CHAPTER 3

PROGRESSIVELY CENSORED SAMPLES FROM WEIBULL
AND INVERSE GAUSSIAN DISTRIBUTIONS

3.1 Introduction

In analysing experimental data the first requirement is often to select a reasonable probability model. The Weibull distribution has been shown in the literature to be applicable to wide variety of response Variables, beginning with a paper by Walloddi Weibull (1939). It is undoubtedly the most commonly encountered model in the area of life testing and reliability. Various inferential problems associated with this distribution have been extensively considered by Dubey (1967), Bain and Antle (1967), Menon (1963) and others. Reviews of estimation procedures for the Weibull model are found in recent textbooks, namely, Bain (1978), Mann et al. (1974) and Nelson (1982). The Inverse Gaussian (IG) distribution is usually suggested as failure model when there is high occurrence of early failures. Though the log-normal distribution is applicable in such situation there are certain
advantages in choosing the inverse Gaussian over the log-normal. For example inverse Gaussian is almost IFR (increasing failure rate) if it is not highly skewed. Chhikara and Folks (1977) have considered the inverse Gaussian distribution as failure model and have shown that its failure rate is non-monotonic, initially increasing and then decreasing. Certain properties of this distribution are studied by Tweedie (1957), Iyengar and Patwardhan (1988) in context of its use as a Statistical tool in reliability theory.

This Chapter deals with the maximum likelihood estimation of parameters with reference to type I Progressively censored samples from Weibull and inverse Gaussian distributions involving different parameters at each stage of censoring where the respective density functions of the Weibull and inverse Gaussian distributions are given by

\begin{equation}
 g(x) = \frac{\beta}{\theta} x^{\beta-1} \exp(-x^{\beta}/\theta), \quad x > 0, \beta > 0, \theta > 0
\end{equation}

\begin{equation}
 h(x) = \left[ \frac{-\lambda - x}{2 \mu^2} \right]^{1/2} \exp \left\{ -\lambda (x - \mu)^2 / 2 \mu^2 x \right\}, \quad x > 0, \mu > 0, \lambda > 0.
\end{equation}

The maximum likelihood equations are derived. These equations, being complicated, do not admit explicit solutions and hence iterative methods are indicated for obtaining the maximum likelihood (ML)
estimates of parameters at each stage of censoring. Asymptotic Variances and Covariances are obtained. A numerical example is given to illustrate the results on the basis of sample data generated from the inverse Gaussian (IG) distribution involving two-stage type I progressive censoring.

3.2 General Model and Likelihood

Let the life \( X \) of an item follow the distribution \( F(x|\Theta) \) with density \( f(x|\Theta) \) for \( \Theta \in \Omega \) where \( \Theta \) is a vector-valued parameter in a real parameter space \( \Omega \). Let \( X \) have the distribution function \( F(x|\Theta_i) \) in the interval \((T_{i-1}, T_i] \) for \( i = 1, 2, \ldots, k \) with \( T_0 = 0 \) and \( T_k = \infty \). Then using the result (1.1.1) of chapter 1 the composite density function \( f(x) \) of \( X \) over the range \((T_o, T_k)\) is given by

