Chapter 5

Radiative Alfvén condensation instability

5.1 Introduction

It is now widely recognized that the plasma edge region plays a crucial role in controlling the bulk stability and confinement properties of a tokamak plasma. This is manifest in macro and microscopic phenomena such as the density limit and edge turbulence, respectively. At the density limit, the total impurity radiation power becomes equal to the total input power to the tokamak discharge. Above the density limit, the discharge undergoes a thermal collapse. When the density is increased towards the density limit, the phenomenon of multifaceted asymmetric radiation from the edge (MARFE) arises. The MARFE, in a sense, is a predisruptive, quasi-steady state of the tokamak discharge, generally achieved by increasing density while keeping the plasma current constant. MARFEs are observed as poloidally asymmetric but toroidally symmetric, dense, cool regions of standing or moving strongly radiating belts having short poloidal and radial extent at the inner edge of tokamak plasmas. Such radiative cooling and condensation are important in astrophysical plasmas as well.

There is general agreement that MARFEs are formed as a result of the radiative condensation instability. If a poloidally asymmetric perturbation in the tokamak edge plasma locally increases the density, the region radiates more copiously (owing to the radiative loss rate being proportional to the square of the local plasma density) and cools; pressure equilibration along the magnetic field lines then brings more density in, giving a positive feedback mechanism driving the instability. We have already seen that
Chapter 5: Radiative Alfvén condensation instability

Radiative condensation instability effects play an important role in the physics of the tokamak edge region. There exist several theoretical investigations of the linear coupling between the radiative instability and various equilibrium and normal modes [1–6]. In most of these studies it has been shown that compressible, electrostatic ion acoustic mode couples via temperature perturbation to the radiative thermal mode to give rise to the aforementioned radiative condensation instability. Taking into account the effect of equilibrium density and temperature gradients on the radiative condensation instability, it was found from a local analysis that the linear coupling of the ion acoustic, drift and radiation modes can lead to radiative condensation instabilities of the drift wave [7, 8]. This analysis was further done in a collision dominated nonuniform plasma, and it was found that the drift waves are further destabilized in the presence of impurity radiation loss [9]. However in all these analysis, electrostatic approximation is used. This assumption is valid in the low β (β = 8πnT/B² is the ratio of kinetic to magnetic pressure) regime, i.e., β ≪ m_e/M. However, as the plasma pressure increases towards the L-H transition regime the electrostatic limit becomes untenable, and the β is at or above m_e/M. Furthermore, as will be shown later in the analysis, the strong inhomogeneities lead to an enhancement of β effects. At these higher values of β, the coupling of the radiative effects to magnetic perturbations in the plasma, like the shear and compressional Alfvén waves, may assume also importance. It is this regime of parameter space that we wish to explore in this chapter. It is to be noted that such plasmas in the high-confinement regime are more relevant to the physics of future fusion reactors.

There have been some studies which have taken into consideration the coupling of magnetic perturbations with the radiative condensation instability. It was found by Field [1] that in the magnetohydrodynamic limit the magnetoacoustic waves are affected by the radiative thermal mode, but the Alfvén waves and the radiative mode are not linearly coupled. However, later it was shown by Shukla et al. that finite gyroradius effects can couple the shear Alfvén waves to the radiative thermal mode to produce an Alfvén condensation instability [10]. However, in these early works several physical effects relevant to the tokamak edge problem have not been taken into account. Thus, when the growth rates of the instabilities become of order ω*, we must include
Chapter 5: Radiative Alfvén condensation instability

the effects of plasma inhomogeneities. Similarly, since many tokamak edge plasmas are typically collisional (connection length larger than mean-free-paths), the use of parallel collisional dynamics of electrons is more appropriate than the one depending on collisionless electron inertia effects.

In this chapter we have reexamined the radiative Alfvén condensation instability, in the local and nonlocal limits, taking into consideration some of the points mentioned above.

The plan of the chapter is as follows. In Sec. 5.2 we derive the basic equations for the radiative condensation instability with the additional terms arising from interaction with the magnetic perturbations. In Sec. 5.3, the local dispersion relation is derived. In Sec. 5.4, we discuss the nonlocal dispersion relation. In Sec. 5.5, the local and nonlocal dispersion relations are numerically studied and a brief summary of the conclusions is presented.

5.2 Theoretical Model and Analysis

The radiative condensation instability with inclusion of coupling to shear Alfvén perturbations is described by the following set of reduced Braginskii equations with the addition of an optically thin radiative energy loss term $L(n,T)$.

$$\frac{\partial n}{\partial t} + \nabla \cdot (nv_E + n\nu_s) - \nabla \cdot \left( \frac{nc}{\Omega_B} \frac{d^\|}{dt} \nabla \perp \phi \right) + \nabla \| (nv\|) = 0$$

$$m_e n \left( \frac{\partial}{\partial t} + v_E \cdot \nabla \right) v_{\|} = -\nabla \| (p_i + p_e)$$

$$\nabla \cdot \left( \frac{nc}{\Omega_B} \frac{d^\|}{dt} \nabla \perp \phi \right) - \nabla \cdot (nv_{se} + n\nu_s) = \frac{1}{c} \nabla \| j\|$$

$$\frac{3}{2} n \frac{d^{(e)}}{dt} T_e - T_e \frac{dn}{dt} - \nabla \| \kappa_{\|} \nabla \| T_e - \nabla \cdot \kappa_{\perp e} \nabla T_e - \frac{5c}{2e} \nabla \cdot \left( \frac{p_e}{B^2} B \times \nabla T_e \right)$$

