CHAPTER VI

AN ORDER-LEVEL INVENTORY MODEL FOR ITEMS WITH EXPONENTIAL DISTRIBUTION DETERIORATION

In the inventory systems of the last four Chapters the problem was to determine the optimum scheduling period so as to minimize the total cost of the system. In Chapters II to IV the demand rate was known and constant whereas in Chapter V it was probabilistic. In Chapter V the order-level was virtually prescribed and shortages were not allowed.

For the order-level inventory system to be discussed in this Section the scheduling period \( T \) is not a decision variable, but is a prescribed constant. Our problem is to determine the order-level \( S \) so as to minimize the total cost of the system. The model for the system is developed under the conditions of shortages permissible and instantaneous delivery. We consider only exponential distribution for the time to deterioration of an item, because in other cases the solution will not be explicit. First a model with known demand rate is developed and then a model with stochastic demands is developed. In the stochastic model we find that no explicit solution of the model is possible.
unless we use some approximations, as done in earlier Chapters. Even after using the approximations, the model allows an explicit solution only for some specific types of demand distributions. It has been shown that both, the deterministic as well as probabilistic, models can be related to those given by Naddor [13].

6.1: The mathematical model for deterministic demand:

The model is developed with the following assumptions:

(i) Demand is deterministic at a constant rate of \( R \) units per time unit.

(ii) Scheduling period \( T \) is a prescribed constant.

(iii) Replenishment rate is infinite; replenishment size is a constant. Lead-time is zero. The fixed lot-size \( q \) raises the on-hand inventory at the beginning of each scheduling period to the order-level \( S \).

(iv) Shortages, if any, are made up as soon as a fresh stock arrives.

(v) The inventory carrying cost \( C_1 \) per unit per time unit, the shortage cost \( C_2 \) per unit per time unit and the unit cost \( C \) are known and constant during the period under considerations.

(vi) There is no repair or replacement of the deteriorated inventory during any given cycle.
(vii) The time to deterioration of an item follows the negative exponential distribution characterized by the p.d.f.

\[(6.1.1) \quad g(t) = \theta \exp(-\theta t), \quad t \geq 0, \quad \theta > 0\]

\[= 0, \quad \text{otherwise}\]

where \(\theta^{-1} = E(t)\) is the mean life of an item. The c.d.f. of \((6.1.1)\) is

\[(6.2.2) \quad G(t) = 1 - \exp(-\theta t), \quad t \geq 0\]

so that the age-specific failure-rate function \(\phi(t)\), as defined by Cox \(\mid 5\), is given by

\[(6.1.3) \quad \phi(t) = g(t) / \{1-G(t)\} = \theta \quad \text{(constant)}.

We analyse a cycle of \(T\) time units, where \(T\) is prescribed constant. At the time \(t = 0\) of a cycle, a batch of \(q\) units enters the inventory system. From this \((q - S)\) units are delivered towards backorders, leaving a balance of \(S\) units as the initial inventory level. Thereafter, as the time passes by, the inventory level gradually decreases mainly due to demands and partly due to deterioration of the items up to a time \(t = t_1\) (say). Thus, units are carried in inventory up to the period \((0, t_1)\) of the cycle. Further demands during the remaining period \((t_1, T)\) of the cycle are backordered. Let \(t_2 = T-t_1\) denote
the duration of the cycle during which shortages occur, then incidentally we have

\[ q = R t_2 + S = RT - R t_1 + S. \]  

Now, let \( Q(t) \) denote the inventory level of the system at time \( t \) \((0 \leq t \leq T)\), then the differential equation that describes the instantaneous states of \( Q(t) \) is given by

\[ \frac{dQ(t)}{dt} + \theta Q(t) = -R, \quad 0 \leq t \leq t_1 \]

\[ \frac{dQ(t)}{dt} = -R, \quad t_1 \leq t \leq T. \]

The solution of the differential equation (6.1.5) is

\[ Q(t) = \exp(-\theta t) \left[ K_1 - \frac{R}{\theta} \left\{ \exp(\theta t) - 1 \right\} \right], \quad 0 \leq t \leq t_1 \]

\[ = R t_1 - R t + K_2, \quad t_1 \leq t \leq T \]

where \( K_1 \) and \( K_2 \) are constants of integration. Using the boundary conditions, \( Q(0) = S \) and \( Q(t_1) = 0 \), we have from (6.1.6)

\[ K_1 = S = \frac{R}{\theta} \left\{ \exp(\theta t_1) - 1 \right\} \quad \text{and} \quad K_2 = 0 \]