\[
\begin{align*}
(3.2.1) \quad f(x) &= \left\{ 
\begin{array}{ll}
  f(x|\Theta_1) & \text{for } T_0 < x \leq T_1 \\
  \frac{1-F(T_1|\Theta_1)}{1-F(T_{i-1}|\Theta_i)} \prod_{j=2}^{i-1} \left\{ \frac{1-F(T_{j-1}|\Theta_j)}{1-F(T_j|\Theta_j)} \right\} f(x|\Theta_i) & T_{i-1} < x \leq T_i, \quad i = 2, 3, \ldots, k-1 \\
  \frac{1-F(T_{k-1}|\Theta_{k-1})}{1-F(T_k|\Theta_{k-1})} \prod_{j=2}^{k-1} \left\{ \frac{1-F(T_{j-1}|\Theta_j)}{1-F(T_j|\Theta_j)} \right\} f(x|\Theta_{k-1}) & \text{for } T_{k-1} < x < T_k.
\end{array}
\right.
\end{align*}
\]
Let \( n \) items whose life time follows the distribution with density (3.2.1) be placed on a life test. Let \( n_i \) be the items which fail during the \( i \)-th stage \((T_{i-1}, T_i), i=1, 2, \ldots, k\) and let \( x^{(i)}_1 < x^{(i)}_2 < \ldots < x^{(i)}_{n_i} \) be the times of failure of the \( n_i \) items. Suppose censoring occurs in \( k \)-stages at times \( T_i > T_{i-1}, i=1, 2, \ldots, k \) and \( r_i \) surviving items are withdrawn (or censored) from further observations at the \( i \)-th stage. Further suppose that all items which remain on test after time \( T_k \) are eliminated and the test is terminated. Here \( r_i \) are either fixed or determined independently of the life span of \( X \). It may be noted that in \( k \)-stage type I progressive censoring with \( k > 1 \) if \( n_i > n - r_i \), the experiment is stopped at the first stage and otherwise the experiment is carried to the second stage.

For \( k \)-stage type I progressive censoring the likelihood function is given by

\[
L \propto \prod_{i=1}^{k} \left\{ \frac{n_i}{1-1} f(x^{(i)}_j) \right\} \prod_{i=1}^{k} \left\{ 1 - F(T_i) \right\}^{r_i}
\]

(3.2.2)
where

\[(3.2.3) \quad F(T_i) = \int_0^{T_i} f(x)dx\]

3.3 ML Estimation of the parameters \(\theta_i\) and \(\beta_i\) of the Weibull Distribution

We define

\[(3.3.1) \quad f(x|\theta_i) = g_i(x) = \frac{\beta_i}{\theta_i} x^{\beta_i-1} \exp\left(-x/\theta_i\right)\]

and

\[(3.3.2) \quad F(x|\theta_i) = G_i(x) = \int_0^x g_i(y)dy = 1 - \exp\left(-x/\theta_i\right)\]

Then using (3.2.1) and (3.2.2) it is easy to see that the likelihood \(L\) can be written as

\[(3.3.3) \quad L \propto \prod_{i=1}^{k} L_i\]

where

\[(3.3.4) \quad L_i = \prod_{j=1}^{n_i} \left\{ g_i(x_j^{(i)}) \right\} \left\{ 1-G_i(T_i) \right\}^{n_i-n_i}\]

for \(i=2,3, \ldots, k\) and \(n_1 = n\), \(n_i = n^{(i-1)} - n_{i-1} - r_{i-1}\) for \(i=2,3, \ldots, k\).

We note from (3.3.4) that \(L_i\) is proportional to a likelihood function due to a censored sample, namely,
from a truncated Weibull distribution.

\[ g_i(x) = \begin{cases} \frac{a^\alpha}{\theta_i} x^{a-1} & \text{for } T_{i-1} < x < \infty, \ i=1,2,\ldots,k \end{cases} \]

\[ 1-G_i(T_{i-1}) \]

Since \( \theta_i \)'s and \( \beta_i \)'s are different we get the estimates of \( \theta_i \) and \( \beta_i \) using single censored results. Hence the ML estimation equations are

\[ \frac{\partial \log L}{\partial \theta_i} = \frac{\partial \log L_i}{\partial \theta_i} = 0 \quad \text{and} \quad \frac{\partial \log L}{\partial \beta_i} = \frac{\partial \log L_i}{\partial \beta_i} = 0 \]

for \( i=1,2,\ldots,k \).

Differentiating (3.3.3) w.r.t. \( \theta_i \) and \( \beta_i \) and equating to zero we get in view of (3.3.6) the equations.

\[ \frac{\partial \log L_i}{\partial \theta_i} = -n_i + \frac{1}{\theta_i} \sum_{j=1}^{n_i} x_j^\beta_i \left( \sum_{j=1}^{n_i} x_j^\beta_i + (n^\alpha - n_i) \beta_i T_i - n^\alpha \beta_i T_{i-1} \right) = 0 \]

and

\[ \frac{\partial \log L_i}{\partial \beta_i} = n_i + \sum_{j=1}^{n_i} \log x_j^\beta_i - \frac{1}{\theta_i} \left\{ \sum_{j=1}^{n_i} x_j^\beta_i \log x_j^\beta_i \right\} + (n^\alpha - n_i) \beta_i \log T_i - n^\alpha \beta_i T_{i-1} \log T_{i-1} = 0 \]

If \( \theta_i \)'s and \( \beta_i \)'s are equal then summing (3.3.7) and (3.3.8) over \( i = 1,2,\ldots,k \) we get the equations as obtained by Cohen (1965) as special cases.
From (3.3.7) we get for $i = 1, 2, \ldots, k$.

(3.3.9) $\theta_i = D(\beta_i)/n_i$ where

(3.3.10) $D(\beta_i) = \left\{ \sum x_j^{(i)} \beta_i + (n^{(i)} - n_i) T_i \right\}$

Substituting the value of $\theta_i$ from (3.3.9) in (3.3.8) we get for $i = 1, 2, \ldots, k$.

(3.3.11) $\frac{\partial \log L_i}{\partial \beta_i} = \frac{n_i}{\beta_i} + \sum \frac{\log x_j^{(i)}}{D(\beta_i)} - \frac{n_i D'(\beta_i)}{D(\beta_i)} = 0$

where $D'(\beta_i)$ denotes the first derivative of $D$ w.r.t. $\beta_i$.

Thus given the estimate of $\beta_i$, the estimate of $\theta_i$ can be obtained from (3.3.9) as

(3.3.12) $\hat{\theta}_i = D(\hat{\beta}_i)/n_i$, $i = 1, 2, \ldots, k$.

Hence it is only necessary to obtain a solution of $\beta_i$ from equation (3.3.11), which will be the ML estimate of $\beta_i$ for $i = 1, 2, \ldots, k$. An iterative solution to this equation can be achieved by Newton-Raphson's method. The method of obtaining the initial solution $\beta_0$ in order to start the method of iteration is given in the next section.

In order to obtain the asymptotic standard errors of the estimates of $\theta_i$ and $\beta_i$, we differentiate (3.3.7)
and (3.3.8) again w.r.t $\theta_i$ and $\beta_i$ and obtain for $i = 1, 2, \ldots, k$.

\[
\begin{align*}
\frac{\sigma^2 \log L_i}{\partial^2 \theta_i^2} &= \frac{n_i}{\theta_i^2} \frac{2D(\beta_i)}{\theta_i^3} \\
\frac{\sigma^2 \log L_i}{\partial^2 \theta_i \partial \beta_i} &= \frac{D'(\beta_i)}{\theta_i^2} \\
\frac{\sigma^2 \log L_i}{\partial^2 \beta_i^2} &= \frac{-n_i}{\beta_i^2} \frac{D''(\beta_i)}{\theta_i}
\end{align*}
\]

(3.3.13)

where $D''(\beta_i)$ denotes the second derivative of $D$ w.r.t $\beta_i$.

If $\hat{\theta}_i$ and $\hat{\beta}_i$ denote the values of the last iteration then estimates of the asymptotic variances and covariance of the ML estimates of $\theta_i$ and $\beta_i$ for $i = 1, 2, \ldots, k$ can be obtained from the matrix

\[
\begin{pmatrix}
- \frac{\sigma^2 \log L_i}{\partial \theta_i^2} & - \frac{\sigma^2 \log L_i}{\partial \theta_i \partial \beta_i} \\
- \frac{\sigma^2 \log L_i}{\partial \theta_i \partial \beta_i} & - \frac{\sigma^2 \log L_i}{\partial \beta_i^2}
\end{pmatrix}
\]

(3.3.14)

\[
\begin{pmatrix}
- \frac{\sigma^2 \log L_i}{\partial \theta_i^2} & - \frac{\sigma^2 \log L_i}{\partial \theta_i \partial \beta_i} \\
- \frac{\sigma^2 \log L_i}{\partial \theta_i \partial \beta_i} & - \frac{\sigma^2 \log L_i}{\partial \beta_i^2}
\end{pmatrix}^{-1}
\]

As a matter of fact, we should have expected values in the matrix but empirical values given above are good enough for all practical purposes.
3.4 Method of obtaining the Initial Estimate $\beta_i^0$

We give the method of obtaining the initial estimate $\beta_i^0$ for solving the equation (3.3.11) by the method of iteration. From (3.2.1), (3.3.1) and (3.3.2) and using the result (3.3.5) it is easy to see that

$$\text{(3.4.1)} \quad P(x > S | x > T_{i-1}) = \frac{1-G_i(S)}{1-G_i(T_{i-1})} \quad \text{for } T_{i-1} < S < T_i$$

and

$$\text{(3.4.2)} \quad P(x > T_i | x > T_{i-1}) = \frac{1-G_i(T_i)}{1-G_i(T_{i-1})}$$

Now $n^{(i)}$ and $n^{(i+1)}$ are the number of items surviving the $i$-th and the $(i+1)$-th stage respectively.

For $T_{i-1} < S < T_i$ let $m_s$ be the number of items surviving beyond $S$. Then using (3.4.1) and (3.4.2) we have

$$\frac{m_s}{n^{(i)}} = \frac{1-G_i^*(S)}{1-G_i^*(T_{i-1})} \quad \text{and} \quad \frac{n^{(i+1)}}{n^{(i)}} = \frac{1-G_i^*(T_i)}{1-G_i^*(T_{i-1})}$$

where $^*$ denotes the estimated values of $G_i$. On account of (3.3.2) we can write (3.4.3) as

$$\text{(3.4.4)} \quad \frac{1}{\beta_i} = \frac{S_i - T_i}{T_i - T_{i-1}}$$

where

$$\frac{1}{\beta_i} = \frac{S_i}{T_i} - \frac{T_i}{T_i - T_{i-1}}$$
(3.4.5) \( l_i = \log \frac{n_{-1}^m}{n_{-1}^{(i+1)}} \) and \( l_2 = \log(n_{-1}^{(i+1)}/n_{-1}^{(i)}) \)

Note that for \( i = 1 \) we can solve (3.4.4) explicitly for \( \beta_1^0 \) since \( T_0 = 0 \). For \( i > 1 \) we construct using (3.4.4) a function as follows:

For \( 1 < a < b < \infty \) define

\[
(3.4.6) \quad Q(\beta_i) = \frac{a^{\beta_i} - 1}{\beta_i} \cdot \frac{1}{b^{\beta_i} - 1} \quad \text{for } \beta_i \neq 0
\]

\[
= \frac{\log a}{\log b} \cdot \frac{l_1}{l_2} \quad \text{for } \beta_i = 0
\]

(3.4.7) \( a = \frac{S}{T_i} \) and \( b = \frac{T_i}{T_{i-1}} \)

Note that the second member on the right of (3.4.6) is obtained using L'Hospital's rule. Using Maclaurin's theorem and neglecting powers greater than the first, the initial estimate of \( \beta_i^0 \) is given by

\[
(3.4.8) \quad \beta_i^0 = \frac{2(\log a - \frac{1}{l_2}\log b)}{(\log b - \log a)\log a} \quad \text{for } i = 2, 3, \ldots, k.
\]

Using this as an initial estimate one can solve the equation (3.3.11) iteratively to obtain the estimate of \( \beta_i \). We can also use the method of hazard
plot to obtain the initial estimate $\beta_i^0$ for solving the equation (3.3.11).

### 3.5 ML Estimation of the parameters of the IG Distribution

In this section we consider the ML estimation of parameters of the IG distribution with reference to $k$-stage type I progressive censoring.

Let

$$f(x|\theta_i) = (\lambda_i x^{-3/2}) \phi\left(\sqrt{-\frac{\lambda_i}{\mu_i}} \left(\frac{x}{\mu_i} - 1\right)\right)$$

where $\phi(y) = \left(\sqrt{2\pi}\right)^{-1} \exp(-y^2/2)$ and

$$
\begin{align*}
\gamma_{ji} &= \sqrt{-\frac{\lambda_i}{T_j}} \left(\frac{T_j}{\mu_i} - 1\right), \\
\delta_{ji} &= -\sqrt{-\frac{\lambda_i}{T_j}} \left(1 + \frac{T_j}{\mu_i}\right)
\end{align*}
$$

(3.5.2)

$$
\Theta(y) = \int_{-\infty}^{y} \phi(z) dz
$$

Then using (3.2.1) and (3.2.2) the likelihood function $L$ in case of type I progressive censoring involving parameters $(\mu_i, \lambda_i)$ at the $i$-th stage is given by

$$L \propto \prod_{i=1}^{k} L_i$$

(3.5.3)

$$L_i = \left(\frac{\lambda_i}{\lambda_i}\right)^{2} \prod_{j=1}^{n_i} \left\{ \left(\frac{x_{(j)}^{(i)}}{x_j}\right)^{-9/2} \phi\left(\sqrt{-\frac{\lambda_i}{x_{(j)}^{(i)}}} \left(\frac{x_{(j)}^{(i)}}{\mu_i} - 1\right)\right) \right\}$$

(3.5.4)

$$L_i = \left\{ 1 - \Theta(\gamma_{ii}) - e^{2\lambda_i / \mu_i} \Theta(\delta_{ii}) \right\}^{n_i - n_i}$$
The maximum likelihood estimating equations are given by

\[ \frac{\partial \log L}{\partial \lambda_i} = \frac{\partial \log L_i}{\partial \lambda_i} = 0 \quad \text{and} \quad \frac{\partial \log L}{\partial \mu_i} = \frac{\partial \log L_i}{\partial \mu_i} = 0 \]

for \( i = 1, 2, \ldots, k \).

After algebraic manipulation we can write the likelihood equations as

\[ \frac{n_i - \lambda_i}{n_i} \bar{x}_i^{(i)} - \frac{2\lambda_i n_i}{\mu_i} + \lambda_i \sum_{j=1}^{2} \frac{1}{x_j^{(i)}} = n^{(i)}Z_{i+1,i} - (n^{(i)} - n_i)Z_{i,i} - n_i \]

\[ \frac{n_i \bar{x}_i^{(i)}}{\mu_i} = n_i \xi_{i-1,i} - (n^{(i)} - n_i)\xi_{ii} \]
where

\[
\begin{align*}
-x^{(i)} & = \sum_{j=1}^{n_i} x_{ij}^{(i)}/n_i \\
Z_{ji} & = (B_{ji} + C_{ji} \delta_{ji} + 4D_{ji}(\lambda_i/\mu_i))/A_{ji} \\
\xi_{ji} & = \left(\phi(Y_{ji})/\sqrt{-\frac{T}{\lambda_i}}\right) - C_{ji}/\sqrt{-\frac{T}{\lambda_i}} + 2D_{ji}/A_{ji} \\
B_{ji} & = r_{ji}(\phi(Y_{ji})), \quad C_{ji} = e^{2\lambda_i/\mu_i} \phi(\delta_{ji}) \\
D_{ji} & = e^{2\lambda_i/\mu_i} \varnothing(\delta_{ji}), \quad A_{ji} = 1 - \varnothing(Y_{ji}) - D_{ji}
\end{align*}
\]

for \( j = i-1, \ i \) and \( i = 1, 2, \ldots, k \).

We note that in type I progressive censoring \( n_i \)'s are random variables. Since \( n^{(i)} \) involves \( n_i \), \( n^{(i)} \) are also random variables. It is easy to see that

\[
E(n_i) = E(n^{(i)}) \left\{ A_{ii} - A_{i-1,i} \right\} / (1 - A_{i-1,i})
\]

for \( i = 1, 2, \ldots, k \).

Since \( n^{(i)} = n \) and \( n^{(i)} = n - \sum_{j=1}^{i-1} n_j - \sum_{j=1}^{i-1} x_j \) for \( i > 1 \)

the expected value of \( n_i \) is given by

\[
E(n_1) = n A_{11}
\]

\[
E(n_2) = \left[ n(1-A_{11}) - x_1 \right] (A_{22} - A_{12})/(1-A_{12})
\]

\[
E(n_i) = \left[ n - \sum_{j=1}^{i-1} x_j \right] \left\{ \frac{A_{ii} - A_{i-1,i}}{1 - A_{ii}} \right\} \left\{ \frac{1 - A_{ii}}{1 - A_{i-1,i}} \right\}
\]

\[
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quasi-
It is easy to verify that

\[
E(n^{(i)}) = \left( n^{(i)} - \lambda_i \right)^{(-1)} \sum_{j=1}^{i-1} \frac{1}{1-A_{ij}} (1-A_{i1})
\]

for \( i = 3, 4, \ldots, k \).

Similarly

\[
\frac{\partial^2 \log L}{\partial \lambda_i^2} = 4\lambda_i^{-2} \left( 2n_i - \frac{\left(n^{(i)} - n_i\right) E_{ii}}{A_{ii}} - n^{(i)} A_{i-1, i} E_{i-1, i} \right)
\]

\[
\frac{\partial^2 \log L}{\partial \mu_i^2} = \lambda_i \mu_i^{-4} \left[ -3n_i x^{(i)} + 2n_i \mu_i^2 + (n^{(i)} - n_i) A_{i-1, i} \right]
\]

\[
\frac{\partial^2 \log L}{\partial \mu_i \partial \lambda_i} = \frac{n_i x^{(i)}}{\lambda_i} - \frac{n_i \mu_i}{\mu_i} + \frac{(n^{(i)} - n_i)}{2\lambda_i^2} \frac{T_i}{\lambda_i} \frac{F_{i-1, i}}{A_{i-1, i}}
\]
where

\[
\begin{align*}
E_{ji} &= A_{ji} Z_{ji} (2 - Z_{ji}) - B_{ji} (1 - \gamma_{ji}^2) - C_{ji} \delta_{ji} (1 - \delta_{ji}^2) \\
&\quad - 4 D_{ji} \alpha_i \mu_i^{-1} \\
F_{ji} &= \phi(\gamma_{ji}) (1 - \gamma_{ji}^2) - C_{ji} \delta_{ji} \gamma_{ji} - C_{ji} \left(1 + \frac{4 \lambda_i}{\mu_i} \right) \\
&\quad + 4 \left(1 + \frac{2 \lambda_i}{\mu_i} \right) D_{ji} \sqrt{\frac{\lambda_i}{T_j}} + A_{ji} Z_{ji} \xi_{ji} \sqrt{\frac{\lambda_i}{T_j}} \\
G_{ji} &= B_{ji} T_j + C_{ji} \delta_{ji} T_j + 4 C_{ji} \sqrt{\frac{\lambda_i T_j}{\mu_i}} - 4 \lambda_i D_{ji} \\
&\quad - \xi_{ji} A_{ji} (2 \mu_i + \lambda_i \xi_{ji})
\end{align*}
\]  

(3.5.16)

for \(j = i - 1, i\) and \(i = 1, 2, 3, \ldots, k\).

The asymptotic variance-covariance matrix of the estimates of \(\mu_i\) and \(\lambda_i\) can be obtained from the relation

\[
\begin{bmatrix}
-\text{E} \left( \frac{\partial^2 \log L}{\partial \mu_i^2} \right) & -\text{E} \left( \frac{\partial^2 \log L}{\partial \mu_i \partial \lambda_i} \right) \\
& \text{E} \left( \frac{\partial^2 \log L}{\partial \lambda_i^2} \right)
\end{bmatrix}^{-1}
\]  

(3.5.17)

The asymptotic covariance between different pairs \((\hat{\mu}_i, \hat{\lambda}_i), i = 1, 2, \ldots, k\) are zero.

It may be noted that the ML equations (3.5.7) and (3.5.8) are not explicitly solvable. We can use the method of iteration to solve them. For using the method of iteration, we need the initial solutions \(\mu_o^i\) and \(\lambda_o^i\). These may be chosen as \(\mu_o^i = \tilde{x}(i)\) and
\[ \lambda_i^{-1} = \frac{1}{n_i - 1} \sum \left( \frac{1}{x_j^{(i)}} - \frac{1}{x^{(i)}} \right) \text{ where } x^{(i)} = \sum_{j=1}^{n_i} \frac{x_j^{(i)}}{n_i}. \]

Writing \( \hat{\lambda}_i = \lambda_i^0 + a_{i1} \) and \( \hat{\mu}_i = \mu_i^0 + a_{i2} \), \( i = 1, 2, 3, \ldots \) where \( a_{i1} \) and \( a_{i2} \) are the corrections to be determined by the iterative process. Using Taylor's theorem and neglecting higher powers of \( a_{i1} \) and \( a_{i2} \) above the first, we get the equations

\[
\begin{align*}
\frac{\partial^2 \text{log} L}{\partial \lambda_i^0} a_{i1} + \frac{\partial^2 \text{log} L}{\partial \lambda_i^0 \partial \mu_i^0} a_{i2} & = - \frac{\partial \text{log} L}{\partial \lambda_i^0} \\
\frac{\partial^2 \text{log} L}{\partial \mu_i^0} a_{i1} + \frac{\partial^2 \text{log} L}{\partial \mu_i^0 \partial \lambda_i^0} a_{i2} & = - \frac{\partial \text{log} L}{\partial \mu_i^0} 
\end{align*}
\]  
(3.5.18)

Solving these equations we can determine the corrections \( a_{i1} \) and \( a_{i2} \). The process can be repeated until the values of \( \hat{\lambda}_i \), \( \hat{\mu}_i \) get stabilised. The coefficients of \( a_{ij} \)'s can be computed from the equations (3.5.13) to (3.5.15).

3.6 Numerical Example

We give a numerical example to illustrate the results given in section 3.5 in case of IG distribution. The following maintenance data were reported on active repair times (in hours) for an airborne communication transceiver due to Von Alven, W.H. (1964).
The corresponding initial estimates for \((\mu_1, \lambda_1)\) and 
\((\mu_2, \lambda_2)\) are \(\hat{\mu}_1^0 = 1.24, \hat{\lambda}_1^0 = 3.0503, \hat{\mu}_2^0 = 5.5308,\)
\(\hat{\lambda}_2^0 = 48.3723.\)

Using Newton's method of iteration and solving the
equations given by (3.5.18).

Hence the first iterates for \((\mu_1, \lambda_1)\) and \((\mu_2, \lambda_2)\) are
\(\hat{\mu}_1^{(1)} = 1.4691, \hat{\lambda}_1^{(1)} = 2.8911, \hat{\mu}_2^{(1)} = 5.6838, \hat{\lambda}_2^{(1)} = 47.4659.\)

Using these as initial estimates and repeating the
procedure of iteration the final estimates of \(\mu_1,\)
\(\lambda_1(i=1,2)\) are \(\hat{\mu}_1 = 1.6873, \hat{\mu}_2 = 5.8013\)
\(\hat{\lambda}_1 = 2.1428, \hat{\lambda}_2 = 42.7567.\)

The corresponding estimated Variance-Covariance
matrices of \(\hat{\mu}_i, \hat{\lambda}_i\) are
\[
\begin{bmatrix}
V(\hat{\mu}_1) & \text{Cov}(\hat{\mu}_1, \hat{\lambda}_1) \\
- & V(\hat{\lambda}_1)
\end{bmatrix}
= \begin{bmatrix}
37.3867 & 3.4557 \\
- & 25.1824
\end{bmatrix}^{-1}
= \begin{bmatrix}
0.02709 & -0.00372 \\
- & 0.04022
\end{bmatrix}
\]

\[
\begin{bmatrix}
V(\hat{\mu}_2) & \text{Cov}(\hat{\mu}_2, \hat{\lambda}_2) \\
- & V(\hat{\lambda}_2)
\end{bmatrix}
= \begin{bmatrix}
8.8520 & 0.5023 \\
- & 0.0547
\end{bmatrix}^{-1}
= \begin{bmatrix}
0.23588 & -2.16603 \\
- & 38.17177
\end{bmatrix}
\]