$$- \frac{0.71}{c} T_e \nabla \| j\| = H - L$$

$$\nabla^2 A\| = -\frac{4\pi}{c} j\|$$
Chapter 5: Radiative Alfvén condensation instability

\[ j_\parallel = neD_\parallel \left[ \frac{1}{n} \nabla_\parallel n + \frac{1.71}{T_e} \nabla_\parallel T_e - \frac{e}{T_e} \nabla_\parallel \phi - \frac{e}{cT_e} \frac{\partial A_\parallel}{\partial t} \right] \]  \hspace{1cm} (5.6)\]

where the parallel current follows directly from the electron force equation, and

\[ \frac{d^{(\alpha)}}{dt} = \frac{\partial}{\partial t} + \mathbf{v}_\alpha \cdot \nabla \]  \hspace{1cm} (5.7)\]

is the convective derivative of species \( \alpha \) with \( \mathbf{v}_\alpha = \mathbf{v}_E + \mathbf{v}_\alpha + \mathbf{v}_{\parallel \alpha} \mathbf{\hat{e}}_2 \). The velocities are the \( \mathbf{E} \times \mathbf{B} \) and the diamagnetic drifts, respectively, and have the definitions \( \mathbf{v}_E = (c/B^2) \mathbf{B} \times \nabla \phi \) and \( \mathbf{v}_\alpha = -(c/q_\alpha nB^2) \mathbf{B} \times \nabla p_\alpha \).

The charge of a species \( \alpha \) is denoted by \( q_\alpha \). The variables \( n, T_e, p, \phi, v_{\parallel \alpha}, \) and \( A_\parallel \) represent the density, electron temperature, pressure, electric potential, the ion parallel velocity, and the parallel component of the vector potential, respectively; \( \kappa_{\parallel c} = 3.2nT_e/m_e\nu_{ci} \) and \( \kappa_{\perp c} \) are the parallel and perpendicular thermal conduction coefficients, respectively; \( H \) is an arbitrary heating function; \( D_\parallel = T_e/0.51m_e \nu_{ce} \) is the parallel electron diffusion and \( \Omega_i = eB/m_i c \) is the ion gyrofrequency. Equation (5.1) is the ion continuity equation. Equation (5.2) is the one-fluid momentum equation. Equation (5.3) is the vorticity equation, which is simply the difference between the electron and ion continuity equations. Equations (5.4) and (5.5) are the electron temperature and the parallel component of the Ampere’s law, respectively. Quasineutrality is assumed.

Before linearizing the equations, it is useful to make a few notable simplifications. We consider the simple slab model of an inhomogeneous plasma in a constant external magnetic field \( \mathbf{B} \mathbf{\hat{e}}_2 \). For simplicity we have assumed that the ions are cold \( (T_i = 0) \). Taking into account a nonvanishing \( A_\parallel \) leads to two additional effects compared to an electrostatic description. First, the parallel electric field contains an inductive part \( c^{-1} \partial_t A_\parallel \). Magnetic induction is important because it affects the linear response of the current to the forces, and hence the speed of waves and their dissipation. Secondly, the parallel gradient gets an additional contribution due to the perturbed magnetic field \( \mathbf{B}_1 \), called 'magnetic flutter'. Therefore, the operator \( \nabla_\parallel \) when operating on the variable \( X \), has the linearized form,

\[ \nabla_\parallel X = \nabla_\parallel^0 X - i\omega_X A_\parallel \]  \hspace{1cm} (5.8)\]
Chapter 5: Radiative Alfvén condensation instability

where, \( \omega_X = k_y \rho_s c_s / L_X \). Here \( k_y = m / a \) is the poloidal wave vector, \( \rho_s = c_s / \Omega_i \) is the ion Larmor radius, \( c_s = \sqrt{\rho_e / m_i} \) is the sound speed, and \( L_X \) is the gradient scale length of the variable \( X \).

We now represent \( n = n_0(x) + n^1, T_e = T_{e0}(x) + T_e^1, \phi = \phi^1, A_\parallel = A_\parallel^1, \) and \( v_\parallel^1 = v_\parallel^1 \), where \( n_0 \) and \( T_{e0} \) denote the local density and temperature equilibrium, and superscript \( 1 \) corresponds to perturbed quantities. Each perturbed function \( f^1 \) is now written as

\[
f^1(x, y, z) = f^1(x, y) \exp \left( i k_\parallel z - i \omega t \right) \tag{5.9}\]

where, the coordinates \( x, y, z \) represent the radial, poloidal, and axial coordinates, respectively. Defining the dimensionless density, temperature, potential, and parallel velocity by \( \tilde{n} = n^1 / n_0, \tilde{T}_e = T_e^1 / T_{e0}, \tilde{\phi} = \phi^1 / T_{e0}, \tilde{A}_\parallel = (c_v A/c_{Te}) A^1, \) \( \tilde{v}_\parallel = v_\parallel^1 / c_s \), where, \( c_v = B/\sqrt{4\pi n_0 m_i} \) is the Alfvén velocity. The linearized set of above Eqs. (5.1) - (5.4) and (5.6) then become

\[
k_\parallel c_s \tilde{v}_\parallel = \omega \tilde{n} - \omega_s \tilde{\phi} \tag{5.10}\]

\[
\omega \tilde{v}_\parallel = k_\parallel c_s \left( \tilde{n} + \tilde{T}_e \right) - \sqrt{\frac{\beta}{2}} (1 + \eta_e \omega_s \tilde{A}_\parallel \right) \tag{5.11}\]

