Chapter 5

Bondage Number, Domatic Number and Saturation Number for Dominating Colour Transversals

We note that \( \gamma_{st} \) of a graph may increase or decrease or remain unchanged on deletion of a vertex or an edge as shown in the examples below:

**Example 1:** Consider \( K_{1,4} \) with centre \( v \). \( \gamma_{st}(K_{1,4}) = 2 \). But \( \gamma_{st}(K_{1,4} - v) = 4 \).

**Example 2:** Consider \( C_5 \). \( \gamma_{st}(C_5) = 3 \). On deleting any one vertex \( v \) from \( C_5 \), we get a path \( P_4 \). Hence \( \gamma_{st}(C_5 - v) = \gamma_{st}(P_4) = 2 \).

**Example 3:** \( \gamma_{st}(C_4) = 2 \). \( C_4 - v \) is \( P_3 \) for any \( v \).

Hence \( \gamma_{st}(C_4 - v) = \gamma_{st}(P_3) = 2 \).

So in Examples 1, 2 and 3 we see that \( \gamma_{st} \) increases, decreases, does not change respectively.

**Example 4:** \( \gamma_{st}(K_{1,4}) = 2 \). But \( \gamma_{st}(K_{1,4} - e) = 2 \), where \( e \) is any edge of the star \( K_{1,4} \). So \( \gamma_{st} \) does not change on removal of an edge.
Example 5: \( \gamma_{st}(C_5) = 3 \). But \( \gamma_{st}(C_5 - e) = \gamma_{st}(P_5) = 2 \), for any edge \( e \) of \( C_5 \). Hence \( \gamma_{st} \) decreases on removal of an edge.

Example 6:

For the graph \( G \) (Figure 5.1), we have \( \gamma_{st}(G) = 3 \) as the \( \gamma_{st} \) set is \( \{v_{10}, v_{11}, v_{12}\} \).

Let \( e = v_8v_{12} \). Then \( \gamma_{st}(G - e) \) increases to 4 because we need an additional dominating vertex for the pendant vertex \( v_8 \) in \( G - e \). Hence in this example, \( \gamma_{st} \) increases when an edge is deleted.

The above examples along with the definition of the bondage number of a graph, motivate us to define a corresponding bondage number for our parameter \( \gamma_{st} \) as defined below:

### 5.1 Bondage number of graphs \( b_{st} \)

**Definition 5.1.1.** Dominating colour transversal bondage number of a graph \( G \) denoted by \( b_{st} \) is defined to be the minimum cardinality of the collection of sets \( E' \subseteq E \) such that \( \gamma_{st}(G) \neq \gamma_{st}(G - E') \). \( b_{st} \) is not defined for \( K_1 \) and \( K_2 \).
Result 5.1.2.

(1) $b_{st}(K_p) = 1$ in view of Corollary 4.1.2.

(2) $b_{st}(K_{m,n}) = m$ if $m \leq n$.

Theorem 5.1.3. $b_{st}(W_p) = 1$ if $p$ is even.

**Proof.** Since $p$ is even, $\chi(W_p) = 4$. Since $W_p$ has a vertex of full degree Theorem 2.2.5 implies that $\gamma_{st} = \chi = 4$. On removal of an edge $e$ from the boundary of $W_p$, we get $\gamma_{st}(W_p - e) = \chi(W_p - e) = 3$ again by Theorem 2.2.5. Hence $b_{st}(W_p) = 1$ if $p$ is even. □

Theorem 5.1.4. When $p$ is odd and $p \geq 11$, $b_{st}(W_p) \leq \frac{p-1}{2}$.

**Proof.** Let the $p$ vertices of $W_p$ be $v, v_1, v_2, \ldots, v_p$, where $v$ is the central vertex. Let $vv_i = e_i$, $i = 1, 2, \ldots, p-1$, be the edges joining the centre and the $p$ vertices on the boundary. Let $G' = W_p - \{e_1, e_3, e_5, \ldots, e_{p-2}\}$.

As the removal of the edges $e_1, e_3, e_5, \ldots, e_{p-2}$ makes $W_p$ triangle free, $G'$ is bipartite without pendant vertices. Thus it is a type I graph and hence $\gamma_{st}(G') = \gamma(G')$.

Let $S$ be a $\gamma$-set of $G'$. If $v \in S$, $v$ dominates $v_2, v_4, v_6, \ldots, v_{p-1}$. To dominate the remaining $\frac{p-1}{2}$ vertices namely $v_1, v_3, \ldots, v_{p-2}$ we need at least $\lceil \frac{p-1}{4} \rceil$ vertices. Hence $\gamma_{st}(G') = \gamma(G') = |S| \geq 1 + \lceil \frac{p-1}{4} \rceil \geq 4$ if $p \geq 11$ . If $v \not\in S$, then $S$ should contain a minimum of $\lceil \frac{p-1}{3} \rceil$ vertices and so $|S| \geq \lceil \frac{p-1}{3} \rceil \geq 4$ if $p \geq 11$. Therefore $\gamma_{st}(G') \geq 4$ in this case also. Thus we have $4 \leq \gamma_{st}(G') \neq \gamma_{st}(W_p) = 3$, which gives $b_{st}(W_p) \leq \frac{p-1}{2}$ if $p$ is odd and $p \geq 11$. □