Here \( T \) is a given constant and \( S \) is a decision variable. Whenever required, we express various quantities and the cost function in terms of \( S \). From (6.1.7) it follows that
(6.1.8) \[ t_1 = \frac{1}{c} \log(1 + \frac{6S}{R}) \]

and then from (6.1.4) the order-quantity \( q \) is given by

(6.1.9) \[ q = RT - \frac{R}{c} \log(1 + \frac{6S}{R}) + S \]

For the system under considerations, the total cost \( K(S) \) per unit of time is

(6.1.10) \[ K(S) = \frac{C}{T} (S - Rt_1) + \frac{1}{2T} C_1 Sl + \frac{1}{2T} C_2 R(T - t_1)^2 \]

In obtaining (6.1.10) we have conveniently assumed that the inventory level \( Q(t) \) is approximately linear in \( t \) over the period \((0, t_1)\) of the cycle. Substituting for \( t_1 \) from (6.1.8) in the cost function (6.1.10) we have

(6.1.11) \[ K(S) = \frac{C}{T} \left[ S - \frac{R}{c} \log(1 + \frac{6S}{R}) \right] + \frac{C_1 S}{2Tc} \log(1 + \frac{6S}{R}) \]

\[ + \frac{C_2 R}{2T} \left[ T - \frac{1}{c} \log(1 + \frac{6S}{R}) \right]^2 \]

For optimum value of \( S \), differentiating (6.1.11) with respect to \( S \) and equating the resulting expression to zero we get

(6.1.12) \[ S(C_1 + 2c) + \frac{1}{c} (C_1 R + 2C_2 R + C_1 S_0) \log(1 + \frac{6S}{R}) - 2C_2 RT = 0 \]

Solving (6.1.12) for \( S \), we obtain the optimum order-level \( S = S_0 \) (say). The optimum lot-size \( q = q_0 \) can be
obtained by substituting \( S = S_0 \) in (6.1.9) and the minimum total cost of the system can be obtained by substituting \( S = S_0 \) in (6.1.11).

Note that when \( \theta = 0 \) there is no deterioration and

\[
\lim_{\theta \to 0} \frac{1}{\theta} \log(1 + \frac{6S}{R}) = \frac{S}{R}
\]

Taking limits as \( \theta \to 0 \) and using (6.1.13), the cost function (6.1.11) becomes

\[
(6.1.14) \quad K(S) = \frac{C_1S}{2RT} + \frac{C_2(RT-S)}{2RT} = \frac{C_1S^2}{2q} + \frac{C_2(q-S)^2}{2q}
\]

because when \( \theta \to 0 \) it follows from (6.1.9) that \( q = RT \).

Further as \( \theta \to 0 \), eqn. (6.1.12) give

\[
(6.1.15) \quad S = S^{(o)} = \frac{C_2}{C_1 + C_2} RT = \frac{C_2}{C_1 + C_2} q
\]

Expressions (6.1.14) and (6.1.15) are the same as those given by Naddor [13, p.66] for a deterministic order-level system for non-deteriorating items.

**First order approximation**:

In order to reduce the complexity of various expressions involved in the above deterministic model, we use series form of the logarithmic terms and ignore the terms with second and higher
order powers of $\theta$ under the assumption that $\theta << T$ such that $eS/R < 1$, so that

$$ (6.1.16) \quad \frac{1}{e} \log(1 + \frac{eS}{R}) = \frac{S}{R} (1 - \frac{eS}{R}) $$

Using (6.1.16) in (6.1.11), the total average cost of the system per unit of time to first order approximation is

$$ (6.1.17) \quad K(S) = \frac{CeS^2}{RT} + \frac{C_1S^2}{2RT} (1 - \frac{eS}{R}) + \frac{C_2R}{2T} (T - \frac{S}{R}) + \frac{C_2S^2e}{RT} (T - \frac{S}{R}) $$

For optimum value of $S$, $\partial K(S)/\partial S = 0$ gives

$$ (6.1.18) \quad 2CeS + C_1S(1 - \frac{3eS}{2R}) - C_2R(T - \frac{S}{R}) + 2C_2Se(T - \frac{3eS}{2R}) = 0 $$

and the optimum lot-size is

$$ (6.1.19) \quad q = q_0 = RT + \frac{eS^2}{R}. $$

where $S_0$ is the solution of (6.1.18). When $\theta = 0$, we find that (6.1.17) reduces to (6.1.14) and (6.1.18) gives (6.1.15).

