Chapter 3

Dominating Colour Transversal Number for Bipartite Graphs

3.1 Bounds and Characterization Theorems for Bipartite Graphs

Theorem 3.1.1. Let $G$ be a connected bipartite graph with bipartition $(X, Y)$; $|X| \leq |Y|$ and $p \geq 3$. Then $\gamma_{st} = \gamma + 1$ if and only if every vertex in $X$ has at least two neighbours which are pendant vertices.

Proof. Let $\gamma_{st} = \gamma + 1$.

Any $\gamma$-set $D$ cannot intersect both $X$ and $Y$. As $|X| \leq |Y|$, $D = X$ is the only $\gamma$-set. Let $x \in X$. If $x$ has only one pendant neighbour, then there
exists $y \in N(x)$ such that $\{D - \{x\}\} \cup \{y\}$ is a $\gamma_{st}$-set with cardinality $\gamma$, which is a contradiction. Hence $x$ has at least two pendant neighbours.

Conversely, if every $x$ in $X$ has at least two pendant neighbours, then $X$ is the unique $\gamma$-set in $G$, which is not an std-set. $X \cup \{y\}$ where $y$ is any element of $Y$, is an std-set with minimum cardinality $(\gamma + 1)$. Hence $\gamma_{st} = \gamma + 1$.

**Definition 3.1.2.** For a bipartite graph $G$, $\gamma_{st} = \gamma$ or $\gamma + 1$. All bipartite graphs for which $\gamma_{st} = \gamma$ are called Type I graphs. Other graphs are Type II graphs.
Note: In Type II graphs vertices at even distances are all support vertices or all non-support vertices. Moreover each support vertex has at least two pendant neighbours. One of the colour classes is the unique $\gamma$-set.

**Definition 3.1.3.** Let $m$, $n$, $k$ be three non-negative integers. By $K_{m,n}^k$ we mean a graph obtained by joining the centres of $K_{1,m}$ and $K_{1,n}$ by a path of length $k$. This graph is called a double star of length $k$ and it contains $(m + n + k + 1)$ vertices when $k \geq 1$. When $k = 0$, $K_{m,n}^k$ is the disjoint union of $K_{1,m}$ and $K_{1,n}$ and it contains $(m + n + 2)$ vertices. In $K_{m,n}^k$, $(m, n)$ is a partition of $p - k - 1$. We note that $K_{0,0}^k$ is $P_{k+1}$ and $K_{0,0}^0$ is $K_2$. Examples are given in Figure 3.2.

![Figure 3.2: Double stars.](image)

**Theorem 3.1.4.** Let $G(\neq K_2)$ be a graph. Then $\gamma_{st}(G) = 2$ if and only if $G$ is a bipartite graph that contains $K_{m,n}^k$ as a spanning subgraph where $k = 0$, 1 or 3.
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**Proof.** Let γ_{st} = 2. Obviously χ = 2. Since γ ≤ γ_{st}, we have γ ≤ 2. When χ = 2 and γ = 1, G becomes a connected bipartite graph with a vertex of full degree. So G becomes the star graph \( K_{1,p-1} = K^1_{0,p-2} \).

When χ = 2 and γ = 2, G cannot have more than two components. If G has two components then G in a union of two disjoint stars namely \( K_{1,m-1} \cup K_{1,n-1} = K^0_{m-1,n-1} \), where \((m, n)\) is a partition of \( p \) and \( m, n \geq 2 \) or \( K_{1,p-2} \cup K_1 = K^0_{p-2,0} \).

Suppose G has exactly one component. Then G is a connected bipartite graph with bipartition \((X, Y)\). As γ_{st} = 2, there exists a γ_{st}-set \( D = \{x, y\} \) such that \( x \in X, y \in Y \) and \( p_{n}(x, D) = Y - \{y\} \) and \( p_{n}(y, D) = X - \{x\} \).

As G is a connected graph, there is a path between x and y. If x and y are adjacent then \( d(x, y) = 1 \); otherwise \( d(x, y) = 3 \). So G must have a double star of length 1 or 3. That is G is a graph that contains a spanning subgraph of \( K^1_{m,n} \) or \( K^3_{m,n} \).

The converse can be verified easily.