Theorem 5.1.5. $b_{st}(W_p) = 3$ if $p$ is odd and $p \geq 11$. 
**Proof.** Let \( p \geq 1 \) and \( p \) is an odd number. Obviously \( b_{st}(W_p) \neq 1 \). Let the vertex set of \( W_p \) be \( \{v, v_1, v_2, \ldots, v_{p-1}\} \). Where \( v \) is the centre of \( W_p \) and \( v_1, v_2, \ldots, v_{p-1} \) are on the boundary. Let \( e_i = vv_i, i = 1, 2, \ldots, p-1 \). When \( p \geq 11 \) we have \( \gamma(W_p - \{e_1, e_4, e_7\}) = 4 \leq \gamma_{st} \). Hence \( b_{st}(W_p) \leq 3 \). Suppose \( e \) and \( e' \) are two edges with \( \gamma_{st}(W') \neq \gamma_{st}(W_p) \), where \( W' = W_p - \{e, e'\} \). As \( p \geq 9, \chi(W') = 3 \). That is \( \gamma_{st}(W') \geq 3 \). We can always obtain a \( \gamma_{st} \)-set in \( W' \) with 3 elements. Hence \( b_{st}(W_p) \neq 2 \). So \( b_{st}(W_p) = 3 \).

**Note:** If \( G \) is a bipartite graph with two components \( G_1 \) and \( G_2 \) then \( \gamma_{st}(G) = \gamma(G_1) + \gamma(G_2) \).

**Theorem 5.1.6.** When \( p \geq 4, b_{st}(P_p) = \begin{cases} 2, & \text{if } p = 4, 7, 10, \ldots \\ 1, & \text{otherwise.} \end{cases} \)

**Proof.** Let \( P_p = v_1 e_1 v_2 e_2 v_3 e_3 \cdots v_{p-1} e_{p-1} v_p \).

Case (i) Let \( p = 3n \) where \( n = 2, 3, \ldots \).

\[
\gamma_{st}(P_p - e_1) = 1 + \gamma(P_{p-1})
\]
\[
= 1 + \left\lceil \frac{p-1}{3} \right\rceil
\]
\[
= 1 + \left\lceil \frac{3n-1}{3} \right\rceil
\]
\[
= 1 - \left\lfloor \frac{n-1}{3} \right\rfloor
\]
\[
= 1 + n > n = \frac{p}{3} = \left\lceil \frac{p}{3} \right\rceil = \gamma_{st}(P_p).
\]

Hence \( b_{st}(P_p) = 1 \) in this case.

Case (ii) Let \( p = 3n + 2 \).

Hence \( \gamma_{st}(P_p - e_1) = 1 + \gamma(P_{p-1}) \)
\[
= 1 + \left\lceil \frac{p-1}{3} \right\rceil
\]
\[
\begin{align*}
&= 1 + \left\lceil \frac{(3n + 2) - 1}{3} \right\rceil \\
&= 1 + (n + 1) \\
&> n + 1 \\
&= \left\lceil \frac{3n + 2}{3} \right\rceil \\
&= \left\lceil \frac{p}{3} \right\rceil = \gamma_{st}(P_p).
\end{align*}
\]
Hence in this case also \( b_{st}(P_p) = 1 \).

**Case (iii)** Let \( p = 3n + 1 \).

\[
\begin{align*}
\gamma_{st}(P_p - \{e_1, e_2\}) &= 2 + \gamma(P_{p-2}) \\
&= 2 + \left\lceil \frac{p - 2}{3} \right\rceil \\
&= 2 + \left\lceil \frac{3n + 1 - 2}{3} \right\rceil \\
&= 2 + n \\
&> n + 1 \\
&= \gamma_{st}(P_p).
\end{align*}
\]
Hence \( b_{st}(P_p) \leq 2 \) in this case.

Suppose \( b_{st}(P_p) = 1 \), then \( \gamma_{st}(P_p - e_i) > \gamma_{st}(P_p) \) for some \( i \). But for \( e_1, e_2, e_{p-1}, e_{p-2}, \gamma_{st}(P_p - e_i) = \gamma_{st}(P_p) \). Hence \( i \neq 1, 2, p - 1, p - 2 \). Let \( P_p - e_i = P_l \cup P_m \) where \( l + m = p, 3 \leq l \leq m \). Now \( \gamma_{st}(P_p - e_i) = \gamma(P_l) + \gamma(P_m) \). If \( l = 3 \), then \( m \geq 4 \).

Therefore \[
\begin{align*}
\gamma_{st}(P_p - e_i) &= 1 + \left\lceil \frac{m}{3} \right\rceil \\
&= 1 + \left\lfloor \frac{3n - 2}{3} \right\rfloor \\
&= 1 + n \\
&= \gamma_{st}(P_p) \quad \text{which is a contradiction.}
\end{align*}
\]
If \( l \geq 4 \), then \( m \geq 4 \). Hence

\[
\gamma_{st}(P_p - e_i) = \gamma(P_l) + \gamma(P_m) \\
= \left\lceil \frac{l}{3} \right\rceil + \left\lceil \frac{m}{3} \right\rceil \\
= \left\lceil \frac{l + m}{3} \right\rceil \quad \text{(simple algebra)} \\
= \left\lceil \frac{3n + 1}{3} \right\rceil \\
= n + 1 \\
= \gamma_{st}(P_p), \quad \text{which is a contradiction.}
\]

So \( b_{st}(P_p) \neq 1 \). Hence \( b_{st}(P_p) = 2 \).

\[\blacksquare\]

Note: One can easily verify that \( b_{st}(C_3) = b_{st}(C_5) = 1 \) and \( b_{st}(C_4) = 3 \).

