Chapter 5

Some special models in Reliability of $k$-out-of-$n$ system with repair under $T$-policy

5.1 Introduction

In this chapter, we consider a $k$-out-of-$n$ system with (i) an unreliable server (ii) activation time for the server which is a positive random variable (iii) positive inactivation time of the server.

First we consider a repair facility which consists of a single server which is subject to failure. Here lifetimes of components are exponentially distributed with rate $\lambda$. Repair time of components are exponentially distributed with parameter $\mu$. Failure of the server and its repair are exponentially distributed with rate $\beta$ and $\gamma$. We call this model 1.

Next a $k$-out-of-$n$ system with repair under $T$-policy with positive activation time of the server is considered. Here, though the server is switched on, it gets activated only after random length of time. Let $U$ be the activation time of the server, i.e., the amount of time required to get activated from the time it is switched on. The activation time is assumed to be exponentially distributed with rate $\theta$. Lifetimes of components and their repair times are exponentially distributed with rate $\lambda$ and $\mu$ respectively. $T$ is assumed to be exponential with rate $\alpha$. This is referred to as model 2.
Finally, we consider a $k$-out-$n$-system with repair with inactivation time of the server. Here, server gets activated on elapse of $T$ time units or the moment $n - k$ units fail, whichever occurs first. In the models discussed earlier the system goes to $(0, 0)$ from $(1, 1)$ on repair of a failed unit. In this model since there is a positive inactivation time the system goes to $(0, 2)$. On the server being switched off, where the status 2 of the server is defined later. From $(0, 2)$ it may go to $(0, 0)$ or $(1, 1)$ depending on whether a failure does not or does occur before inactivation. That is, though the server is switched off, he does not get inactivated. Denote by $W'$ the time required for the server to get inactivated from the moment it is switched off. $W'$ is assumed to be exponentially distributed with rate $\eta$. Lifetimes of the components and repair times are assumed to be exponential with rate $\lambda$ and $\mu$ respectively. These are the assumptions underlying model 3.

In all the three models, we obtain system state probabilities and some characteristics. Also we investigate the system reliability in the case of a cold system above.

Section 5.2 considers modelling and analysis of $k$-out-of-$n$ system with unreliable server. Section 5.3 gives the stationary probability distribution and some numerical illustrations. Section 5.4 is devoted to some system state characteristics. Section 5.5 analyses the model of $k$-out-of-$n$ system with an activation time for the server. Section 5.6 gives system state probabilities of this model and section 5.7 deals with some performance measures. Finally in Section 5.8 we analyse the last mentioned model.

### 5.2 Modelling and analysis

**Model 1**

Here we assume that the server is switched on only if there is at least one failed unit the system. Let $X(t)$ = number of failed units at time $t$.

\[ Y'(t) = \begin{cases} 
0 & \text{if server is inactive (but not in failed state)} \\
1 & \text{if server is active at time } t \\
2 & \text{if server is activated but down at } t 
\end{cases} \]

{(X(t), Y(t), t \in \mathbb{R}_+)} is a Markov chain on the state space.

\[
E = \{(i, 0)/0 \leq i \leq n-k+1\} \cup \{(i, 1)/1 \leq i \leq n-k+1\} \cup \{(i, 2)/1 \leq i \leq n-k+1\}.
\]

Let \(P_{ij}(t) = P((X(t), Y(t)) = (i, j)/(X(0), Y(0)) = (0, 0))\). State transition are as follows,

\[
\begin{align*}
(0,0) & \overset{\lambda}{\rightarrow} (1,0) \overset{\lambda}{\rightarrow} (2,0) \overset{\ldots}{\rightarrow} (n-k,0) \overset{\mu}{\rightarrow} (n-k+1,0) \\
(0,0) & \overset{\gamma}{\rightarrow} (1,1) \overset{\gamma}{\rightarrow} (2,1) \overset{\ldots}{\rightarrow} (n-k,1) \overset{\beta}{\rightarrow} (n-k+1,1) \\
(1,0) & \overset{\lambda}{\rightarrow} (1,1) \overset{\lambda}{\rightarrow} (1,2) \overset{\ldots}{\rightarrow} (n-k,2) \overset{\beta}{\rightarrow} (n-k+1,2)
\end{align*}
\]

### 5.3 Stationary Probability Distribution

The infinitesimal generator of the Markov chain is given [refer next page]. We write the states lexicographically ie. \((0, 0), (1, 0), (1, 1), (1, 2), (2, 0), \ldots, (n-k+1, 1), (n-k+1, 2)\).