\[
\omega \rho_s^2 \nabla^2 \tilde{\phi} = -k_\parallel n_0 \tag{5.12}\]

\[
\chi_{\perp e} \nabla^2 \tilde{T}_e = -\frac{3}{2} i \omega \tilde{T}_e + i \omega \tilde{n} + i \left( \frac{3}{2} \eta_e - 1 \right) \omega_s \tilde{\phi} + \chi_{\parallel e} k_\parallel^2 \tilde{T}_e - \frac{3}{2} \eta_e \frac{\chi_{\parallel e} c_s}{\rho_s} \omega_s \tilde{\phi} - 0.71 i k_\parallel \frac{D_{\parallel e}}{c_{Te}} + \frac{\gamma_n}{\gamma_T} - \gamma_T \tilde{T}_e \tag{5.13}\]

\[
j_\parallel^1 = \frac{\rho_n e D_{\parallel e}}{c_s} \left[ k_\parallel c_s \left( \tilde{n} + 1.71 \tilde{T}_e - \tilde{\phi} \right) - \sqrt{\frac{\beta}{2}} (\omega_s + 1.71 \eta_e \omega_s - \omega) \tilde{A}_\parallel \right] \tag{5.14}\]

where, \( k_\parallel^{-1} \sim q R, \beta = 8 \pi n_0 T_{e0} / B^2, \eta_e = L_n / L_{Te}, \chi_{\parallel (1)e} = k_\parallel (\omega_e / n_0, \gamma_n = 2 L_n / n_0 T_{e0}, \gamma_T = -1 / n_0 (\partial L / \partial T_e), \) and in evaluating \( \gamma_n \) and \( \gamma_T \) we have used the fact that the radiation loss term in the coronal limit can be written as \( L \propto n^2 F(T_e) \), where \( F(T_e) \) is the temperature dependent part of the radiation loss function, and \( \partial F / \partial T_e < 0 \). The
third and fifth term on the right side of Eq. (5.13) represent the \( \mathbf{E} \times \mathbf{B} \) convection, and modification of the parallel heat flux due to ‘magnetic flutter’, respectively.

Now, substituting for \( j_{\parallel} \) from Eq. (5.5) into Eq. (5.3), we get a relation between the perturbed vector potential \( \vec{A}_\parallel \) and the potential \( \vec{\phi} \), as

\[
\vec{A}_\parallel = \sqrt{\frac{\beta}{2}} \left( \frac{\omega}{k_{\|}c_s} \right) \vec{\phi}
\]  

(5.15)

From Eqs. (5.10), (5.11), and (5.15), we get the following relation between the perturbed density, temperature, and potential:

\[
\tilde{n} = A_1 \tilde{T}_e + A_2 \vec{\phi}
\]  

(5.16)

with,

\[
A_1 = \frac{k_{\|}^2 c_s^2}{\omega^2 - k_{\|}^2 c_s^2}, \quad A_2 = \frac{\omega \omega_s}{\omega^2 - k_{\|}^2 c_s^2} \left[ 1 - \frac{\beta}{2} (1 + \eta_e) \right]
\]  

(5.17)

Using Eqs. (5.14) - (5.16) in Eq. (5.12) and Eq. (5.13), we get the coupled equation describing the radiative condensation instability in the presence of finite magnetic perturbations.

\[
\nabla^2 \vec{\phi} + \frac{i}{\rho_s^2} \left( \frac{k_{\parallel}^2 D_{\parallel \theta \rho}}{\omega} \right) A_1 \vec{\phi} + \frac{i}{\rho_s^2} \left( \frac{k_{\parallel}^2 D_{\parallel \theta \phi}}{\omega} \right) A_2 \tilde{T}_e = 0.
\]  

(5.18)

\[
\nabla^2 \tilde{T}_e - \frac{1}{\chi_\perp \epsilon} \left[ -\frac{3}{2} \frac{i \omega}{\omega_e} + \frac{i \omega}{\omega_e} A_1 + k_{\|}^2 \chi_{\| \phi} \right] + 0.71 k_{\|}^2 D_{\parallel \theta \phi} A_1 + \gamma_e A_1 \tilde{T}_e
\]

\[
- \frac{1}{\chi_\perp \epsilon} \left[ \frac{i \omega A_2 + i \left( \frac{3}{2} \eta_e - 1 \right)}{\omega_e} \omega_s - \frac{\beta}{2} \frac{\eta_e}{\epsilon_s} \omega_s \omega_s + 0.71 k_{\|}^2 D_{\parallel \theta \phi} A_1 + \gamma_e A_2 \right] \tilde{T}_e
\]

\[= - \frac{\gamma_e}{\chi_\perp \epsilon} \tilde{T}_e
\]  

(5.19)

where,

\[
A_3 = 1.71 + A_1
\]  

(5.20)

\[
A_4 = A_2 - 1 - \frac{\beta}{2} \left[ \frac{\omega (\omega + 1.71 \eta_e \omega - \omega)}{k_{\|}^2 c_s^2} \right]
\]  