**Theorem 5.1.7.** Let \( p \geq 6 \). Then

\[
b_{st}(C_p) = \begin{cases} 
3, & \text{if } p = 3n + 1 \\
2, & \text{otherwise.} 
\end{cases}
\]

**Proof.** Let \( p \geq 6 \) and \( C_p = v_1 e_1 v_2 e_2 \ldots v_{p-1} e_{p-1} v_p e_p v_1 \).

\[
\gamma_{st}(C_p - e_i) = \gamma_{st}(P_p) \quad \text{for every } i \\
= \left\lceil \frac{p}{3} \right\rceil = \gamma_{st}(C_p).
\]

Hence \( b_{st}(C_p) \geq 2 \).

Now \( \gamma_{st}(C_p - \{e_1, e_2, e_3\}) = 2 + \gamma(P_p - 2) \)

\[
= 2 + \left\lceil \frac{p - 2}{3} \right\rceil > 2 + \left\lceil \frac{p}{3} \right\rceil \quad \text{by induction.}
\]

Hence \( b_{st}(C_p) \leq 3 \).
Let $e_i$ and $e_j, \ i \neq j$ be any two edges of $C_p$. Then
\[
\gamma_{st}(C_p - \{e, e^1\}) = \gamma_{st}(P_l \cup P_m)
= \gamma(P_l) + \gamma(P_m)
= \left\lceil \frac{l}{3} \right\rceil + \left\lceil \frac{m}{3} \right\rceil
\]
where $l + m = p$ and $l$ and $m$ are integers.

If $p = 3n + 1$, we have
\[
\left\lceil \frac{l}{3} \right\rceil + \left\lceil \frac{m}{3} \right\rceil = \left\lceil \frac{l + m}{3} \right\rceil
= \left\lceil \frac{p}{3} \right\rceil
= \gamma_{st}(C_p).
\]
Therefore $b_{st}(C_p) = 3$ if $p = 3n + 1$.

If $p \neq 3n + 1$, we have
\[
\left\lceil \frac{l}{3} \right\rceil + \left\lceil \frac{m}{3} \right\rceil \neq \left\lceil \frac{l + m}{3} \right\rceil
= \left\lceil \frac{p}{3} \right\rceil
= \gamma_{st}(C_p).
\]
Hence $b_{st}(C_p) = 2$ if $p = 3n + 1$.

\section*{5.2 Domatic number of graphs $d_{st}$}

\textbf{Definition 5.2.1.} A partition $\{V_1, V_2, ..., V_n\}$ of $V(G)$ is a dominating colour transversal domatic partition of $G$ if each $V_i$ is an std-set. The dominating colour transversal domatic number $d_{st}$ of $G$ is the maximum order of a dominating colour transversal domatic partition of $G$. We note that $\gamma_{st} \cdot d_{st} \leq p$ for any graph $G$. 

Remark 5.2.2. For any graph that has a vertex of full degree or an isolated vertex, $d_{st} = 1$. In particular, $d_{st} = 1$ for a star $K_{1,p-1}$.

Theorem 5.2.3. If $G$ is a connected bipartite graph that is not a star, $d_{st}(G) \geq 2$.

Proof. Let $(X, Y)$ be the bipartition of $G$. As $G$ is not a star, $|X|, |Y| \geq 2$. If $G$ is the complete bipartite graph $K_{m,n}$, then $d_{st} = \min(m,n) \geq 2$. If $G \neq K_{m,n}$ then there exists a vertex say $x \in X$ such that $N(x) \neq Y$. Then the sets $S_1 = (X - \{x\}) \cup N(x)$ and $S_2 = (Y - N(x)) \cup \{x\}$ are std-sets of $G$. Hence $d_{st} \geq 2$.

Corollary 5.2.4. If $G$ is a tree that is not a star, then $d_{st} = 2$.

Proof. By Theorem 5.2.3, $d_{st} \geq 2$. But $d_{st} \leq 1 + 1 = 2$ and hence $d_{st} = 2$.

Result 5.2.5. It is easily checked that

(i) $d_{st} \leq d \leq 1 + \delta$

(ii) $1 \leq d_{st} \leq p/\gamma_{st} \leq p/\chi$

(iii) $d_{st} \leq p/2, p \geq 2$

(iv) For a tree, $\delta = 1$ and hence $d_{st} \leq 2$.

Theorem 5.2.6. If $G$ is a connected unicyclic graph then $d_{st}(G) \leq 3$.