**Theorem 3.1.5.** In a bipartite graph G, γ_{st} = 3 if and only if G is any one of the following graphs.

(a) Union of three stars.

(b) Union of a star and a graph containing a spanning subgraph of \( K^2_{m,n} \).
(c) A type II graph containing a spanning subgraph of $K^2_{m,n}$

(d) A type I graph with two vertices $x_1, x_2 \in X$ and one vertex $y \in Y$ such that $d(x_1) + d(x_2) \geq |Y| - 1$ and $d(y) \geq |X| - 2$.

**Proof.** Assume that $\gamma_{st}(G) = 3$.

If $\gamma = 1$, we have a vertex of full degree and hence by Theorem 2.2.5 $\chi = \gamma_{st}$ which is not true. When $\gamma = 2$, $G$ has at most two components and in each component there is a vertex of full degree. In this case also $\gamma_{st} = 2$ which is a contradiction.

If there is only one component, then $G$ is connected. Since $\gamma = \chi = 2$ and $\gamma_{st} = 3$, $G$ is a type II graph that contains a spanning subgraph of $K^2_{m,n}$ where $(m, n)$ is a partition of $(p - 3)$.

If $\gamma = 3$, there can be at most three components and in case the number of components is maximum, $G$ is a union of three disjoint stars. When there are two components, $G$ is a union of a star and a graph containing a spanning subgraph of $K^2_{m,n}$. If $G$ is connected, then $G$ is a Type I graph with two vertices $x_1, x_2 \in X$ (or $Y$) and one vertex $y_1 \in Y$ (or $X$) such that $d(x_1) + d(x_2) \geq |Y| - 1$ and $d(y_1) \geq |X| - 2$. Hence the result.

**Corollary 3.1.6.** Let $G$ be a bipartite graph. Then $\gamma(G) = 2$ if and only if $G$ contains $K^k_{m,n}$ as a spanning subgraph where $k = 0, 1, 2, 3$.  

Proof. Let $\gamma(G) = 2$. Since $G$ is bipartite, $\gamma_{st}(G) = 2$ or 3. When $\gamma_{st} = 2$, by Theorem 3.1.4 $G$ contains $K_{m,n}^k$ as a spanning subgraph where $k = 0, 1, 3$. When $\gamma_{st} = 3$, by Theorem 3.1.4 $G$ contains $K_{m,n}^k$ as a spanning subgraph where $k = 2$.

The converse is obvious.

\section{Trees}

In this section, we investigate $\gamma_{st}$ for trees that attain various bounds involving order, $\Delta$, $\gamma$ and $\gamma_c$.

Remark: For any graph $G$, $\gamma \leq p - \Delta$, and for bipartite graphs $\gamma_{st} = \gamma$ (or) $\gamma + 1$. Therefore we have $p - \Delta + 1$ is an upper bound for $\gamma_{st}$. In the following theorems we investigate those trees for which $\gamma_{st} = p - \Delta + 1$ or $p - \Delta$.

**Theorem 3.2.1.** For any tree, $T$, $\gamma_{st} \leq p - \Delta + 1$. Equality is attained if and only if $T$ is a star $K_{1,p-1}$.

Proof. Obviously $\gamma_{st} \leq p - \Delta + 1$. If $\gamma_{st} = p - \Delta + 1$ then $\gamma = p - \Delta$ and $\gamma_{st} = \gamma + 1$. But $\gamma = p - \Delta$ if and only if $T$ is a wounded spider by Theorem 1.1.41. Again $\gamma_{st} = \gamma + 1$ implies $T$ is a Type II graph. Hence $T$ is a star $K_{1,p-1}$.
Theorem 3.2.2. If $T$ is tree, then $\gamma_{st}(T) = p - \Delta$ if and only if $T$ is a wounded spider which is not a star.

Proof. Let $T$ be a wounded spider that is not a star. Let $v$ be the vertex of $T$ with $d(v) = \Delta$ (Ref. Figure 3.3).

Take $N(v) = \{u_1, u_2, u_3, \ldots, u_k\}$ and $V - N[v] = \{v_1, v_2, \ldots, v_l\}$ where each $v_i$ is a pendant vertex that is adjacent to $u_i$.