**Method of obtaining numerical solution:**

Eqn. (6.1.12) can be solved by using the Newton-Raphson iterative procedure described in Section 3.2 by taking

$$ h(S) = S(C_1 + 2Ce) + \frac{1}{e} (C_1R + 2C_2R + C_1Se) \log(1 + \frac{eS}{R}) - 2C_2RT $$

and
\[ h'(s) = 2(C_1 + C_2) + C_1 \log(1 + \frac{6s}{R}) + \frac{2C_2^2R}{R + ES} \]

and as the initial value of the iterative procedure we use the value \( S = S^{(o)} \) given by (6.1.15).

Eqn. (6.1.13) in the case of first order approximation is a quadratic equation in \( S \) and can be solved easily.

**A numerical example:**

The model developed above is illustrated numerically by taking a hypothetical example using the following constants:

- \( C = \$80.00 \) per unit,
- \( C_1 = \$1.00 \) per unit per month,
- \( C_2 = \$9.00 \) per unit per month,
- \( R = 200 \) units per month,
- \( T = \) one month.

The lifetime or the time to deterioration of an item is assumed to follow exponential distribution with mean \( \theta = 20 \) months i.e. \( \theta = 1/20 = 0.05 \).

A computer program was written to solve eqn. (6.1.12) iteratively by using the Newton-Raphson iterative procedure until \( S_{m+1} - S_m < 0.0001 \).

The optimum order-level for the illustrative model is found to be \( S_0 = 129.8227 \) units with optimum lot-size \( q_0 = 202.0622 \) units and corresponding minimum cost \( K(S_0) = \$323.86 \).
Note that with the optimum policy, the number of units that deteriorate per month is, on an average, 2.06.

To check the optimality of the solution obtained, the total variable cost per unit of time was computed for $S = 127 \ (1) \ 131$. They are shown in the following table:

<table>
<thead>
<tr>
<th>Order level S (units)</th>
<th>Total cost per month $K(S)$ (in $)</th>
</tr>
</thead>
<tbody>
<tr>
<td>127</td>
<td>324.1272</td>
</tr>
<tr>
<td>128</td>
<td>323.9726</td>
</tr>
<tr>
<td>129</td>
<td>323.8846</td>
</tr>
<tr>
<td>129.8227</td>
<td>323.8621</td>
</tr>
<tr>
<td>130</td>
<td>323.8631</td>
</tr>
<tr>
<td>131</td>
<td>323.9081</td>
</tr>
</tbody>
</table>

6.2: The mathematical model for probabilistic demand:

The inventory model for an order-level system with probabilistic demands is developed with the same assumptions as those given in Section 6.1, except the following modifications in assumptions (1) and (iii).

The demand $x$ during any prescribed scheduling period $T$ follows a probability distribution with p.d.f. $f(x)$ ($0 < x < \infty$). This demand of $x$ units is assumed to occur in a uniform pattern during $T$. At every replenishment, a variable quantity is ordered so that the inventory level reaches to $S$.
units in the beginning of the cycle. The problem is to find optimum value of the order-level $S$.

Two typical situations may arise in the system depending on the demand and deterioration of units during the cycle time $T$, and the order level $S$. We first consider the two situations separately.

(1) When demand plus deterioration do not exceed $S$ i.e. when shortages do not occur:

Let $Q_x(t)$ denote the inventory level of the system at time $t$ ($0 \leq t \leq T$) when a demand of $x$ units occurs during the cycle of $T$ time units; then during that period the demand rate is $x/T$ and the differential equation governing the system is given by

$$\frac{dQ_x(t)}{dt} + 6Q_x(t) = -\frac{x}{T} , \quad 0 \leq t \leq T .$$

The solution of the first order linear differential equation (6.2.1) is

$$Q_x(t) = \left[ K_3 - \frac{x}{T} \int_0^t \exp(\epsilon t) \, dt \right] \exp(-\epsilon t) , \quad 0 \leq t \leq T .$$

where $K_3$ is the constant of integration. Using the boundary condition $Q_x(0) = S$, we have $K_3 = S$ and then, from (6.2.2)

$$Q_x(t) = \left[ S - \frac{x}{dT} \{ \exp(\epsilon t) - 1 \} \right] \exp(-\epsilon t) ; \quad 0 \leq t \leq T .$$
Since shortages are not allowed we must have $Q_x(T) \geq 0$ i.e.

$$x \leq \frac{S\cdot eT}{\exp(eT) - 1} = S^* \quad \text{(say)}$$

The number of units that deteriorate during the cycle is given by

$$Q_1(x) = S - x - Q_x(T)$$

$$= \left\{1 - \exp(-eT)\right\} (S + \frac{x}{eT}) - x, \quad x \leq S^*$$

Assuming that $Q_x(t)$ is approximately linear in $t$, the average number of units in inventory per unit of time during the cycle is

$$I_{11}(x) = \frac{1}{2} \left[ Q_x(0) + Q_x(T) \right]$$

$$= \frac{1}{2} S \left\{1 + \exp(-eT)\right\} - \frac{x}{2eT} \left\{1 - \exp(-eT)\right\}, \quad x \leq S^*$$

Average number of shortages during the cycle are clearly

$$I_{21}(x) = 0, \quad x \leq S^*$$

When demand plus deterioration exceeds $S$ i.e. when shortages occur:

In this case suppose that the system carries inventory during a period $(0, t_1)$ and runs with shortages during the period
(t₁, T) of the cycle; then the differential equation governing the system is given by
\[ \frac{dQ_x(t)}{dt} + \zeta_x(t) = -\frac{x}{T}, \quad 0 \leq t \leq t_1 \]
\[ \frac{dQ_x(t)}{dt} = -\frac{x}{T}, \quad t_1 \leq t \leq T. \]

The solution of the differential equation (6.2.8), after adjusting for the constant of integration, is
\[ \zeta_x(t) = \left[ S - \frac{x}{eT} \{\exp(\theta t) - 1\} \right] \exp(-\theta t), \quad 0 \leq t \leq t_1 \]
\[ = \frac{x}{T} (t_1 - t), \quad t_1 \leq t \leq T. \]

Since \( \zeta_x(t_1) = 0 \), (6.2.9) gives
\[ S = \frac{x}{eT} \{\exp(\theta t_1) - 1\} \]
or
\[ (6.2.10) \quad t_1 = \frac{1}{\theta} \log(1 + \frac{STe}{x}) \]

Further, since \( t_1 < T \), we also have
\[ x > \frac{ST}{\exp(T) - 1} = S^* \]

The number of units that deteriorate during the cycle is given by
Assuming $Q_x(t)$ to be roughly linear in $t$ over the entire period $(0, T)$, the average number of units in inventory per time unit during the cycle is given by

(6.2.12) $I_{12}(x) = \frac{1}{2} S \frac{t_1}{T} = \frac{S}{2T} \log(1 + \frac{STb}{x})$, $x > S^*$

Finally, the average shortages during the cycle are

(6.2.13) $I_{22}(x) = -\frac{1}{2} Q_x(T) \frac{(T-t_1)}{T}$

$$= \frac{x}{2T^2} \left[ T - \frac{1}{6} \log(1 + \frac{STb}{x}) \right]^2, \ x > S^*$$

From (6.2.5) and (6.2.11), the expected number of units that deteriorate during a cycle time is

(6.2.14) $D(S) = \int_0^S D_1(x) f(x) \, dx + \int_{S^*}^\infty D_2(x) f(x) \, dx$

$$= \int_0^{S^*} \left\{ \left[ 1 - \exp\left( -\frac{b}{6T} \right) \right] \left( S + \frac{x}{6T} \right) - x \right\} f(x) \, dx$$

$$+ \int_{S^*}^{\infty} \left[ S - \frac{x}{6T} \log(1 + \frac{STb}{x}) \right] f(x) \, dx$$

From (6.2.6) and (6.2.12), the average expected number of units in inventory per unit of time during a cycle is given by
Finally, from (6.2.7) and (6.2.13), the average expected shortages during the cycle time $T$ are given by

\begin{align*}
(6.2.16) \quad I_2(S) &= \int_0^S I_{21}(x) f(x) \, dx + \int_{S^*}^\infty I_{22}(x) f(x) \, dx \\
&= \frac{1}{2T^2} \int_{S^*}^\infty x \left\{ T - \frac{1}{e} \log(1 + \frac{ST_e}{x}) \right\}^2 f(x) \, dx
\end{align*}

From (6.2.14), (6.2.15) and (6.2.16) the average expected total cost of the system during a cycle is given by

\begin{align*}
(6.2.17) \quad K(S) &= \frac{C}{T} D(S) + C_1 I_1(S) + C_2 I_2(S) \\
&= \frac{C}{T} \int_0^{S^*} \left[ (1-\exp(-\theta T)) \left( S + \frac{x}{\theta T} \right) - x \right] f(x) \, dx \\
&\quad + \frac{C}{T} \int_{S^*}^\infty \left[ S - \frac{x}{\theta T} \log(1 + \frac{ST_e}{x}) \right] f(x) \, dx \\
&\quad + \frac{1}{2} \int_0^{S^*} \left[ S \{1+\exp(-\theta T)\} - \frac{x}{\theta T} \{1-\exp(-\theta T)\} \right] f(x) \, dx \\
&\quad + \frac{C_1 S}{2T^2} \int_{S^*}^\infty \log(1 + \frac{ST_e}{x}) f(x) \, dx \\
&\quad + \frac{C_2}{2T^2} \int_{S^*}^\infty x \left\{ T - \frac{1}{e} \log(1 + \frac{ST_e}{x}) \right\}^2 f(x) \, dx
\end{align*}
Now, in practice, for obtaining optimal value of $S$ and total expected minimum cost of the system we are required to evaluate various integrals in $\frac{\partial K(S)}{\partial S} = 0$ and in the cost function (6.2.17). It is found that in the resulting expression and in (6.2.17), some of the integrals cannot be explicitly evaluated even for the uniform density for demand. To overcome this difficulty, we use the series form of the exponentials and logarithms in the cost function (6.2.17) and ignore the terms with second and higher order powers of $\theta$ under the assumption that $\theta \ll T$ such that $ST\theta/x < 1$ for $x > S^*$. With this approximation, the cost function (6.2.17) now becomes

\[
(6.2.18) \quad K(S) = (C\theta + C_1 - \frac{1}{2} C_1 \theta T) \int_0^\infty (S - \frac{1}{2} x) f(x) \, dx
\]

\[
+ \frac{1}{2} S^2 (C\theta + C_1 + C_2 \theta T) \int_0^\infty \frac{f(x)}{S(1 - \frac{1}{2} \theta T)} \, dx
\]

\[
+ \frac{C_2}{2} \int_0^\infty \frac{(x-S)^2}{S(1 - \frac{1}{2} \theta T)} f(x) \, dx
\]

\[
- \frac{S^3 \theta T}{4} (C_1 + 2C_2) \int_0^\infty \frac{f(x)}{S(1 - \frac{1}{2} \theta T)} \, dx
\]

For optimum value of $S$, $\frac{\partial K(S)}{\partial S} = 0$ gives to first order approximation in $\theta$
(6.2.19) \((C\theta + \frac{1}{2} C_1 \theta T + C_1 + C_2) F[S(1 - \frac{1}{2} \theta T)] + (C\theta + C_2 \theta T + C_1 + C_2) S \int_{S(1 - \frac{1}{2} \theta T)}^{\infty} \frac{f(x)}{x} dx + \frac{3}{4} S^2 \theta T (C_1 + 2C_2) S \int_{S(1 - \frac{1}{2} \theta T)}^{\infty} \frac{f(x)}{x^2} dx - C_2 = 0\)

where

\[
F(x) = \int_{0}^{x} f(x) dx
\]

When \(\theta = 0\) there is no deterioration and in that case (6.2.18) reduces to

(6.2.20) \(K(S) = C_1 \int_{0}^{S} (S - \frac{1}{2} x) f(x) dx + \frac{1}{2} C_1 S^2 \int_{S}^{\infty} \frac{f(x)}{x} dx + \frac{1}{2} C_2 \int_{S}^{\infty} \frac{(x - S)^2}{x} f(x) dx\)

and eqn. (6.2.19) gives

(6.2.21) \(F(S) + S \int_{S}^{\infty} \frac{f(x)}{x} dx = \frac{C_2}{C_1 + C_2}\)

(6.2.20) is the standard cost function and (6.2.21) is the standard formulae for finding the optimum order-level \(S = S^{(o)}\) for a probabilistic order-level system for non-deteriorating items, as given by Naddor [13, p.132].
The integrals involved in (6.2.18) and (6.2.19) can be evaluated explicitly for some specific types of functions \( f(x) \) only; for such functions we can obtain optimum order-level \( S = S_o \) by solving eqn. (6.2.19). This is illustrated in the following example.

**Example:**

Let the demand density be

\[
f(x) = \begin{cases} \frac{1}{2}x^2 \exp(-x), & x > 0 \\ \text{otherwise} \end{cases}
\]

For this demand density, the cost function (6.2.18) to first order approximation in \( \theta \) becomes

\[
K(S) = (C_0 - \frac{1}{2}C_1\theta + C_1 + C_2)(S - \frac{3}{2})
\]

\[
+ \frac{1}{8}[4(C_0 + C_1 + C_2)(S+3) - C_1\theta(S+6) - 2C_2\theta(2S-3)]\exp(-S)
\]

and optimum order-level \( S = S_o \) is a solution of

\[
(C_0 - \frac{1}{2}C_1\theta + C_1 + C_2) - \frac{1}{8} \left[ (4C_0 + C_1 + C_2)(S + 2) - C_1\theta(S^2 + 4S - 6) - 2C_2\theta(2S - 5) \right] \exp(-S) = 0
\]