Proof. Since $G$ is unicyclic, it contains a unique cycle which is either odd or even. Let $G = C_p$. We note that $d_{st}(C_3) = d_{st}(C_5) = 1$ and $d_{st}(C_4) = 2$. Now consider the case when $p \geq 6$. $\gamma_{st}(C_p) = \left\lceil \frac{p}{3} \right\rceil \geq \frac{p}{2}$, hence $d_{st} \leq \frac{p}{\gamma_{st}} \leq \frac{p}{\frac{p}{2}} = 3$. In
case \( G \neq C_p \), then \( G \) has a pendant vertex and therefore \( d_{st}(G) \leq 1 + \delta = 2 \). Hence the theorem.

\[
\text{Corollary 5.2.7. For a connected unicyclic graph } G, \ d_{st} = 3 \text{ if and only if } G = C_p \text{ where } p \equiv 0 \pmod{3}.
\]

\textbf{Proof.} Assume \( d_{st} = 3 \). When \( p \leq 5 \), \( G \) will have a vertex of full degree implying \( d_{st} = 1 \), a contradiction. Hence \( p \geq 6 \). \( G \) cannot have a pendant vertex since in that case \( d_{st} \leq 2 \). Hence \( G = C_p \). Since \( \gamma_{st} = \left\lceil \frac{p}{3} \right\rceil \) and \( d_{st} = 3 \), \( p \) has to be a multiple of three.

Conversely if \( p \equiv 0 \pmod{3} \) and \( G = C_p \) where \( C_p = v_1, v_2, v_3, \ldots, v_p, v_1 \), then the sets \( V_1 = \{v_1, v_4, v_7, \ldots\} \), \( V_2 = \{v_2, v_5, v_8, \ldots\} \) and \( V_3 = \{v_3, v_6, v_9, \ldots\} \) form three \( \gamma_{st} \)-sets and hence \( d_{st} = 3 \).

\[
\text{Corollary 5.2.8. If } G \text{ is a connected unicyclic graph with } p \geq 6, \text{ then } d_{st}(G) = 2 \text{ if and only if } G \neq C_p \text{ where } p \equiv 0 \pmod{3}.
\]

\textbf{Proof.} Let \( d_{st}(G) = 2 \) and \( p \geq 6 \). By Corollary 5.2.7 \( G \neq C_p \) where \( p \equiv 0 \pmod{3} \). Conversely let \( G = C_p \) where \( p \equiv 0 \pmod{3} \) and \( G = C_p \) where \( p \equiv 0 \pmod{3} \). Then let \( G = C_p \) where \( p \equiv 0 \pmod{3} \) or \( G \) contains a pendant vertex. In both the cases we can always fix a \( \gamma_{st} \)-set \( D \) such that \( V - D \) also an std-set of \( G \). Hence \( d_{st}(G) = 2 \).

\[
\text{Theorem 5.2.9. For a nontrivial graph } G \text{ with } \text{diam}(G) \leq 2 \text{ and } \chi(G) = \omega(G) \text{ we have } \gamma_{st} \leq \delta + \omega - 1 \text{ and the bound in sharp.}
\]

\textbf{Proof.} If \( \text{diam}(G) = 1 \), then \( G = K_p \) for which \( \gamma_{st} = p \) and hence

\[
\delta + \omega - 1 = (p - 1) + p - 1 = 2p - 2 \geq p = \gamma_{st}.
\]

Let \( \text{diam}(G) = 2 \). Let \( v \) be a vertex of minimum degree \( \delta \). Since
diam(G) = 2, \ N(v) \ is \ a \ dominating \ set \ for \ G. \ Also \ since \ \chi(G) = \omega(G),
\ \text{every maximum clique say } S \ \text{in } G \ \text{is a transversal for every } \chi-\text{partition of } G. \ \text{Let } S = \{v_1, v_2, \ldots, v_k\} \ \text{where } k = \chi(G) = \omega(G). \ \text{Take } \{V_1, V_2, \ldots, V_k\} \ \text{a } \chi-\text{partition of } G. \ \text{Obviously } S \cap V_i = \{v_i\} \ \text{say for every } i. \ \text{Let } x \in N(v). \ \text{Then } x \in V_i \ \text{for some } i. \ \text{Hence } N(v) \cup (S - \{v_i\}) \ \text{in an std-set for } G. \ \text{So } \gamma_{st} \leq \delta + \omega - 1. \ \text{The bound is sharp for the graph given in Figure 5.2.}

\[ \delta = 1, \ \omega = 4; \ \chi = 4; \ \gamma_{st} = 4. \]

\[ \begin{array}{c}
\text{Figure 5.2: Sharp bound.}
\end{array} \]

**Theorem 5.2.10.** For any graph \( G \), \( d_{st} + \gamma_{st} \leq p + 1 \). Equality is attained if and only if \( G = K_p \) or \( K_p \).

**Proof.** For any graph, \( d_{st} \gamma_{st} \leq p \).

**Case (i):** Let \( d_{st}, \gamma_{st} \geq 2 \). Then \( d_{st} + \gamma_{st} \leq d_{st} \gamma_{st} \leq p \). Hence \( d_{st} + \gamma_{st} \leq p + 1 \).

Thus the inequality is true in this case.

**Case (ii):** Let \( \gamma_{st} = 1 \). Then \( p = 1 \). Hence \( d_{st} = 1 \). Therefore \( d_{st} + \gamma_{st} \leq p + 1 \) is true in this case also.

**Case (iii):** Let \( d_{st} = 1 \). Since \( \gamma_{st} \leq p \) we have \( d_{st} + \gamma_{st} \leq p + 1 \). Hence \( d_{st} + \gamma_{st} \leq p + 1 \) in this case also. From cases (i) (ii) and (iii), we conclude that \( d_{st} + \gamma_{st} \leq p + 1 \) for any graph.