(5.21)
5.3 Local Instability Analysis

In this section we examine the local dispersion relation of the radiative condensation instability, modified by the presence of equilibrium density and temperature gradients as well as finite magnetic perturbations. In this local limit we treat $\nabla^2_\perp$ as $-k^2_\perp$. Then from Eq. (5.18) we get the following relation between perturbed potential $\tilde{\phi}$ and the perturbed electron temperature $\tilde{T}_e$

$$\tilde{\phi} = B_1 \tilde{T}_e$$  \hspace{1cm} (5.22)

where,

$$B_1 = \frac{ik^2||D||e A_3}{k^2_|| \rho^2_o \omega - ik^2_\perp D||e A_4}$$  \hspace{1cm} (5.23)

Using the relation given by Eq. (5.22) in Eq. (5.19), we get the following local dispersion relation:

$$i\omega \left[ -\frac{3}{2} + A_1 + A_2 B_1 + i\frac{3}{2} \eta_e \chi'||_e \omega^2 B_1 \right] + \gamma_D + i \left( \frac{3}{2} \eta_e - 1 \right) \omega B_1 + 0.71 k^2||D||e (A_3 + A_4 B_1) + \gamma_n (A_1 + A_2 B_1) - \gamma_T = 0$$  \hspace{1cm} (5.24)

where $\gamma_D = k^2|| \chi'||_e + k^2_\perp \chi'_{\perp}$. Using this dispersion relation, we study the different limiting cases for the various modes, i.e., acoustic mode, radiative mode, drift mode, and Alfvén mode.

(i) In the limit of a homogeneous plasma, neglect of the magnetic perturbations, and $\gamma k^2_|| \rho^2_o \ll k^2||D||e$, Eq. (5.24) reduces to the following cubic equation (Ref. Stringer)

$$\frac{3}{2} \omega^3 + i (\gamma_D - \gamma_T) \omega^2 - \frac{5}{2} k^2|| D||e \omega + i (\gamma_n + \gamma_T - \gamma_D) k^2_|| c^2_e = 0$$  \hspace{1cm} (5.25)

The roots of this equation correspond to two acoustic modes and one radiative mode. For $\omega \gg k|| c_\perp$, one finds a purely growing mode with the growth rate $\gamma = 2(\gamma_T - \gamma_D)/3$, corresponding to the thermal instability, whereas in the opposite limit, the growth rate is $\gamma = 2(\gamma_n + \gamma_T - \gamma_D)/5$. This is the standard dispersion relation of the MARFE instability. The first two terms are the condensation and radiation effects, respectively, which are destabilizing and the last term represents the thermal conduction stabilization. For $\omega \sim k|| c_\perp > \gamma_n, \gamma_T, \gamma_D$ we get the acoustic mode whose frequency is given
Chapter 5: Radiative Alfvén condensation instability

by

\[ \omega_b = \pm \sqrt{\frac{5}{3} k_\parallel c_s} \]

and the growth rate is

\[ \gamma = \frac{2}{15} \left( \gamma_T - \frac{3}{2} \gamma_n - \gamma_D \right) \]

This shows that the acoustic modes can become unstable for \( \gamma_T > 1.5 \gamma_n + \gamma_D \).

(ii) In case of retaining the plasma inhomogeneity, but neglecting the magnetic perturbations, and taking the limit \( \gamma k_\parallel^2 \rho_s^2 \ll k_\parallel^2 D_{||}\omega_c \), we get

\[
\frac{3}{2} \omega^2 + \left[ i (\gamma_D - \gamma_T) - \frac{3}{2} (1.71 \eta_e + 1) \omega_s \right] \omega^2 \\
+ \left[ i (1.71 \gamma_n + \gamma_T - \gamma_D) \omega_s - \frac{5}{2} k_\parallel^2 c_s^2 \omega \right] \\
+ \left[ i (\gamma_n + \gamma_T - \gamma_D) + 0.71 \left( \frac{3}{2} \eta_e - 1 \right) \omega_s \right] k_\parallel^2 c_s^2 = 0 \quad (5.26) \\
\]

Such a dispersion relation has been derived earlier by Shukla et al. We analyze this relation for \( \omega \sim \omega_s > \gamma_n, \gamma_T, \gamma_D, k_\parallel c_s \). The frequency of the mode is

\[ \omega = (1.71 \eta_e + 1) \omega_s \]

and the growth of the mode is

\[ \gamma \approx \frac{1.14 [ \eta_e (\gamma_T - \gamma_D) - \gamma_n ]}{1 + 1.71 \eta_e} \]

This shows that parallel and perpendicular thermal conduction (\( \gamma_D \)) and condensation (\( \gamma_n \)) provide the stabilization of the mode.

(iii) We now investigate the full dispersion relation Eq. (5.24), i.e., including the finite beta terms. These terms lead to the coupling of the Alfvén waves to the radiative condensation instability. However, Eq. (5.24) is quite complex and so to bring out this coupling more clearly, we rederive a simplified local dispersion relation as follows:

In the limit \( \omega \gg k_\parallel c_s \) (valid for the Alfvén mode under consideration), the ion continuity equation yields

\[ \tilde{n} = \frac{\omega_s}{\omega} \tilde{\phi} \quad (5.27) \]
Now substituting for $A_\parallel$ from Eq. (5.15) in Eq. (5.14), and taking the limit $\gamma k_\parallel^2 \rho_e^2 \ll k_\parallel^2 D_{\parallel}\omega_e$, we obtain a relation between $\hat{n}$, $\tilde{T}_e$, and $\phi$, as

$$\hat{n} + 1.71 \tilde{T}_e = \left[ 1 + \frac{\omega \left( \omega_\ast + 1.71 \eta_e \omega_\ast - \omega \right)}{k_\parallel^2 v_A^2} \right] \phi$$

(5.28)

Using Eq. (5.27) for replacing $\phi$ into Eq. (5.28), we get

$$\left[ \omega_\ast - \omega - \frac{\omega^2 (\omega_\ast + 1.71 \eta_e \omega_\ast - \omega)}{k_\parallel^2 v_A^2} \right] \hat{n} = -1.71 \omega \tilde{T}_e$$

(5.29)

This relation between $\hat{n}$ and $\tilde{T}_e$ is different from the one obtained from the parallel pressure balance condition, (i.e., $\hat{n} = -\tilde{T}_e$). Here the $E \times B$ convection brings in more density in the region where the temperature drops.

The electron temperature equation is simply written as

$$\frac{3}{2} \omega \tilde{T}_e = \gamma_0 \hat{n} - \gamma_\gamma \tilde{T}_e$$

(5.30)

Thus from Eqs. (5.29) and (5.30) we get the following quartic dispersion relation

$$\omega^4 - \left[ (1 + 1.71 \eta_e) \omega_\ast + \frac{2}{3} \gamma_\gamma \right] \omega^3 - \left[ k_\parallel^2 v_A^2 - \frac{2}{3} (1 + 1.71 \eta_e) \omega_\ast \gamma_\gamma \right] \omega^2$$

$$+ \left[ \omega_\ast + \frac{2}{3} \gamma_\gamma \right] k_\parallel^2 v_A^2 \omega - i \left[ 1.14 \gamma_\gamma + \frac{2}{3} \gamma_\gamma \right] \omega \omega_\ast - i = 0$$

(5.31)

For $\omega \sim k_\parallel v_A > \omega_\ast$, $\gamma_\gamma$, $\gamma_\gamma$ the frequency of this mode is given by

$$\omega_0 = \pm k_\parallel v_A$$

(5.32)

and the growth rate is

$$\gamma = 0.57 \left( \gamma_\gamma - \eta_e \gamma_\gamma \right) \frac{\omega_\ast}{k_\parallel v_A}$$

(5.33)

This growth has some interesting features. First, the mode grows because of the condensation effect ($\gamma_\gamma$) while the thermal instability ($\gamma_\gamma$) tries to stabilize it. However, this stabilizing term can contribute to growth in the regime when the temperature drop causes a decrease in the radiation (i.e., $\partial L/\partial T_e > 0$). For example the coronal radiation rate for carbon impurity shows this behavior for $50 \text{ eV} < T_e < 100 \text{ eV}$. Second, the mode growth is dependent on the diamagnetic drift frequency, thus for larger density gradients (i.e., smaller $L_n$) the growth rate will be higher.
5.4 Nonlocal Analysis

In this section we examine the slab radial eigenvalue problem Eqs. (5.18) and (5.19) and derive a nonlocal dispersion relation. We start by reviewing briefly the equilibrium plasma temperature in the presence of sources and sinks as discussed by Drake. The equilibrium is described by the heat energy equation

$$\frac{1}{x} \frac{\partial}{\partial x} \left( \kappa_{\perp} x \frac{\partial T}{\partial x} \right) = L - H$$  \hspace{1cm} (5.34)

The radiation loss function $L$ is assumed to have the simple form $L = L_0 \Theta(T_l - T)$, where, $\Theta$ is the Heaviside step function with $T_l$ as the cutoff temperature above which $L = 0$. The solution of Eq. (5.34) can be obtained in separate regions $0 < x < x_L$ and $x_L < x < a$ as

$$T = T_c - \frac{H x^2}{4 \kappa_{\perp} L_c} \quad , \quad x < x_L$$  \hspace{1cm} (5.35)

$$T = T_L \left[ 1 - \frac{\ln(x/x_L)}{\ln(a/x_L)} \right] + (L_0 - H)(a^2 - x_L^2)$$

$$\times \frac{1}{4 \kappa_{\perp} L_c} \left[ \frac{x^2 - x_L^2}{a^2 - x_L^2} - \frac{\ln(x/x_L)}{\ln(a/x_L)} \right] \quad , \quad x_L < x < a$$  \hspace{1cm} (5.36)

where, $T = T_c$ at $x = 0$ and $T = 0$ at $x = a$ are the boundary conditions used. For the case of $T_c \gg T_L$ and taking into account the discontinuity of $\partial T/\partial x$ at $x = x_L$, one can solve for narrow radiative thickness $\Delta x = a - x_L \ll a$. It is given by

$$\frac{\Delta x}{a} = \frac{T_L L_c}{T_c L_0} \left[ 1 - \left( 1 - \frac{L_0}{L_c} \right)^{\frac{1}{2}} \right]$$  \hspace{1cm} (5.37)

where, $H \approx 4 \kappa_{\perp} T_i / a^2$ and $L_c \equiv HT_c / 2T_L$. For the sake of completeness we shall assume that the equilibrium has moderate density gradients sustained by ionization sources, inward pinch velocity and density diffusion effects; we shall not, however, explicitly take account of these sources and sinks in our subsequent calculations since their contribution to MARFE instability is relatively weak. It may be noted from the above analysis that a sharp radiation loss function (step profile used above) leads to a temperature profile which is a relatively slower function of position. In our subsequent eigenmode analysis of the instability, we shall take account of the slower density and
Chapter 5: Radiative Alfvén condensation instability

Temperature inhomogeneities only implicitly through the inclusion of \( \omega_* \) effects. This assumes that the structure of the eigenfunction is dictated by the sharp variation of the radiation loss function and not by the slower equilibrium variation of density and temperature.

We now rewrite Eqs. (5.18) and (5.19) using Eq. (5.35) and the form of radiation loss function as

\[
\frac{d^2 \phi}{dx^2} - C_1 \phi + C_2 T_e = 0,
\]

\[
\frac{d^2 \tilde{T}_e}{dx^2} - C_3 \tilde{T}_e - C_4 \phi = -\gamma_{T0} \delta (x - x_L) \tilde{T}_e
\]

where,

\[
C_1 = k_y^2 - \frac{i}{\rho_*^2} \left( \frac{k_y^2 D_{||0e}}{\omega} \right) A_1
\]

\[
C_2 = \frac{i}{\rho_*^2} \left( \frac{k_y^2 D_{||0e}}{\omega} \right) A_3
\]

\[
C_3 = \frac{1}{\chi_{||e}} \left[ \frac{3}{2} i \omega + i \omega A_1 + k_y^2 \chi_{||e} + k_y^2 \chi_{\perp e} + 0.71 k_y^2 D_{||0e} A_3 + \gamma_{n0} A_1 \Theta (x - x_L) \right]
\]

\[
C_4 = \frac{1}{\chi_{\perp e}} \left[ i \omega A_2 + i \left( \frac{3}{2} \eta_e - 1 \right) \omega_* - \frac{\beta}{2} \eta_e \chi_{\perp e} \omega_* + 0.71 k_y^2 D_{||0e} A_1 + \gamma_{n0} A_2 \Theta (x - x_L) \right]
\]

and, \( \gamma_{n0} = 2L_0/n_0 T_e \), \( \gamma_{T0} = 2L_0/HT_e \).

In deriving Eq. (5.39) we have neglected the cylindrical modifications to the perpendicular thermal diffusivity, since the mode is localized in the edge region. From the choice of the radiation loss it is apparent that the coupled Eqs. (5.38) and (5.39) have to be solved in two regions, \( x < x_L \) and \( x > x_L \), respectively.

We define new variables \( \xi_1 \) and \( \xi_2 \) such that,

\[
\xi_1 = \tilde{T}_{e1} + \alpha_1 \phi_1,
\]

\[
\xi_2 = \tilde{T}_{e2} + \alpha_2 \phi_2
\]
where, \( \dot{T}_e (x < x_L) \equiv \dot{T}_{e1} \), \( \dot{T}_e (x > x_L) \equiv \dot{T}_{e2} \), \( \dot{T}_e (x < x_L) \equiv \dot{T}_{e1} \), \( \dot{T}_e (x > x_L) \equiv \dot{T}_{e2} \), respectively. \( \alpha_1 \) and \( \alpha_2 \) are constants. Multiplying Eq. (5.38) by \( \alpha_1(2) \) and adding to Eq. (5.39), we get the following equations:

\[
\frac{d^2 \xi_1}{dx^2} - k_1^2 \xi_1 = 0, \quad x < x_L \tag{5.45}
\]

\[
\frac{d^2 \xi_2}{dx^2} + k_2^2 \xi_2 = 0, \quad x > x_L \tag{5.46}
\]

where,

\[
k_1^2 = C_{31} - \alpha_1 C_2, \quad k_2^2 = \alpha_2 C_2 - C_{32} \tag{5.47}
\]

The constants \( \alpha_1 \) and \( \alpha_2 \) are given by,

\[
\alpha_1 = \frac{C_{31} - C_1 \pm [(C_{31} - C_1)^2 - 4C_2 C_{41}]^{1/2}}{2C_2} \tag{5.48}
\]

\[
\alpha_2 = \frac{C_{32} - C_1 \pm [(C_{32} - C_1)^2 - 4C_2 C_{42}]^{1/2}}{2C_2} \tag{5.49}
\]

where,

\[
C_{31} = \frac{1}{\chi_{ee}} \left[ -\frac{3}{2} \omega + i\omega A_1 + k_2^2 \chi_{ee} + k_2^2 \chi_{ee} + 0.71k_2^2 D_{||ee} A_1 \right] \tag{5.50}
\]

\[
C_{41} = \frac{1}{\chi_{ee}} \left[ i\omega A_2 + i \left( \frac{3}{2} \eta_e - 1 \right) \omega e - \frac{3}{2} \eta_e \chi_{ee} \omega e + 0.71k_2^2 D_{||ee} A_3 \right] \tag{5.51}
\]

\[
C_{32} = C_{31} + \gamma_{\omega} A_1 \tag{5.52}
\]

\[
C_{42} = C_{41} + \gamma_{\omega} A_2 \tag{5.53}
\]

The solutions of Eqs. (5.45) and (5.46), subject to the conditions \( \xi_1(2) (x = x_L) = \xi_1'A \) and \( \xi_2 (x = a) = 0 \), respectively, are given by,

\[
\xi_1 = \xi_1'A \exp [k_1 (x - x_L)] \tag{5.54}
\]
\[ \xi_2 = \xi_L^* \frac{\sin[k_2(a-x)]}{\sin[k_2(a-x_L)]} \]  

(5.55)

Using Eq. (5.44) and the fact that \( \hat{T}_{e1(2)}(x = x_L) = \hat{T}_L \) and \( \hat{\phi}_{1(2)}(x = x_L) = \hat{\phi}_L \), we have

\[ \begin{align*}
\xi_L^1 &= \hat{T}_L + \alpha_1 \hat{\phi}_L, \\
\xi_L^2 &= \hat{T}_L + \alpha_2 \hat{\phi}_L
\end{align*} \]

(5.56)

Looking at Eq. (5.39) we realise that there exists a discontinuity at \( x = x_L \), and thus a proper matching would require integrating Eq. (5.39) across this discontinuity, which yields the following condition

\[ \frac{dT_{e2}}{dx}|_{(x=x_L)} - \frac{dT_{e1}}{dx}|_{(x=x_L)} = -\gamma_T \Phi_{e1}^L \]

(5.57)

However the solutions that we possess are in the variable \( \xi \). But Eq. (5.44) can be readily recast in terms of the variable \( \hat{T}_r \), as

\[ \begin{align*}
\hat{T}_{r1} &= \xi_1 - \alpha_1 \hat{\phi}_1, \\
\hat{T}_{r2} &= \xi_2 - \alpha_2 \hat{\phi}_2
\end{align*} \]

(5.58)

Now in order to get the desired solutions in the variable \( \hat{T}_r \), we need to find the solutions in the variable \( \hat{\phi} \) in both the regions. So we manipulate Eq. (5.38) such that we have the following equations

\[ \begin{align*}
\frac{d^2 \hat{\phi}_1}{dx^2} + \delta_1 \hat{\phi}_1 &= -C_2 \xi_1, & x < x_L \\
\frac{d^2 \hat{\phi}_2}{dx^2} + \delta_2 \hat{\phi}_2 &= -C_2 \xi_2, & x > x_L
\end{align*} \]

(5.59, 5.60)

where,

\[ \begin{align*}
\delta_1^2 &= C_1 + \alpha_1 C_2, & \delta_2^2 &= C_1 - \alpha_2 C_2
\end{align*} \]

(5.61)

Substituting for \( \xi_1 \) and \( \xi_2 \) from Eqs. (5.54) and (5.55) into Eqs. (5.59) and (5.60), respectively, we obtain the following solutions

\[ \hat{\phi}_1 = \hat{\phi}_L \exp[\delta_1 (x - x_L)] + \frac{C_2 \xi_L^{|x-x_L|}}{\delta_1^2 - \delta_2^2} \left[ \exp \{ \delta_1 (x - x_L) \} - \exp \{ \delta_2 (x - x_L) \} \right] \]

(5.62)
\[ \tilde{\phi}_2 = \phi_L \frac{\sin \left[ \delta_2 (a - x) \right]}{\sin \left[ \delta_2 (a - x_L) \right]} - \frac{C_2 \xi_L^2}{k^2 - \delta_2^2} \left[ \frac{\sin \left\{ \delta_2 (a - x) \right\}}{\sin \left\{ \delta_2 (a - x_L) \right\}} - \frac{\sin \left\{ k_2 (a - x) \right\}}{\sin \left\{ k_2 (a - x_L) \right\}} \right] \]  

(5.63)

Since \( \tilde{\phi} \) is continuous, matching its first derivatives at \( x = x_L \) gives the value of the constant \( \tilde{\phi}_L \) as

\[ \tilde{\phi}_L = \sigma T_L \]  

(5.64)

where,

\[ \sigma = \frac{\sigma_1}{\sigma_2} \]  

(5.65)

and,

\[ \sigma_1 = C_2 \left[ \frac{1}{k_1 + \delta_1} - \frac{k_2 \cot \{ k_2 (a - x_L) \} - \delta_2 \cot \{ \delta_2 (a - x_L) \}}{k_1^2 - \delta_2^2} \right] \]  

(5.66)

\[ \sigma_2 = \delta_1 + \delta_2 \cot \{ \delta_2 (a - x_L) \} - \frac{\alpha_1 C_2}{k_1 + \delta_1} + \frac{\alpha_2 C_2}{k_1^2 - \delta_2^2} \left[ k_2 \cot \{ k_2 (a - x_L) \} \right. \\
\left. - \delta_2 \cot \{ \delta_2 (a - x_L) \} \right] \]  

(5.67)

Now we substitute the value of \( \tilde{\phi}_L \) from Eq. (5.64) into Eqs. (5.62) and (5.63), and after a little algebra we can reconstruct the \( T \) solutions on both sides of \( x = x_L \) as

\[ \tilde{T}_{e1} = \left[ \exp \left\{ k_1 (x - x_L) \right\} - \exp \left\{ \delta_1 (x - x_L) \right\} - \exp \left\{ k_2 (x - x_L) \right\} \right] \\
\left\{ \sigma a_1 + \frac{\alpha_1 C_2 (1 + \sigma a_1)}{k_1^2 - \delta_2^2} \right\} T_L \]  

(5.68)

\[ \tilde{T}_{e2} = \left[ \frac{\sin \left\{ k_2 (a - x) \right\}}{\sin \left\{ k_2 (a - x_L) \right\}} - \frac{\sin \left\{ \delta_2 (a - x) \right\}}{\sin \left\{ \delta_2 (a - x_L) \right\}} - \frac{\sin \left\{ k_2 (a - x) \right\}}{\sin \left\{ k_2 (a - x_L) \right\}} \right] \\
\left\{ \sigma a_2 - \frac{\alpha_2 C_2 (1 + \sigma a_2)}{k_1^2 - \delta_2^2} \right\} T_L \]  

(5.69)