Clearly $p = k + l + 1$ where $k = \Delta$. The set $\{v, v_1, v_2, v_3, \ldots v_l\}$ is a minimum std-set of $T$ and hence $\gamma_{st} = p - \Delta$.

Conversely, let $T$ be a tree with $\gamma_{st} = p - \Delta$. By Theorem 3.2.1, $T$ cannot be a star. Let $v$ be a vertex with $d(v) = \Delta$. Take $(X,Y)$ as the unique bipartition of $T$ with $v \in X$. Naturally $N(v) \subseteq Y$. Suppose there exists a vertex $u \in Y - N(v)$. Let $D = M \cup \{v\}$, where $M$ is a maximal
independent set in \( (V - N[v]) \) containing \( u \). Since \( M \) is maximal, \( M \cup \{v\} \) becomes a dominating set and also an std-set of \( G \). As \( u \in Y \), there exists an \( x(\neq v) \in X \) such that \( u \) is adjacent to \( x \). Therefore \( |M| \leq (|V - N[v]| - 1) \).

\[
|M \cup \{v\}| = |M| + 1 \\
\leq (|V - N[v]| - 1) + 1 \\
\leq p - (\Delta + 1) - 1 + 1 \\
= p - \Delta - l \\
< p - \Delta \\
= \gamma_{st}.
\]

This is a contradiction to the fact \( M \cup \{v\} \) is an std-set. Hence \( Y = N(v) \) and \( X = V - N(v) \). \( T \) cannot be a spider. Hence \( T \) is a wounded spider that is not a star.

**Remark:** For a bipartite graph, \( \gamma_{st} \leq \gamma + 1 \); since \( \gamma \leq \frac{p}{2} \) we have \( \gamma_{st} \leq \frac{p}{2} + 1 \) for trees and this bound is sharp as proved below.

**Theorem 3.2.3.** If \( T \) is a tree, then \( \gamma_{st} = \frac{p}{2} + 1 \) if and only if \( T \) is \( K_2 \).

**Proof.** Let \( \gamma_{st} = \frac{p}{2} + 1 \). Since \( \gamma \leq \frac{p}{2} \), we have \( \gamma = \frac{p}{2} \). By Theorem 1.1.38, \( T \) is \( C_4 \) or \( H^+ \) where \( H \) is a connected graph. If \( H \neq K_1 \), then \( T = H^+ \) is a Type I graph and so \( \gamma_{st} = \frac{p}{2} \), a contradiction. Hence \( H = K_1 \) and \( T = H^+ = K_2 \).
Conversely if $T = K_2$, then $\gamma_{st} = \frac{p}{2} + 1$.

**Corollary 3.2.4.** For a tree $T$, $\gamma + \gamma_{st} \leq p + 1$. The bound is attained if and only if $T = K_2$.

**Proof.** Since $\gamma \leq \frac{p}{2}$ and $\gamma_{st} \leq \frac{p}{2} + 1$, we have $\gamma + \gamma_{st} \leq p + 1$. When $\gamma + \gamma_{st} = p + 1$, we have $\gamma = \frac{p}{2}$ and $\gamma_{st} = \frac{p}{2} + 1$. Hence by Theorem 3.2.3, $T = K_2$. Converse is trivial.

**Corollary 3.2.5.** For a tree $T$, $\chi + \gamma_{st} \leq \frac{p}{2} + 3$. The bound is attained if and only if $T = K_2$.

**Proof.** Since for a tree, $\chi \leq 2$ and $\gamma_{st} \leq \frac{p}{2} + 1$, we have $\chi + \gamma_{st} \leq \frac{p}{2} + 3$ and if the bound is attained then $\gamma_{st} = \frac{p}{2} + 1$. Hence by Theorem 3.2.3, $T = K_2$. The converse is obvious.

**Theorem 3.2.6.** If $T$ is a tree, then $\gamma_{st} = \frac{p}{2}$ if and only if $T$ is $K_{1,3}$ or $H^+$ where $H$ is a non-trivial tree.

**Proof.** Let $\gamma_{st} = \frac{p}{2}$. Then either $\gamma = \frac{p}{2} - 1$ or $\gamma = \frac{p}{2}$. If $\gamma = \frac{p}{2}$, then by Theorem 1.1.38, $G$ is $H^+$ where $H$ is a non-trivial tree. Now assume $\gamma = \frac{p}{2} - 1$. Since $\gamma_{st} = \frac{p}{2}$, we have $\gamma_{st} = \gamma + 1$ and hence $T$ is a Type II graph. Also $p$ is even and $p \geq 4$. Let $T = G(X, Y)$ with $|X| \leq |Y|$. By Theorem 3.1.1 $X$ is the unique $\gamma$-set and hence $|X| = \frac{p}{2} - 1$. Also every vertex in $X$
has at least two pendant neighbours in $Y$ which implies that $|X| \leq \frac{p}{3}$ and $|Y| \geq 2|X| = 2\left(\frac{p}{2} - 1\right) = p - 2$. Since $|X| \leq \frac{p}{3}$, we have $\frac{p}{2} - 1 \leq \frac{p}{3}$, which means that $p \leq 6$. This implies that $p = 4$ or 6. When $p = 4$, $|X| = 1$ and $|Y| = 3$ and consequently the tree is $K_{1,3}$. When $p = 6$, $|X| = 2$, and $|Y| = 4$. Since every vertex in $X$ has at least two pendant neighbours in $Y$, the only possibility is that $T$ is a forest namely $K_{1,2} \cup K_{1,2}$ which contradicts the fact that $T$ is a tree. Hence $p = 6$ is not possible and we conclude $T = K_{1,3}$ or $H^+$ where $H$ is a non trivial tree. The converse can easily be verified.

**Theorem 3.2.7.** If $T(\neq K_2)$ is a tree, then $\gamma_c + \gamma_{st} \leq \frac{3p}{2} - 2$. Equality is attained if and only if $T$ is $P_4$.

**Proof.** Since $T \neq K_2$, we have by Theorem 3.2.3, $\gamma_{st} \leq \frac{p}{2}$. Also by Theorem 1.1.51, $\gamma_c = p - l(T) \leq p - \Delta \leq p - 2$. Hence $\gamma_{st} + \gamma_c \leq \frac{p}{2} + (p - 2) = \frac{3p}{2} - 2$.

Consider the equality $\gamma_{st} + \gamma_c = \frac{3p}{2} - 2$.

**Case(i)** Let the tree $T$ be a type I graph with $\gamma_{st} = \gamma$. If $\gamma_c < p - 2$, $\gamma + (p - 2) > \frac{3p}{2} - 2$ implying that $\gamma > \frac{p}{2}$ which is impossible. Hence $\gamma_c = p - 2$ in which case $\gamma_{st} = \gamma = \frac{p}{2}$. Now $\gamma_c = p - 2$ implies that $T$ is a path and $\gamma_{st} = \frac{p}{2}$ implies that $T = H^+$ where $H$ is a non trivial tree. Hence we conclude that $T = P_4$.

**Case(ii)** Assume that the tree $T$ is a type II graph that is $\gamma_{st} = \gamma + 1$. Again if $\gamma_c < p - 2$, we get $(\gamma + 1) + (p - 2) > \frac{3p}{2} - 2$ which means $\gamma > \frac{p}{2} - 1$ and hence $\gamma \geq \frac{p}{2}$. But $\gamma \leq \frac{p}{2}$ is always true for $T$ and hence $\gamma = \frac{p}{2}$. Hence as in case I, we have $T = P_4$ but $P_4$ is not a type II graph. Hence no solution exists in this case. Combining the two cases, we get $T = P_4$ is the only graph. Converse is obvious.

\[\square\]
Corollary 3.2.8. If \( T(\neq K_2) \) is a tree then \( \gamma_{st} + \chi \leq \frac{p}{2} + 2 \). The bound is attained if and only if \( T = K_{1,3} \) or \( H^+ \) where \( H \) is a non trivial tree.

**Proof.** Since \( T \) is a tree which is not \( K_2 \), by Theorem 3.2.3, \( \gamma_{st} \leq \frac{p}{2} \). Hence \( \gamma_{st} + \chi \leq \frac{p}{2} + 2 \). The bound is attained when \( \gamma_{st} = \frac{p}{2} \) and hence by Theorem 3.2.6, \( T = K_{1,3} \) or \( H^+ \) where \( H \) is a non-trivial tree. \( \blacksquare \)