Suppose \( d_{st} + \gamma_{st} = p + 1 \). Since \( \gamma_{st} \cdot d_{st} \leq p \), we get either \( \gamma_{st} = 1 \) and \( d_{st} = p \) or \( d_{st} = 1 \) and \( \gamma_{st} = p \). In the first case since \( \gamma_{st} = 1 \) we have \( G = K_1 \) and also \( p = 1 \). In the latter case, \( \gamma_{st} = p \) which implies that \( G = K_p \) or
Konig. Converse is obvious. Hence equality is attained if and only if \( G = K_p \) or \( \overline{K_p} \).

**Theorem 5.2.11.** (Nordhaus–Gaddum Inequality) For any non-trivial graph \( G \),

\[
2 \leq \text{d}_{st} + \overline{\text{d}_{st}} \leq p \quad \text{and} \quad 1 \leq \text{d}_{st} \cdot \overline{\text{d}_{st}} \leq p^2/4.
\]

The bounds are sharp and the upper bounds are attained if and only if

\[
G = P_4, C_4, \overline{C_4}, K_2, \overline{K_2}, K_2 \cup K_2
\]

**Proof.** For any non-trivial graph we have \( 1 \leq \text{d}_{st} \leq p/\gamma_{st} \leq p/2 \). Therefore \( 1 \leq \text{d}_{st} \leq p/2 \). Also \( 1 \leq \overline{\text{d}_{st}} \leq p/2 \). Adding and multiplying we get

\[
2 \leq \text{d}_{st} + \overline{\text{d}_{st}} \leq p \quad \text{and} \quad 1 \leq \text{d}_{st}\overline{\text{d}_{st}} \leq p^2/4
\]

Let \( \text{d}_{st} + \overline{\text{d}_{st}} = p \). Since \( \text{d}_{st} \leq p/2 \) and \( \overline{\text{d}_{st}} \leq p/2 \) we must have \( \text{d}_{st} = \overline{\text{d}_{st}} = p/2 \). Hence \( p \) is even.

If \( p = 2 \), then \( G = K_2 \) or \( \overline{K_2} \). Let \( p \geq 4 \). If \( \delta = 0 \) or \( (p - 1) \) then \( \text{d}_{st} = \overline{\text{d}_{st}} = 1 \neq p/2 \), a contradiction. Hence \( 0 < \delta(G), \delta(\overline{G}) < p - 1 \).

Hence \( \chi, \overline{\chi} \geq 2 \). Since \( \chi \leq \gamma_{st} \leq p/d_{st} = 2 \) we have \( \chi \leq 2 \). Therefore \( \chi = 2 \) and \( \gamma_{st} = 2 \). Similarly \( \overline{\chi} = 2 \) and \( \overline{\gamma_{st}} = 2 \). Hence \( G \) and \( \overline{G} \) are both bipartite graphs. Let \((X, Y)\) be a bipartition of \( G \) if \( |X| \geq 3 \), then in \( \overline{G} \) we get a clique of size 3 or more which is not possible. Hence \( |X| \leq 2 \). Also \( |Y| \leq 2 \). Therefore \( p = 4 \) and \( |X| = |Y| = 2 \). The various possibilities for \( G \) are \( C_4, \overline{C_4}, P_4, K_2 \cup K_2 \).

If \( \text{d}_{st}\overline{\text{d}_{st}} = p^2/4 \) then \( \text{d}_{st} = \overline{\text{d}_{st}} = p/2 \) and hence \( G \) is the same as above. \( \blacksquare \)

**Notation:**

Let \( \mathcal{G} \) be the collection of graphs \( K_2, \overline{K_2}, K_2 \cup K_2, C_4, \overline{C_4}, P_4 \).
Theorem 5.2.12. Let $G$ be a graph not in $\mathcal{G}$ then $d_{st} + \overline{d}_{st} \leq p - 1$. The equality is attained if and only if $G \simeq K_3, \overline{K}_3, P_3, \overline{P}_3, K_{3,3}, K_{3,3} - X$, where $X$ is a matching of $K_{3,3}$.

Proof. In view of the above theorem $d_{st} + \overline{d}_{st} \leq p - 1$. Assume that $d_{st} + \overline{d}_{st} = p - 1$.

Case 1: Let $p$ be even. Then obviously $p \geq 4$. $d_{st} + \overline{d}_{st} = p - 1$ implies that $d_{st} = p/2$ and $\overline{d}_{st} = p/2 - 1$ or $d_{st} = p/2 - 1$ and $\overline{d}_{st} = p/2$. Let us assume that $d_{st} = p/2$ and $\overline{d}_{st} = p/2 - 1$. (The other choice can be similarly dealt with.) As $p \geq 4$ and $d_{st} = p/2$, we have $d_{st} \neq 1$. Hence $0 < \delta(G) < p - 1$.

This implies that $\chi \geq 2$. Also $\chi \leq \gamma_{st} \leq \frac{p}{d_{st}} = 2$. So $\chi = \gamma_{st} = 2$. Hence $G$ is a bipartite graph with say $(X, Y)$ as bipartition. As $d_{st} = p/2$ and $\gamma_{st} = 2$ we have $|X| = |Y|$. Since $|X| = p/2$ we have $\overline{\chi} \geq p/2$. This implies that $\overline{\gamma}_{st} \geq p/2$. Hence $\overline{d}_{st} = 1$ or 2. So $p = 4$ or 6. No bipartite graph on 4 vertices satisfies the given conditions. Hence $p = 6$. In this case $G \simeq K_{3,3}$ or $K_{3,3} - X$, where $X$ is a matching of $K_{3,3}$.

Case 2: Let $p$ be odd. Since $d_{st} + \overline{d}_{st} = p - 1$, we have $d_{st} = \overline{d}_{st} = \frac{p-1}{2}$.

When $p = 3$ $G \simeq K_3, \overline{K}_3, P_3$ or $\overline{P}_3$. Take $p \geq 5$. If $\delta = 0$ or $p - 1$ then $d_{st} = \overline{d}_{st} = 1 \neq \frac{p-1}{2}$, a contradiction. Hence $0 < \delta(G), \delta(\overline{G}) < p - 1$. This implies that $\chi, \overline{\chi} \geq 2$. Also $\chi \leq \gamma_{st} \leq \frac{p}{d_{st}} = \frac{2p}{p-1} < 3$. So $\chi \leq \gamma_{st} \leq 2$. Hence $\gamma_{st} = \overline{\chi} = 2$. Similarly we have $\gamma_{st} = \overline{\chi} = 2$. Let $(X, Y)$ be a partition of $G$. Since $\gamma_{st} = 2$ and $d_{st} = \frac{p-1}{2}$, $|X|, |Y| \geq \frac{p-1}{2}$. Also $|X| + |Y| = p$ and $p$ is odd. Hence $|X| \neq |Y|$. Without loss of generality, take $|X| > |Y| \geq \frac{p-1}{2}$.

This implies that $\overline{\chi} \geq \frac{p+1}{2}$. So $\gamma_{st} \geq \frac{p+1}{2} > \frac{p}{2}$ and $\overline{d}_{st} = 1$ which is again a contradiction, since $\overline{d}_{st} = \frac{p-1}{2} \neq 1$. Hence $p \geq 5$ does not arise. The converse is obvious.
Example: For the Petersen Graph $P$, $d_{st} = 2$. Refer Figure 2.1.

Since $\gamma_{st} = 4$, $p = 10$, $d_{st} \leq \frac{p}{\gamma_{st}}$ we have $d_{st} \leq \frac{10}{4}$. Therefore $d_{st} \leq 2$. A transversal domatic partition of $G$ is $\{\{u_1, u_2, u_5, v_4\}, \{u_3, u_4, v_1, v_2, v_3, v_5\}\}$. Hence $d_{st} = 2$.

5.3 Saturation Number of Graphs $\eta(G)$

In this section, we define a saturation number $\eta(G)$ analogous to the dom-saturation number defined by Acharya [1]. We determine $\eta(G)$ for standard graphs.

Definition 5.3.1. Let $G$ be a graph. If for every $v \in V$, there exists a $\gamma_{st}$-set $D$ such that $v \in D$, then $G$ is said to be a class I graph with saturation number $\eta(G) = \gamma_{st}$. Otherwise, $G$ is said to be a class II graph with saturation number $\eta(G) = \gamma_{st} + 1$.

We note that $\eta(G)$ is the smallest positive integer $k$ such that every vertex of $G$ is in a std-set of cardinality $k$.

Result 5.3.2.

(1) $d_s \leq d_{sg} \leq \eta(G)$ since every std-set is a dominating set as well as a global dominating set of $G$.

(2) Petersen graph is a class I graph.

(3) $C_p$ is a class I graph.

(4) $W_p$ is a class I graph.

(5) In fact all vertex transitive graphs are class I graphs.
Proposition 5.3.3. $H^+$ is a class I graph for any graph $H$.

Proof. $H^+ = H \circ K_1$. For $H^+$, $V(H)$ and $V(H^+) - V(H)$ are $\gamma_{st}$-sets. Hence $H^+$ is a class I graph. ■

Proposition 5.3.4. For $p \geq 4$, $P_p$ is a class I graph if and only if $p \equiv 1 \pmod 3$.

Proof. For $P_4$, the proposition is trivially true. Take $p \geq 5$. Since $P_p$ is a type I graph, $\gamma_{st} = \gamma$. By Theorem 1.1.58, $ds = \gamma + 1$ if and only if $p \equiv 1 \pmod 3$. We know that $\gamma_{st} \geq \eta(G) \geq ds = \gamma_{st} + 1$ if and only if $p \equiv 1 \pmod 3$. Therefore $P_p$ is a class I graph if and only if $p \equiv 1 \pmod 3$. ■

Proposition 5.3.5. Let $T$ be a caterpillar. $T$ is of class I if and only if $T$ is a type II graph or every support in $T$ is adjacent to exactly one pendant vertex and for any two consecutive supports $u$ and $v$, $d(u, v) \equiv 1 \pmod 3$.

Proof. Let $T$ be class I caterpillar. Then $\eta(T) = \gamma_{st}(T)$. Since $T$ is bipartite, $T$ must be type I or type II graph. If it is type I, then $\gamma_{st} = \gamma$. Suppose $T$ has a support vertex that has more than one pendant vertex or for any two consecutive supports $u$ and $v$ in $T$, $d(u, v) \not\equiv 1 \pmod 3$, then by Theorem 1.1.61, $ds(T) = \gamma + 1 = \gamma_{st} + 1$. Hence $\eta(T) = \gamma_{st} + 1$, a contradiction.

Conversely, if $T$ is a type II graph. Then $T = (X, Y)$ is a bipartition with $|X| \leq |Y|$ and therefore $X \cup \{y\}$ is a $\gamma_{st}$-set for every $y \in Y$. Hence every vertex of $T$ belongs to a $\gamma_{st}$-set, which means that $T$ is a class I graph.

Now assume that every support in $T$ is adjacent to exactly one pendant vertex and for any two consecutive supports $u$ and $v$, $d(u, v) \equiv 1 \pmod 3$. Let $k$ denote the number of supports in $T$. The result is proved by induction on $k$. 
When $k = 1$, $T = K_2$ and so the result is true for $k = 1$. Assume the result for all caterpillars with fewer than $k$ supports. Let $T$ be a caterpillar with $k$ supports as shown in Figure 5.3.

![Figure 5.3: Caterpillar.](image)

Let $v_1, v_2, \ldots, v_k$ be the $k$ supports and $y_1, y_2, \ldots, y_k$ be the corresponding pendant neighbours. $d(v_1, v_2) = 3m + 1$. Consider the path $P = y_1v_1x_1x_2\cdots x_{3m}$. By Proposition 5.3.4, $\eta(P) = \gamma_{st}(P) = \gamma(P)$. The caterpillar $T' = T - P$ has $(k - 1)$ supports. So by inductive hypothesis, $\eta(T') = \gamma_{st}(T') = \gamma(T')$ since $T'$ is type I graph.

\[
\eta(T) \leq \eta(T') + \eta(P) \\
= \gamma(T') + \gamma(P) \\
\leq \gamma_{st}(T)
\]

Hence $\eta(T) = \gamma_{st}(T)$. Therefore $T$ is a class I graph. ■
Open problems

1. Given any three positive integers $a$, $b$, $c$ with $a \leq b \leq c$, is it possible to obtain a graph $G$ such that $\gamma(G) = a$, $\chi(G) = b$, $\gamma_{st}(G) = c$?

2. Given any three positive integers $a$, $b$, $c$ with $a \leq b \leq c$, is it possible to obtain a graph $G$ such that $\chi(G) = a$, $\gamma(G) = b$, $\gamma_{st}(G) = c$?

3. Characterize all graphs $G$ which satisfy

   (i) $\gamma(G) = \chi(G) = \gamma_{st}(G)$

   (ii) For connected bipartite graphs $\gamma < \chi < \gamma_{st}$ is not possible since $\gamma_{st} = \gamma$ or $\gamma + 1$. Is it possible to obtain a characterization for graphs satisfying $\gamma < \chi = 3 < \gamma_{st}$?

4. Characterize all graphs that satisfy $\chi < \gamma = \gamma_{st}$.

5. For any graph without isolated vertices, $\gamma \leq \frac{p}{2}$ Hence given $p$, is it possible to obtain a graph with $p$ vertices with $\gamma = a$ and $\gamma_{st} = b$ where $1 \leq a \leq \frac{p}{2}$ and $a \leq b$?
Applications

Two important applications in communication network and parallel algorithms have been postulated for global domination. Finding a minimum global dominating set for an arbitrary graph $G$ is very challenging, since the structure of $\overline{G}$ is difficult to analyze. Since every std-set of $G$ is a global dominating set of $G$, we can consider a $\gamma_{st}$-set as a representative of $\gamma_g$-set, especially for the class of graphs for which $\gamma_{st} = \gamma_g$.

If $G$ is a graph, a dominating set $D$ in $G$ is a representation in some sense namely “adjacency” of all the vertices in $V$, whereas a $\gamma_{st}$-set $S$ is not only a representation of the adjacency of the vertices but also of the non-adjacency of vertices as $S$ is both a dominating set and a transversal of some $\chi$-partition of $V$. In a graph we see that once the vertices are specified and adjacency is defined, colouring is automatic and we do not have a choice over the $\chi$-partitions as all the $\chi$-partitions possible in a graph are fixed. Hence in real life situations, it is usually difficult to choose the vertex set $V$ and define adjacency in such a way that colouring leads to $\chi$-partitions of our choice. Hence we try to overcome this difficulty by attacking the problem in the reverse order by choosing a suitable vertex set $V$, and a partition of $V$ and then defining adjacency, carefully avoiding adjacency among vertices in each equivalence class. Though this method leads to dealing with uniquely colourable graphs, still it has the advantage of representations of
two attributes in the set of all vertices.

To be specific, consider the example of all students in a college. We want to find a representation of the student community in such a way that two attributes “religion” and “community” are represented. Let the students belong to the religions, Christianity, Hinduism, Buddhism, Sikhism, Islam, Jainism and atheism (7 groups) and assume that the communities represented among the students are OC, BC, MBC, SC and ST (5 groups). From the large community of students of the college, we seek a small representative body in such a way that all the 7 groups of religion and all the 5 groups of community are represented. To do this, we see that the vertex set $V$ is the collection of all students in the college. Next we partition $V$ into 5 groups representing the five communities. “Adjacency” is judiciously defined in such a way that students of the same community are not “adjacent” and for this definition the $\gamma_{st}$-set that we find will give us a representation of students without leaving out any religion or community.