We now write the infinitesimal generator in the following form,

\[
\begin{pmatrix}
0 & 1 & 2 & 3 & \ldots & (n-k) & (n-k+1) \\
0 & -\lambda & \lambda \beta & 0 & 0 & 0 & 0 \\
1 & S^o & -\lambda I & \lambda I & 0 & 0 & 0 \\
2 & 0 & 0 & S^o B^o & S - \lambda I & \lambda I & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(n-k) & 0 & 0 & 0 & S^o B^o & S - \lambda I & \lambda I \\
(n-k+1) & 0 & 0 & 0 & 0 & S^o B^o & S
\end{pmatrix}
\]

where \(S^o = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}, S^o B^o = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 0 \end{pmatrix}\).
The infinitesimal generator of the Markov chain

\[
\begin{pmatrix}
(0, 0) & (1, 0) & (1, 1) & (2, 0) & (2, 1) & (2, 2) & (n - k, 0) & (n - k, 1) & (n - k, 2) & (n - k + 1, 0) & (n - k + 1, 1) & (n - k + 1, 2) \\
-\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -(\lambda + \mu) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mu & 0 & -(\lambda + \mu + \gamma) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \gamma & -(\lambda + \gamma) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -(\lambda + \mu + \gamma) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & (\lambda + \mu + \gamma) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -(\lambda + \gamma) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -(\lambda + \mu + \gamma) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -(\lambda + \gamma) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -(\lambda + \mu + \gamma) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -(\lambda + \gamma) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -(\lambda + \mu + \gamma) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -(\lambda + \gamma) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -(\lambda + \mu + \gamma) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -(\lambda + \gamma) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -(\mu + \gamma) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu \\
\end{pmatrix}
\]
In this case, the stationary probability vectors are given by $\Pi_i = \pi_0 e_i R^i$, $1 \leq i \leq n - k$ and $\Pi_{n-k+1} = \pi_0 \beta S^{-1} R^{n-k}$, where $R = \lambda (I - \lambda B^{o_o} - S)^{-1}$ and $B^{o_o} \subseteq e_i \beta$. $\beta$ is the initial probability vector (See Neuts (1981)).

$$B^{o_o} \subseteq e_i \beta = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \beta_1 & \beta_2 & \beta_3 \end{bmatrix}$$

$$R = \lambda \begin{bmatrix} \lambda (1 - \beta_1) + \alpha & -(\lambda \beta_2 + \alpha) & -\lambda \beta_3 \\ -\lambda \beta_1 & \lambda (1 - \beta_2) + \mu + \beta & -\lambda \beta_3 + \beta \\ -\lambda \beta_1 & -(\lambda \beta_2 + \gamma) & \lambda (1 - \beta) + \gamma \end{bmatrix}^{-1}$$

and $\pi_0 = \{\beta (\sum_{i=1}^{n-k} R^i - \lambda R^{n-k} S^{-1}) e_i \}^{-1}$.

### 5.3.1 Numerical illustration

For given values of parameters, we obtain the stationary probability vectors as follows.

$\lambda = 6, \mu = 10, \alpha = 8, \beta = 5, \gamma = 9, \beta_1 = 1/3, \beta_2 = 1/2, \beta = 1/6, n = 15, k = 5.$

we get

$$R = \begin{bmatrix} 0.545 & 0.164 & 0.109 \\ 0.121 & 0.503 & 0.224 \\ 0.182 & 0.455 & 0.636 \end{bmatrix}$$

Here $n - k = 10$

$\pi_0 = 0.098; \quad \Pi_1 = (0.036, 0.04, 0.027)$

$\Pi_2 = (0.029, 0.038, 0.03); \quad \Pi_3 = (0.026, 0.037, 0.031)$

$\Pi_4 = (0.024, 0.037, 0.031); \quad \Pi_5 = (0.023, 0.037, 0.031)$

$\Pi_6 = (0.023, 0.036, 0.03); \quad \Pi_7 = (0.022, 0.036, 0.030)$

$\Pi_8 = (0.022, 0.035, 0.029); \quad \Pi_9 = (0.022, 0.035, 0.029)$

$\Pi_{10} = (0.021, 0.034, 0.029); \quad \Pi_{11} = (0.014, 0.045, 0.012)$
5.4 Some system state characteristics

5.4.1 Distribution of first passage time to break down state

Consider the Markov chain on the state space \{ (0,0), (1,0), \ldots, (n-k+1,0), (1,1), (2,1), \ldots, (n-k+1,1), (1,2), (2,2), \ldots, (n-k+1,2) \}. Consider the class \{ (1,2), (2,2), \ldots, (n-k+1,2) \}. To find the distribution of the time required to reach \( (i,2) \) for \( 1 \leq i \leq n-k+1 \). Let \( (i,l) \), where \( l = 0 \) or \( 1 \) be any of the transient states. The infinitesimal generator of \( (X(t),Y(t)) \) be denoted by

\[
\begin{bmatrix}
Q & Q^n \\
0 & 0
\end{bmatrix}
\]

Here \( Q \) is the matrix obtained by deleting the rows and columns corresponding to the states in the class \{ (1,2), (2,2), \ldots, (n-k+1,2) \}. The distribution of the time required to reach \( (i,2) \) is of phase type given by

\[
F_i(x) = 1 - Q_1 \exp(Q_1 x) \xi
\]

with \( Q_1 \) the initial probability vector.

5.4.2 Distribution of the time from server activation till all failed units are repaired

We have to find the distribution of time required to reach \( (0,0) \) starting from \( (i,1) \) without visiting \( (i,2) \) for \( 1 \leq i \leq n-k+1 \). Let \( Q_1 \) be the matrix obtained by deleting the rows and columns corresponding to the states \{ (1,2), (2,2), \ldots, (n-k+1,2) \} and state \( (0,0) \). the distribution of time required to reach \( (0,0) \) starting from \( (i,1) \) is of phase type given by

\[
F_2(x) = 1 - Q_2 \exp(Q_2 x) \xi
\]

where \( Q_2 \) is the initial probability vector. Then, the distribution of the time required to reach \( (0,0) \) starting from \( (i,1) \) before going to \( (i,2) \) is given by

\[
F_2(x) \sum_{i=1}^{n-k+1} \left( \frac{1 - \left( \lambda/\mu \right)^i}{1 - \left( \lambda/\mu \right)^{i+1}} \right)
\]

5.4.3 Distribution of time server remains continuously in the system

We need to find the distribution of the time required to reach \( (0,0) \) starting from some \( (i,1) \), at which an \( (i,0) \) to \( (i,1) \) transition took place, for \( 1 \leq i \leq n-k+1 \). Let \( Q_2 \) be the matrix obtained by deleting the row and column corresponding to the state \( (0,0) \).
Distribution of the time required to reach \( (0,0) \) is given by \( F_3(x) = 1 - \alpha_3 \exp(Q_2x) \).

5.4.4 Distribution of cycle length

To find the distribution of the time required to reach \( (0,0) \) starting from \( (0,0) \).

Consider the infinitesimal generator of the transition probabilities. Regard \( (0,0) \) as an absorbing state. Then distribution of the time required to reach \( (0,0) \) starting from any of the transient states \( (i,1) \) is \( F_3(x) = 1 - \alpha_3 \exp(Q_2x) \). Distribution of the time required to reach \( (0,0) \) starting from \( (0,0) \) is given by \( P(S_i < T < S_{i+1}) E_{i,\lambda} \ast E_{1,\alpha} \ast F_3(x) \) where \( E_{i,\lambda} \) is an Erlang distribution of order \( i \), parameter \( \lambda \). \( S_i \) is the time till \( i \) failures take place.

