwise neutral density waves. With the estimated star-formation rates, it appears that the amplification of waves associated with this process can be significant against their damping. The effect of a transfer from the stellar to the gaseous fluid, in fact, contributes a damping rate, which is much smaller as compared to the amplification rate. Thus, a net growth of the density wave is possible.
The above analysis has been done in the approximation of tightly wound spiral waves and hence, the boundary effects were ignored. The results are valid only for density waves with wavelength, $\lambda$, much smaller than the characteristic lengths, for instance, the radius of the disk, $R$.

An attempt has been made here to understand the full content of a self-consistent, boundary value problem, for gravitating disk systems in a one component fluid model. A similar analysis using a two fluid model for the gas and stars with mass and momentum exchange is, in principle, possible. However, we have confined to a one fluid model for simplicity. The problem is posed in an eigen-value form and the discrete spectrum of allowed modes has been obtained for a large number of cases in a variety of model disks. In principle, an infinite number of unstable, as well as, real (neutral) modes are permissible in the self-gravitating, flat disks. The real modes show spoke (or wagon-wheel) like structure, whereas, the unstable modes exhibit spiral features. These spirals occur as a natural consequence of the collective effects in gravitating disks and, thus explain the "problem of the origin" of spirals in disk galaxies. The "persistence problem" which requires the amplification of the waves is also explained since the spiral modes are allowed only for complex eigen-modes.

The global, linear model analysis, mentioned above, yields results which can be considered somewhat intermediate between the "local" analysis of density waves and the computer
simulations, which consider the evolution of the system using Newtonian equations for each particle comprising the disk. The cold disks are generally found to be most unstable, while thermal dispersion tends to stabilize them. It has been found that for the global modes, the thermal-dispersion required to suppress all unstable modes is larger than that required for the local density waves of Lin and Shu. On the other hand, the global unstable modes are quenched much before the Ostriker-Peebles criterion is reached. This criterion represents the critical thermal energy required to suppress the unstable modes which distort disks into bar-shaped systems.

The non-axisymmetric modes which, in fact, represent spiral modes, are affected as the central condensation of the disk increases. We find that the growth-rates of open (or large wavelength) modes are lower as the central density increases, so much so, that some open modes are completely stabilized. However, relatively higher modes show opposite trend. It is to be noted that a similar effect of central condensation has been found on symmetric ($m=0$) modes by Hunter (1985). However, some authors (cf. Takehara (1976)) have reported the absence of stabilizing effect by the central density on $m \neq 0$ modes. It note that the role of thermal pressure and central concentration are quite different; then as growth rates of small scale perturbations are significantly affected in moderately hot disks, the central condensation stabilizes only the open modes.
The resulting patterns for density perturbations, in the disks, show 'leading' spirals in most of the cases; however, in some warm disks, there is an indication of 'trailing' patterns as well. It is not clear, as yet, as to what are the parameters, which decide the exact nature of the spiral patterns.

The growth rates for ring-, one-armed, two-armed and multi-armed patterns are comparable. Thus, the occurrence of bisymmetric (m=2) structures in most of the galaxies is not understood from the linear theory. The exchange of energy between various modes would perhaps settle this question which would require a nonlinear analysis of the problem.

In all cold disks, eigenvalues with large growth rates occur. In fact, for many small wavelength modes, $\omega \gg \nu$, and thus these represent violent instabilities in the system. Such explosive modes cannot grow indefinitely in the disks, since the energy reservoir is finite. Their evolution and saturation can be studied through a nonlinear calculation. However, it is likely that large halos enveloping the disks, or central, rigid, bulges would suppress such explosive instabilities to a certain extent.

In order to understand the morphological sequence of disk galaxies, from tight patterns (Sa) to open spirals (Sb) we have considered disk-bulge systems. We find that the pitch angle of spirals appears to depend on the disk to bulge ratio.
(both on $M_d/M_b$ and $a/b$, where $M_d$ and $M_b$ represent the masses of the disk and the bulge, respectively, and $a,b$ represent the radii of the spherical bulge and the flat disk, respectively). We have studied the eigenmodes of the annular, gaseous disks with varying bulge contents. No explosive modes are found for these annular disks. The number of unstable modes, is reduced considerably as compared to the case of pure disks. The effect of pressure is, again, to stabilize the unstable modes of the system. The dominant (or principle) mode show regular spiral patterns, which open up and occupy large fractions of disk areas as the central bulge component diminishes. This, in essence, explain the transition from Sa to Sc type as the amorphous bulge content decreases in the galactic systems. The growth rate of the principle mode increases as the disk component dominates over the bulge. However, it acquires more or less a constant growth-rate when a particular $M_d/M_b$ ratio is reached. The other unstable modes in the system have exceedingly small growth-rates with $\omega_i \ll \omega_r$, hence, grow very slowly. This would suggest that the 'principle mode' dominates in the system. Again, only a nonlinear analysis governing the evolution of the dominant mode would settle the problem.

We may remark here that the eigenvalue problem, in the form discussed above, removes several difficulties encountered by the 'local density wave theory'. However, a full nonlinear study of the problem of spiral structure of galaxies, based on density wave concept, is required to study the evolution of the system. Though there have been some attempts to study the
nonlinear density waves with \( \lambda \ll R \), nonlinear analysis for the global eigenmodes of disk galaxies, taking dissipative processes into account is called for. In fact, such a study is important to understand the secular transport of energy, angular momentum and the mass in the system. This global readjustment of the system could lead to the so-called 'exponential disks' found in majority of disk galaxies. The generation of global spirals shaped solitons leading to shocks (if the dissipative processes are included) and consequent formation of stars on a global front due to the compression by shocks, all require a consistent, nonlinear study of the density waves.

There have been several attempts to fit spiral patterns over many optical disk galaxies by choosing free parameters such as pattern-velocity and the amplitude of the wave from observational constraints. However, a similar fit for global spiral modes has not been carried out. It would be very interesting to model some galaxies with massive, spiral structure on the basis of the approach, discussed in Chapters III and IV. The allowed modes for such models representing observed galaxies can be obtained and the patterns associated with various modes constructed to compare with the observed spiral structures.

The problem of the dynamics and stability of flat, self-gravitating disk galaxies is extremely exciting and needs much work in future in order to understand many yet unsolved problems.
References:


