CHAPTER 4

PROGRESSIVELY CENSORED SAMPLES FROM MIXTURE OF TWO EXPONENTIAL DISTRIBUTIONS WITH CHANGING PARAMETERS

4.1 Introduction

Statistical distributions which can be expressed as super positions of (usually simpler) component distributions are termed as mixture distributions or compound distributions. Mixtures with a finite number of components are termed as finite mixtures. Finite mixture distributions arise in variety of disciplines, ranging from atomic physics to psychiatry and microbiology.

Medgyessi (1961) analyses absorption spectra in terms of normal mixtures, Gregor (1969) applies a mixture of normal distributions to data arising from measuring the content of DNA in the nuclei of liver cells of rats. Normal mixtures have also been used in many studies designed to investigate the robustness of certain statistical techniques when the data are not normally distributed. Some examples of this are given
by Subrahmaniam and Messeri (1975), Hyrenius (1950) and Weibull (1950).

Mixtures of distributions are also important in the area of life testing and reliability analysis. Here the observations are the times to failure of a sample of items. Often failure can occur for more than one reason and the failure distribution for each reason can be adequately approximated by a simple density function such as the negative exponential. The overall failure distribution is then a mixture. For details of the mixtures of distribution we refer to a survey paper by Gupta and Huang (1981).

Suppose that an item exhibits k modes of failures $m_1, m_2, \ldots, m_k$ then in mixed population model only one of the k possible modes generates a random life time that causes part failure. The selection of this mode is governed by the parameter $p_i$, where $p_i$ can be interpreted as the probability of randomly selecting a device that is predestined to fail because of mode $i$.

The general form of finite exponential mixture is

$$f(x) = \sum_{i=1}^{c} \frac{p_i}{\theta_i} e^{-x/\theta_i}$$

for $x \geq 0$

$$= 0$$

otherwise
where \( \sum_{i=1}^{c} p_i = 1, p_i > 0 \) and \( \theta_i > 0 \) for \( i = 1, 2, \ldots, c \).

In life testing failure data, the observed samples are often censored or truncated. This kind of problem was studied by Mendenhall and Hader (1958) for estimating the parameters of mixed exponentially distributed failure time distributions from type I censored life test data. Rider (1961) has considered the method of moments applied to a mixture of two exponential distributions. Also by Tallis and Light (1968), Padgett and Tsokos (1978) and others considered the problem of estimation for mixture data. Lee and Sinha (1976) consider the mixture when components have Weibull distributions and estimated the parameters by the method of maximum likelihood.

In this chapter we have considered the problem investigated by Mendenhall and Hader, in a different way, by taking type I progressive censoring with changing failure rate at each stage of censoring for mixture of two exponential distributions. As the surviving items are checked and overhauled after each stage of censoring and defects if any are repaired wherever possible it is reasonable to assume that the failure rates as well as the mixing proportion under go change. The method of M L is used to estimate the
parameters at each stage of censoring. A numerical example is given to illustrate the results of type I progressively group-censored samples considered in section 4.4.

4.2 Mixed failure model

A population is postulated to be composed of two sub populations, representing failure types, mixed in proportion $p : (1-p)$, $0 \leq p \leq 1$. For simplicity of notation suppose that $q = 1-p$.

Each unit of the population conceptually contains a tag which indicates the sub population to which the unit belongs and hence defines the way in which that particular unit will fail. The information on the tag, i.e. the cause of failure is obtained only after failure has occurred.

The failure times for the $i$th sub population $i=1,2$ are assumed to have a cumulative failure probability distribution defined by

$$F_i(t) = 1 - e^{-t/\theta_i} \quad (0 \leq t < \infty)$$

(4.2.1)

If $p$ is the proportion of units belonging to sub population with $i = 1$, then the cumulative distribution function (CDF) for the population has the CDF and
Density given respectively by

\[
(4.2.2) \begin{cases} 
F(t) = pF_1(t) + qF_2(t) \\
q(t) = pf_1(t) + qf_2(t)
\end{cases}
\]

\[
(4.2.3) G_i(t) = 1 - F_i(t) \text{ and } G(t) = 1 - F(t)
\]

The probability function \( G(t) \) is the probability that a unit will survive to time \( t \) and is called the survival function.

Due to restrictions on time available for testing, the experiments frequently desires to conclude the life test after predetermined length of time has elapsed or after a predetermined number of units have failed. For sufficiently large samples censoring is done through several stages and this leads to progressive censoring.

Now we consider the sampling with type I progressive censoring with two stages of censoring.

A sample of \( n \) units is drawn from the population as defined in (4.2.2) and placed on a life test. The test is terminated at fixed time \( T_z \) after the second stage of censoring at which time \( n_1 + n_2 \) units have failed. \( n_i \) (\( i=1,2 \)) indicate the number of failed units at \( i \)th stage of censoring and let \( T_i \) be the times of
censoring at the ith stage. Out of these \( n_i \) units failed in ith stage; suppose \( k_i \) units have failed from first subpopulation and \( (n_i - k_i) \) from second subpopulation during the ith stage of censoring.

Let \( m^{(i)} \) be the fixed number of surviving items withdrawn or censored from the test immediately after the ith stage here we assume that the cause of failure is (retrospectively) known and the distribution is truncated. This means that one can study the failed items and determine to which category they belong, but one can not tell to which category the non-failed item belong.

Thus the composite density function of \( X \) over the range \((0, \infty)\) for two stage type I progressive censoring is given by

\[
(4.2.4) \quad f(x) = \begin{cases} 
    p^{(1)} f_1^{(1)}(x^{(1)}) + q^{(1)} f_2^{(1)}(x^{(1)}), & 0 < x \leq T_1 \\
    C[p^{(2)} f_1^{(2)}(x^{(2)}) + q^{(2)} f_2^{(2)}(x^{(2)})], & T_1 < x < \infty
\end{cases}
\]

where \( C \) can be determined from the condition \( P(0 < x \leq T_1) + C[p(T_1 < x < \infty)] = 1 \) and hence, using (4.2.1) and (4.2.2) the p.d.f and c.d.f of \( X \) are given by
(4.2.5) $f(x) =$

\[
\begin{cases}
  p_1^{(1)} e^{-x/\theta_1} + q_1^{(1)} e^{-x/\theta_2}, & 0 < x \leq T_1 \\
  p_1^{(2)} e^{-T_1/\theta_1} + q_1^{(2)} e^{-T_1/\theta_2}, & x > T_1
\end{cases}
\]

and

(4.2.6) $1 - F(x) =$

\[
\begin{cases}
  p_1^{(1)} e^{-x/\theta_1} + q_1^{(1)} e^{-x/\theta_2}, & 0 < x \leq T_1 \\
  p_1^{(2)} e^{-T_1/\theta_1} + q_1^{(2)} e^{-T_1/\theta_2}, & x > T_1
\end{cases}
\]

Let $n$ items whose failure distribution is given by (4.2.5) or (4.2.6) be placed on a test. Let $n_1$ & $n_2$ be the number of items that fail in the first and second stages of censoring respectively. Out of $n$, suppose $K$ correspond to failures due to exponential population with failure rate $\theta_i$ for $i = 1, 2$. Let $m^{(1)}$ and $m^{(2)} = n - n_1 - m_1 - n_2$ be the number of items
censored after times $T_1$ and $T_2$. Let $x_{j_1}^{(i)}$ and $x_{j_2}^{(i)}$ be the time of failure of the $j$th unit in the $i$th stage of censoring from first and second subpopulation respectively.

We now consider the Mendenhall and Hader's approach to construct the likelihood function. Hence for two stage type I progressive censoring, the likelihood is given by

$$L \propto \prod_{i=1}^{2} \left[ \prod_{j=1}^{k_i} \left\{ p_i^{(i)} f_i(x_{j1}^{(i)}) \right\} \right] \prod_{j=1}^{n_i-k_i} \left\{ q_i^{(i)} f_2(x_{j2}^{(i)}) \right\} \prod_{i=1}^{2} \left[ 1 - F(T_i) \right]^{m_i}$$

thus $L$ can be written as

$$L = L_1 L_2$$

where

$$(4.2.7) \quad L_1 = (p_1^{(i)}) (q_1^{(i)}) (\theta_1^{(i)}) \exp \left\{ - \sum_{j=1}^{k_1} \frac{x_{j1}^{(i)}}{\theta_1^{(i)}} \right\} \exp \left\{ - \sum_{j=1}^{n_1-k_1} \frac{x_{j2}^{(i)}}{\theta_2^{(i)}} \right\}$$

$$- (n_1-k_1) \quad (\theta_2^{(i)}) \quad \exp \left\{ - \sum_{j=1}^{x_{j2}^{(i)}} \frac{x_{j2}^{(i)}}{\theta_2^{(i)}} \right\}$$

$$\left[ p_1^{(i)} G(T_1/\theta_1^{(i)}) + q_1^{(i)} G(T_1/\theta_2^{(i)}) \right]^{n_1-k_1}$$

and

$$L_2 = \prod_{i=1}^{2} \left[ 1 - F(T_i) \right]^{m_i}$$
(4.2.10) \[ L_z = (p^{(2)}) (q^{(2)}) \exp \left\{ \frac{k_z}{\theta_1^{(2)}} (\theta_2^{(2)}) \exp \left\{ -\sum_{j=1}^{\frac{k_z}{\theta_1^{(2)}}} \right\} \right\} \]

\[ -(n_z - k_z) \exp \left\{ \frac{n_z - k_z}{\theta_2^{(2)}} \sum_{j=1}^{\frac{n_z - k_z}{\theta_2^{(2)}}} \right\} \]

\[
\left[ P^{(2)} G(T_1/\theta_1^{(2)}) + q^{(2)} G(T_1/\theta_2^{(2)}) \right]^{-n_z^{(2)}}
\]

\[
\left[ P^{(2)} G(T_2/\theta_1^{(2)}) + q^{(2)} G(T_2/\theta_2^{(2)}) \right]^{n_z^{(2)} - n_z}
\]

where \( n^{(1)} = n, n^{(2)} = n - n_1 - m^{(1)} \) and \( m^{(2)} = n^{(2)} - n_z \)

and

\[
\begin{align*}
G(T_i/\theta_j^{(i)}) &= e^{-T_i/\theta_j^{(i)}} \\
G(T_{i-1}/\theta_j^{(i)}) &= e^{-T_{i-1}/\theta_j^{(i)}} \\
G(x_j/\theta_j^{(i)}) &= e^{-x_j/\theta_j^{(i)}}
\end{align*}
\]

for \( i = j = 1, 2 \) and \( T_0 = 0 \).

### 4.3 Maximum likelihood estimation of parameters

In this section we shall derive the M.L estimating equations and discuss the methods of solving them in order to obtain the M.L estimates of the parameters \( (p_i, \theta_1^{(i)}, \theta_2^{(i)}) \) involved in the \( i \)th stage of type I censoring for \( i = 1, 2 \).

From the equation (4.2.9) we get
\( \frac{\partial \log L_1}{\partial p^{(1)}} = \frac{k_1}{p^{(1)}} - \frac{(n_1 - k_1)}{q^{(1)}} \)

\( + (n-n_1) \left\{ \frac{G(T_{1}/\theta^{(1)}_{1}) - G(T_{1}/\theta^{(1)}_{2})}{p^{(1)} G(T_{1}/\theta^{(1)}_{1}) - q^{(1)} G(T_{1}/\theta^{(1)}_{2})} \right\} \)

\( \frac{\partial \log L_1}{\partial \theta_{1}^{(1)}} = -\frac{k_1}{\theta_{1}^{(1)}} + \frac{k_1 x_{1}^{(1)}}{(\theta_{1}^{(1)})^2} \)

\( + \frac{(n-n_1) p^{(1)} G(T_{1}/\theta^{(1)}_{1}) T_{1}}{\left\{ p^{(1)} G(T_{1}/\theta^{(1)}_{1}) + q^{(1)} G(T_{1}/\theta^{(1)}_{2}) \right\} (\theta_{1}^{(1)})^2} \)

and

\( \frac{\partial \log L_1}{\partial \theta_{2}^{(1)}} = -\frac{n - k_1}{\theta_{2}^{(1)}} + \frac{(n - k_1) x_{1}^{(1)}}{(\theta_{2}^{(1)})^2} \)

\( + \frac{(n-n_1) q^{(1)} G(T_{1}/\theta^{(1)}_{2}) T_{1}}{\left\{ p^{(1)} G(T_{1}/\theta^{(1)}_{1}) + q^{(1)} G(T_{1}/\theta^{(1)}_{2}) \right\} (\theta_{2}^{(1)})^2} \)

Let us write

\( \beta^{(1)} = \frac{p^{(1)} G(T_{1}/\theta^{(1)}_{1})}{p^{(1)} G(T_{1}/\theta^{(1)}_{1}) + q^{(1)} G(T_{1}/\theta^{(1)}_{2})} \)

Equating (4.3.1) to (4.3.3) to zero we get,

\( \tilde{p}^{(1)} = \frac{k_1}{n} + \beta^{(1)} \left\{ \frac{n-n_1}{n} \right\} \)

\( \tilde{\theta}_{1}^{(1)} = \tilde{x}_{1}^{(1)} + \frac{\beta^{(1)} (n-n_1) T_{1}}{k_1} \)

\( \tilde{\theta}_{2}^{(1)} = \tilde{x}_{2}^{(1)} + \frac{(n-n_1)(1-\beta^{(1)}) T_{1}}{(n_1 - k_1)} \)
Where 

\[ x_1^{(1)} = \sum_{j=1}^{n-k} \frac{x_{1j}^{(1)}}{k_j}, \quad x_2^{(1)} = \sum_{j=1}^{n-k} \frac{x_{2j}^{(1)}}{(n_k_k_j)} \]

The estimates of \( p_1^{(1)}, \theta_1^{(1)} \) and \( \theta_2^{(1)} \) have to be obtained from solution of the simultaneous equations (4.3.4) to (4.3.7).

On substituting the equations for \( \hat{p}^{(1)}, \hat{\theta}_1^{(1)} \) and \( \hat{\theta}_2^{(1)} \) from (4.3.5) to (4.3.7) in equation (4.3.4), yields an equation involving only single parameter, \( \beta^{(1)} \), which will be,

\[ \hat{\beta}^{(1)} = \frac{1}{1 + q^{(1)}} \left\{ \frac{G(T / \hat{\theta}_1^{(1)})}{\hat{p}^{(1)}} - \frac{G(T / \hat{\theta}_2^{(1)})}{\hat{p}^{(1)}} \right\} \]

The above equation in \( \hat{\beta}^{(1)} \) will be of the form

\[ \hat{\beta}^{(1)} = g(\hat{\beta}^{(1)}), \text{ function of } \hat{\beta}^{(1)}, \quad 0 \leq \hat{\beta}^{(1)} \leq 1. \]

It is relatively easy to obtain the initial solution \( \hat{\beta}_0^{(1)} \) by considering the graph of \( g(\hat{\beta}^{(1)}) - \hat{\beta}^{(1)} \) versus \( \hat{\beta}^{(1)} \) and solution of \( g(\hat{\beta}^{(1)}) - \hat{\beta}^{(1)} = 0 \). Then using iteration for the equation (4.3.9) solution for \( \hat{\beta}^{(1)} \) can be obtained. Now for second stage of censoring the ML equations from the equation (4.2.10) are
\[
\frac{\partial \log L_{(2)}}{\partial \theta_{(2)}} = \frac{k_2}{p^{(2)}} - \frac{(n_z - k_z)}{q^{(2)}},
\]
+(n^{(2)} - n_z) \left\{ \frac{G(T_{(2)} / \theta^{(2)}) - G(T_{(2)} / \theta_{(2)})}{p^{(2)} G(T_{(2)} / \theta_{(2)}) - q^{(2)} G(T_{(2)} / \theta_{(2)})} \right\}

- n^{(2)} \left\{ \frac{G(T_{(1)} / \theta^{(2)}) - G(T_{(1)} / \theta_{(2)})}{p^{(2)} G(T_{(1)} / \theta_{(1)}) - q^{(2)} G(T_{(1)} / \theta_{(2)})} \right\}

\[
\frac{\partial \log L_{(2)}}{\partial \theta_{(2)}} = \frac{-k_2}{\theta^{(2)}} + \frac{k_z}{(\theta^{(2)})^2}
\]
+(n^{(2)} - n_z) \left\{ \frac{p^{(2)} G(T_{(2)} / \theta^{(2)}) T_{(2)}}{p^{(2)} G(T_{(2)} / \theta_{(2)}) + q^{(2)} G(T_{(2)} / \theta_{(2)})} \right\}\left\{ \theta^{(2)} \right\}^2

- \left\{ \frac{p^{(2)} G(T_{(1)} / \theta^{(2)}) T_{(1)}}{p^{(2)} G(T_{(1)} / \theta_{(2)}) + q^{(2)} G(T_{(1)} / \theta_{(2)})} \right\}\left\{ \theta^{(2)} \right\}^2

\[
\frac{\partial \log L_{(2)}}{\partial \theta_{(2)}} = \frac{-(n_z - k_z)}{\theta^{(2)}} + \frac{(n_z - k_z) x^{(2)}}{(\theta^{(2)})^2}
\]
+(n^{(2)} - n_z) q^{(2)} \left\{ \frac{G(T_{(2)} / \theta^{(2)}) T_{(2)}}{p^{(2)} G(T_{(2)} / \theta_{(2)}) + q^{(2)} G(T_{(2)} / \theta_{(2)})} \right\}\left\{ \theta^{(2)} \right\}^2

- n^{(2)} q^{(2)} \left\{ \frac{G(T_{(1)} / \theta^{(2)}) T_{(1)}}{p^{(2)} G(T_{(1)} / \theta_{(2)}) + q^{(2)} G(T_{(1)} / \theta_{(2)})} \right\}\left\{ \theta^{(2)} \right\}^2

Writing

\[
\beta^{(2)} = \frac{p^{(2)} G(T_{(2)} / \theta_{(1)})}{p^{(2)} G(T_{(2)} / \theta_{(2)}) + q^{(2)} G(T_{(2)} / \theta_{(2)})}
\]
and

\[
\beta^{(2)} = \frac{p^{(2)} G(T_{(2)} / \theta_{(1)})}{p^{(2)} G(T_{(2)} / \theta_{(2)}) + q^{(2)} G(T_{(2)} / \theta_{(2)})}
\]
(4.3.14) \[ \beta^{(12)} = \frac{p^{(2)} G(T_1 / \theta^{(1)})}{p^{(2)} G(T_1 / \theta^{(2)}) + q^{(2)} G(T_1 / \theta^{(2)})} \]

and

equating equations (4.3.10) to (4.3.12) to zero we get

(4.3.15) \[ \beta^{(12)} = \frac{k^2 + (n^{(2)} - n_2) \beta^{(2)}}{n^{(2)}} \]

(4.3.16) \[ \hat{\theta}_1^{(2)} = x_1^{(2)} + \frac{(T_2 - T_1)(n^{(2)} - n_2) \beta^{(2)} - T_1 k_2}{k_2} \]

(4.3.17) \[ \hat{\theta}_2^{(2)} = x_2^{(2)} + \frac{\left\{ (n^{(2)} - n_2)T_2 - (n^{(2)} - k_2)T_1 \right\}}{n_2 - k_2} \]

From the equation (4.3.14) we get

(4.3.18) \[ p^{(2)} = \frac{\beta^{(12)}}{\left\{ \frac{G(T_1 / \theta^{(1)})}{G(T_1 / \theta^{(2)})} \right\} (1 - \beta^{(12)}) + \beta^{(12)}} \]

and again using (4.3.15) we finally get,

(4.3.19) \[ \hat{p}^{(2)} = \frac{k^2 + (n^{(2)} - n_2) \beta^{(2)}}{\left\{ \frac{G(T_1 / \hat{\theta}_1^{(2)})}{G(T_1 / \hat{\theta}_2^{(2)})} \right\} \left\{ (n^{(2)} - k_2) - (n^{(2)} - n_2) \beta^{(2)} \right\} + k_2 + (n^2 - n_2) \beta^{(2)}} \]

Using (4.3.16) & (4.3.17) in (4.3.18) we get \( \hat{p}^{(2)} \) as a function of \( \hat{\beta}^{(2)} \) and finally substituting \( \hat{\theta}_1^{(2)} \), \( \hat{\theta}_2^{(2)} \)
and \( p^{(2)} \) as a function of \( \beta^{(2)} \) in equation (4.3.13) we can get the equation involving only parameter \( \beta^{(2)} \) as
\[
(4.3.20) \quad \beta^{(2)} = h(\beta^{(2)})
\]
Using the graph of \( h(\beta^{(2)}) - \beta^{(2)} \) versus \( \beta^{(2)} \) and equation \( h(\beta^{(2)}) - \beta^{(2)} = 0 \) we get initial solution for \( \beta^{(2)} = \beta_o^{(2)} \).

Solving the equation (4.3.20) iteratively we can obtain estimate of \( \beta^{(2)} \) say \( \hat{\beta}^{(2)} \). Using the estimate of \( \beta^{(2)} \) in (4.3.16) and (4.3.17) we get the estimates \( \hat{\theta}_1^{(2)} \) and \( \hat{\theta}_2^{(2)} \) and hence from equation (4.3.19) we get the estimate \( \hat{p}^{(2)} \). Asymptotic standard errors of these estimates can be obtained in the usual way.

4.4 Maximum likelihood estimation from progressively group - censored samples

In this section, the maximum likelihood estimation of parameters of model (4.2) using group censoring is considered. Life tests are, sometimes performed in group censoring so that only the total number of failures rather than times of failures are observed during each stage \( (T_{i-1}, T_i) \), where \( T_i (i = 1,2) \) are the fixed points of censoring. The number of failures observed by each sub population at each stage is, of course, a random variable. The likelihood function
for the type I progressively group-censored sample of the model (4.2) is

\[(4.4.1) \ L \propto \prod_{i=1}^{z} \left\{ \int_{T_{i-1}}^{T_{i}} p^{(i)} f_{1}^{(i)}(x_{1}^{(i)})dx_{1}^{(i)} \right\}^{k_{i}} \]

\[\left\{ \int_{T_{i-1}}^{T_{i}} q^{(i)} f_{2}(x_{2}^{(i)})dx_{2}^{(i)} \right\}^{n_{i}-k_{i}} \cdot \prod_{i=1}^{z} \left\{ 1 - F(T_{i}) \right\}^{m^{(i)}} \]

Where \(m^{(2)} = n^{(2)} - n_{2}\) and \(n^{(2)} = n - n_{1} - m^{(1)}\).

As usual the likelihood function \(L\) is

\[(4.4.2) \ L \propto L_{1} L_{2} \]

where

\[(4.4.3) \ L_{1} = (p^{(1)})^{k_{1}} (q^{(1)})^{n_{1}-k_{1}} \left[ 1 - G \left( \frac{T_{1}}{\theta_{1}} \right) \right]^{k_{1}} \]

\[\cdot \left[ 1 - G \left( \frac{T_{1}}{\theta_{2}} \right) \right]^{n_{1}-k_{1}} \]

\[\cdot \left[ p^{(1)} G \left( \frac{T_{1}}{\theta_{1}} \right) + q^{(1)} G \left( \frac{T_{1}}{\theta_{2}} \right) \right]^{n_{1}} \]

and

\[(4.4.4) \ L_{2} = (p^{(2)})^{k_{2}} (q^{(2)})^{n_{2}-k_{2}} \left[ G \left( \frac{T_{1}}{\theta_{1}} \right) - G \left( \frac{T_{2}}{\theta_{2}} \right) \right]^{k_{2}} \]

\[\cdot \left[ G \left( \frac{T_{1}}{\theta_{2}} \right) - G \left( \frac{T_{2}}{\theta_{2}} \right) \right]^{n_{2}-k_{2}} \]

\[\cdot \left[ p^{(2)} G \left( \frac{T_{1}}{\theta_{1}} \right) + q^{(2)} G \left( \frac{T_{1}}{\theta_{2}} \right) \right]^{-n^{(2)}} \]

\[\cdot \left[ p^{(2)} G \left( \frac{T_{2}}{\theta_{1}} \right) + q^{(2)} G \left( \frac{T_{2}}{\theta_{2}} \right) \right]^{n^{(2)}-n_{2}} \]

73
Differentiating loglikelihood equation for $L_1$ with respect to $p^{(1)}$, $\theta_1^{(1)}$ and $\theta_2^{(2)}$ and equating them to zero, using $\beta^{(1)}$ as defined in (4.3.4), we get

$(4.4.5) \hat{p}^{(1)} = \frac{k_1}{n} + \beta^{(1)} \left[ \frac{n - n_1}{n} \right]$

$(4.4.6) \hat{\theta}_1^{(1)} = T_1 \left[ \log \left\{ \frac{k_1 + (n-n_1)\beta^{(1)}}{n - n_1(1 - \beta^{(1)})} \right\} \right]^{-1}$

and

$(4.4.7) \hat{\theta}_2^{(1)} = T_1 \left[ \log \left\{ \frac{(n_1 - k_1) + (n-n_1)(1-\beta^{(1)})}{n - n_1(1 - \beta^{(1)})} \right\} \right]^{-1}$

Using (4.4.5) to (4.4.7) in (4.3.4) we get an equation involving only parameter $\beta^{(1)}$ of the form

$(4.4.8) \beta^{(1)} = t(\hat{\beta}^{(1)})$

Using the graph of $t(\hat{\beta}^{(1)}) - \hat{\beta}^{(1)}$ versus $\hat{\beta}^{(1)}$ and equation $t(\hat{\beta}^{(1)}) - \hat{\beta}^{(1)} = 0$, initial solution for $\beta^{(1)} = \hat{\beta}_0^{(1)}$ can easily be obtained and then by iteration procedure, finally we can obtain the estimate $\hat{\beta}^{(1)}$.

Using this estimate of $\hat{\beta}^{(1)}$ in (4.4.5) to (4.4.7) we get the estimates $\hat{p}^{(1)}$, $\hat{\theta}_1^{(1)}$ and $\hat{\theta}_2^{(1)}$.

In a similar way for the second stage of censoring using log likelihood equation from (4.4.4), the estimates

$(4.4.9) n^{(2)}\beta^{(12)} = k_2 + (n^{(2)} - n_2)\beta^{(2)}$
Using (4.4.10) & (4.4.11) in (4.3.18) we get,

\[
\hat{\theta}_1^{(2)} = (T_2 - T_1). \left[ \log \left( \frac{k_2 + (n_2^{(2)} - n_2^2)\hat{\beta}^{(2)}}{(n_2^{(2)} - n_2^2)\hat{\beta}^{(2)}} \right) \right]^{-1}
\]

\[
\hat{\theta}_2^{(2)} = (T_2 - T_1). \left[ \log \left( \frac{(n - k_2)^2 + (n_2^{(2)} - n_2^2)(1 - \hat{\beta}^{(2)})}{(n_2^{(2)} - n_2^2)(1 - \hat{\beta}^{(2)})} \right) \right]^{-1}
\]

Again using (4.4.10) to (4.4.12) in equation (4.3.13), we get the equation having only parameter \(\hat{\beta}^{(2)}\) as

\[
\hat{p}^{(2)} = \frac{k_2 + (n_2^{(2)} - n_2^2)\hat{\beta}^{(2)}}{G(T_2/\hat{\beta}^{(2)}) - G(T_1/\hat{\beta}^{(2)}) + k_2 + (n_2^{(2)} - n_2^2)\hat{\beta}^{(2)}}
\]

Again using (4.4.10) to (4.4.12) in equation (4.3.13), we get the equation having only parameter \(\hat{\beta}^{(2)}\) as \(\hat{\beta}^{(2)} = u(\hat{\beta}^{(2)})\). Using the method suggested above all the estimates of \(\hat{p}^{(2)}\), \(\hat{\theta}_1^{(2)}\) and \(\hat{\theta}_2^{(2)}\) can be obtained. The asymptotic standard errors of the estimates can be obtained in the usual manner.

4.5 Numerical Example

The data taken from Hoel and Walburg (1972) represent the ages at death of conventional mice. The data is little modified with respect to only number of deaths rather than actual times at death.

<table>
<thead>
<tr>
<th>Age (At death)</th>
<th>No. of death (first type)</th>
<th>No. of death (second type)</th>
<th>No. of withdrawn</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-500</td>
<td>3</td>
<td>15</td>
<td>2</td>
</tr>
<tr>
<td>500-900</td>
<td>24</td>
<td>54</td>
<td>2</td>
</tr>
</tbody>
</table>

75
Using the notations from section 4.2, 

Here \( K = 3, \quad n_1 - K = 15, \quad m_1 = 2, \quad T_1 = 500, \quad n = n^{(1)} = 100 \)

\( K = 24, \quad n_2 - K = 54, \quad m_2 = 2, \quad T_2 = 900, \quad n^{(2)} = 80. \)

For the first stage of group censoring, using the equations (4.3.4), (4.4.5), (4.4.6) & (4.4.7) the initial value of \( \hat{\beta}^{(1)} = \hat{\beta}_0^{(1)} \) will be 0.6 and 0.7 which satisfies the equation (4.4.8) very closely with positive and negative errors. Hence using the method of false position or iteration procedure for the equation, the final solution will be

\[
\hat{\beta}^{(1)} = 0.641555, \quad \hat{p}^{(1)} = 0.5560751
\]

\[
\hat{t}_1^{(1)} = 9015.6077, \quad \hat{t}_2^{(1)} = 1212.6177
\]

For second stage of group censoring using the equations (4.4.9), (4.4.10), (4.4.11) and (4.4.12) the initial value of \( \hat{\beta}^{(2)} = \hat{\beta}_0^{(2)} \) will be 0.3 and 0.4 which satisfies the equation (4.4.13) very closely with positive and negative errors. Hence using the method of false position or iteration procedure for the equation, the final solution will be

\[
\hat{\beta}^{(2)} = 0.309, \quad \hat{p}^{(2)} = 0.3061353
\]

\[
\hat{t}_1^{(2)} = 108.55569, \quad \hat{t}_2^{(2)} = 108.37985
\]