A generalization of the above application could be the following:

Consider the cells in the human body where each cell has say $n$ attributes, $a_1, a_2, \ldots, a_n$. Out of these $n$ attributes choose $k$ attributes and denote them by $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$. Define two cells to be ‘adjacent’ if and only if they both possess a combination of the $k$ chosen attributes. In this method, the partition need not be fixed beforehand as in the first example above and the colouring or non-adjacency will once again reflect a combination of the presence or absence of the chosen $k$ attributes. The dominating colour transversal set, that is the $\gamma_{st}$-set for this graph shall provide us a representative group of cells with a combination of attributes $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$.

We give an example using alkaloids, which are naturally occurring chemical compounds containing basic nitrogen atoms and are used as medications
Applications

and recreational drugs. Alkaloids are usually classified by their common molecular precursors, based on the metabolic pathway used to construct the molecule. One such classification is the “Pyridine” group of alkaloids which consists of piperine, conine, trigonelline, arecoline, guvacine, cytosine, labelline, nicotine, anabasine, sparteine and pelletierine. The chemical formula and the solubility of this group of alkaloids in water, alcohol, ether, chloroform and benzene are given in Table 5.1.

Table 5.1: Pyridine group: Solubility of certain alkaloids.

<table>
<thead>
<tr>
<th>No.</th>
<th>Alkaloids</th>
<th>Chemical formula</th>
<th>Solubility</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Water</td>
</tr>
<tr>
<td>1</td>
<td>Piperine</td>
<td>C_{17}H_{19}O_{2}N</td>
<td>S.S</td>
</tr>
<tr>
<td>2</td>
<td>Conine</td>
<td>C_{23}H_{7}C_{5}H_{10}N</td>
<td>V.S.S</td>
</tr>
<tr>
<td>3</td>
<td>Trigonelline</td>
<td>C_{7}H_{7}O_{2}NH_{2}O</td>
<td>V.S.</td>
</tr>
<tr>
<td>4</td>
<td>Arecoline</td>
<td>C_{9}H_{13}O_{2}N</td>
<td>S</td>
</tr>
<tr>
<td>5</td>
<td>Guvacine</td>
<td>C_{6}H_{9}NH_{2}O</td>
<td>S</td>
</tr>
<tr>
<td>6</td>
<td>Cytosine</td>
<td>C_{11}H_{14}ON_{2}</td>
<td>S</td>
</tr>
<tr>
<td>7</td>
<td>Lobelline</td>
<td>C_{22}H_{27}O_{2}N</td>
<td>V.S.S</td>
</tr>
<tr>
<td>8</td>
<td>Nicotine</td>
<td>C_{10}H_{14}N_{2}</td>
<td>S</td>
</tr>
<tr>
<td>9</td>
<td>Anabasine</td>
<td>C_{10}H_{14}N_{2}</td>
<td>S</td>
</tr>
<tr>
<td>10</td>
<td>Sparteine</td>
<td>C_{13}H_{26}N_{2}</td>
<td>V.S.S</td>
</tr>
<tr>
<td>11</td>
<td>Pelletierine</td>
<td>C_{8}H_{15}ON</td>
<td>S.S</td>
</tr>
</tbody>
</table>

S—soluble, i—insoluble, S.S—slightly soluble
V.S—very soluble, V.S.S—very slightly soluble.
Example 1: Let us take the alkaloids as vertices of a vertex set \( V \) represented by their serial number in the tabular column. Define adjacency among these vertices as follows:

Two alkaloids are adjacent if and only if

- (i) they contain oxygen and are soluble in water
- (ii) they are insoluble in chloroform and benzene
- (iii) they are soluble in alcohol and ether
- (iv) they contain a minimum of fifteen carbon atoms.

The graph for these eleven vertices with adjacency as defined above is drawn in Figure 6.1.

The graph is disconnected as it has an isolated vertex.

For this graph a minimum of four colours are needed and hence \( \chi = 4 \). One minimum dominating set is \( \{3, 5, 10\} \) and \( \gamma = 3 \). One \( \gamma_{st} \)-set is \( \{1, 3, 5, 10\} \) and \( \gamma_{st} = 4 \). The \( \gamma_{st} \)-set besides giving a set satisfying the important properties outlined in the adjacency defined above, also gives a representation for all alkaloids that do not have the properties mentioned in the
adjacency above, since representation for each colour implies representation for non-adjacency.

**Example 2:** For the same vertex set of 11 alkaloids, define adjacency as follows: Two alkaloids are adjacent if and only if either (i) they have a maximum of 15 hydrogen atoms and are soluble in alcohol or ether or (ii) they have one nitrogen atom and are soluble in chloroform or benzene or (iii) they have two nitrogen atoms and are insoluble in chloroform or benzene.

The graph is given in Figure 6.2.

The graph is disconnected and has two components. $\chi = 6$ and $\gamma = 2$. We can easily check that $\gamma_{st} = 6$. A $\gamma_{st}$-set shall contain representations for the properties mentioned in the adjacency as well as for the absence of those properties.
List of Publications