Using Eqs. (5.57), (5.68) and (5.69), we get the following dispersion relation

\[ (1 + \sigma a_1) \left[ k_1 + \frac{\alpha_1 C_2}{k_1 + \delta_1} \right] - \sigma a_1 k_1 - \sigma a_2 k_2 \cot \{ \delta_2 (a - x_L) \} + (1 + \sigma a_2) \times \\
\left[ k_2 \cot \{ k_2 (a - x_L) \} - \frac{\alpha_2 C_2}{k_1^2 - \delta_2^2} \left( k_2 \cot \{ k_2 (a - x_L) \} - \delta_2 \cot \{ \delta_2 (a - x_L) \} \right) \right] \]  

= \gamma_\alpha \]  

(5.70)
Multiplying throughout by $\Delta x$ [See Eq. 5.37], Eq. (5.70) can be rewritten as

$$
(1 + \sigma_1) \left[ \hat{k}_1 + \frac{\delta_{12}}{k_1 + \delta_1} \right] + (1 + \sigma_2) \left[ \hat{k}_2 \cot \hat{k}_2 - \frac{\delta_{22}}{k_2^2 - \delta_2^2} \left( \hat{k}_2 \cot \hat{k}_2 - \delta_2 \cot \delta_2 \right) \right]
$$

$$
- \alpha_1 \sigma_1 \delta_1 - \alpha_2 \sigma_2 \cot \delta_2 = 1 - \left( 1 - \frac{L_0}{L_c} \right)^{1/2}
$$

(5.71)

where

$$
\hat{k}_{1(2)} = k_{1(2)} \Delta x, \quad \delta_{1(2)} = \delta_{1(2)} \Delta x,
$$

$$
\alpha_{12} = \alpha_1 C_2 (\Delta x)^2, \quad \alpha_{22} = \alpha_2 C_2 (\Delta x)^2
$$

5.5 Numerical Results and Discussion

In this section we present the numerical results obtained from the local and nonlocal dispersion relation given by Eqs. (5.24) and (5.71), respectively. The frequency in both these equations is normalized as

$$
\bar{\omega} = \left( \frac{\bar{\omega}^2}{2 \chi_{\perp \epsilon}} \right) \omega \left( \frac{T_1}{T_c} \right)^2
$$

(5.72)

The local and nonlocal dispersion relations are solved numerically using a standard root-finding algorithm. Parameters relevant to the typical tokamak edge plasmas are used, with $T_\perp = 10$ eV, $T_c = 1$ keV, $n = 6 \times 10^{13}$ cm$^{-3}$, $R/a \approx 4$, $B = 1.5$ T, $L_n = 2$ cm, $\eta = 2.5$, $\chi_{\perp \epsilon} \simeq 10^4$ cm$^2$ s$^{-1}$, and $q = 4$. The poloidal mode number $m$ is taken to be 1. In case of the local dispersion relation we have assumed that $\partial L_\perp / \partial L_\parallel \sim \epsilon$, where $\epsilon < 1$. Such a physical situation arises when the equilibrium is near the peak of the radiation curve. The parameter varied is $\beta$.

In Figs. 5.1 and 5.2 we plot the real and imaginary parts of the normalized frequency as a function of $\beta$. In all these plots the values from the local dispersion relation are denoted by the broken lines, while the solid lines represent values obtained from the nonlocal dispersion relation. The real parts are displayed by Figs. 5.1(a), 5.1(c), and 5.1(e), respectively. The plots of the imaginary parts are given in Figs. 5.1(b), 5.1(d), and 5.1(f), respectively. Figs. 5.1(a) - 5.1(d) correspond to the acoustic modes. The drift-radiative mode is displayed in Figs. 5.1(e) and 5.1(f). It is observed that the acoustic modes are damped, while the drift-radiative modes are unstable and have a
Figure 5.1: Plots showing the effect of $\beta$ on the normalized frequency $\Delta$. The solid and broken lines represent solutions of the nonlocal and local dispersion relations, respectively. Here (a)-(d) correspond to the acoustic modes, and (e)-(f) denote the drift-radiative mode, respectively.

Figure 5.2: Plots showing the effect of $\beta$ on the normalized frequency $\Delta$. The solid and broken lines represent solutions of the nonlocal and local dispersion relations, respectively. Here (a)-(d) correspond to the Alfvén condensation mode.
real part of frequency. Also all these modes do not have any significant dependence on \( \beta \).

In Figs. 5.2(a) and 5.2(b), we plot the real part of the frequency, while the imaginary parts are plotted in Figs. 5.2(b) and 5.2(d). These roots of the dispersion relation correspond to the Alfvén-condensation mode. It is readily seen that these modes have a much higher real frequency and is inversely proportional to \( \beta \). The growth rate of this mode is found to be smaller than the drift-radiative mode. The growth is observed to be directly proportional to \( \beta \).

To summarize, we have considered the effect of radiation on Alfvén waves. It is shown that drift effects because of plasma inhomogeneity can couple the Alfvén waves to the radiative condensation mode. Specifically, the growth rate of the resulting instability is found to be directly proportional to the diamagnetic drift frequency and growth due to the density dependence of the radiation loss function. Thus the growth rate will be large for sufficiently strong plasma density and temperature gradients and radiative losses. Physically, this happens because the magnetic perturbations try to bend the field lines. The plasma tries to resist this change by creating an electric field. The potential perturbation because of this electric field then \( E \times B \) convects the density into the region of low-temperature and thereby increase the radiative loss. This mechanism is different from the conventional case of the condensation instability, where the parallel pressure balance condition brings about the condensation effect. The present instability can be of importance in both astrophysical and tokamak plasmas.

### 5.6 Bibliography


Chapter 5: Radiative Alfvén condensation instability