5.4.5 Expected time server remains continuously in the system

From the distribution of time server remains continuously in the system (Section 5.4.3), we can compute the expected time as \( -\alpha_3 Q_2^{-1} \). To find the inverse is a difficult task. So, we go for an iterative procedure.

Let \( T_{i1} \) denote the time to reach \( (i-1,1) \) starting from \( (i,1) \) and \( T_{i2} \) denote the time to reach \( (i,1) \) starting from \( (i,2) \). Th possible transitions are:

\[
\begin{align*}
(i,1) &\rightarrow (i+1,1) \rightarrow (i+1,2) \rightarrow (i,1) \rightarrow (i-1,1) \\
(i,1) &\rightarrow (i,2) \rightarrow (i,1) \rightarrow (i-1,1) \\
(i,1) &\rightarrow (i-1,1)
\end{align*}
\]

Thus

\[
E(T_{i1}) = \frac{1}{\lambda + \mu + \beta} \left( \frac{\mu}{\lambda + \mu + \beta} \right) + \frac{1}{\lambda + \mu + \beta} \left( \frac{\beta}{\lambda + \mu + \beta} \right)
\]

\[
= \frac{1}{\lambda + \mu + \beta} \left( \frac{1}{\lambda + \mu + \beta} + E(T_{i2}) + E(T_{i1}) \right)
\]

\[
E(T_{i2}) = \frac{1}{\lambda + \mu + \beta} \left( \frac{\gamma}{\lambda + \mu + \beta} \right) + \frac{1}{\lambda + \mu + \beta} \left( E(T_{i+1,1}) + E(T_{i+1,2}) + E(T_{i1}) \right)
\]

where

\[
E(T_{i2}) = \frac{\lambda \mu}{(\lambda + \gamma)(\lambda + \mu + \beta)^2} + \frac{1}{\lambda + \gamma} \left( E(T_{i+1,1}) + E(T_{i+1,2}) + E(T_{i1}) \right)
\]

\[
= \frac{1}{\lambda + \gamma} \left( E(T_{i+1,1}) + E(T_{i+1,2}) + E(T_{i1}) \right)
\]

\[
i = 1, 2, \ldots, n - k + 1
\]
Hence

\[ E(T_{i1}) = \frac{(\lambda + \beta + \gamma)}{\mu(\lambda + \gamma)} + \frac{\beta\lambda(\lambda + \mu + \gamma)}{\mu(\lambda + \mu + \beta)^2(\lambda + \gamma)} \]

which gives

\[ E(T_{i1}) = \frac{(\lambda + \beta + \gamma)}{\mu\gamma} + \frac{\beta\lambda(\lambda + \mu + \gamma)}{\mu(\lambda + \gamma)(\lambda + \mu + \beta)^2} + \frac{\lambda^2(\lambda + \beta + \gamma)}{\gamma(\lambda + \gamma)(\lambda + \mu + \beta)^2} \]

Recursively, we get

\[ E(T_{i1}) = \left( \frac{\lambda + \beta + \gamma}{\mu\gamma} \right) + \frac{\beta\lambda(\lambda + \mu + \gamma)}{\mu(\lambda + \gamma)(\lambda + \mu + \beta)^2} \]

\[ \frac{1 - (\lambda/\mu)^{n-k+1-i}}{1 - (\lambda/\mu)} + \frac{\lambda^2(\lambda + \beta + \gamma)}{\gamma(\lambda + \gamma)(\lambda + \mu + \beta)^2} \frac{(1 - (\lambda/\mu)^{n-k+1-i})}{1 - (\lambda/\mu)} \]

\[ - \frac{\lambda(\lambda + \beta + \gamma)}{\gamma(\lambda + \mu + \beta)^2(\lambda + \gamma)} \frac{(1 - (\lambda/\mu)^{n-k+1-i})}{1 - (\lambda/\mu)} (\lambda/(\lambda + \gamma))^{n-k+1-i} \] for \( i = 1, 2, \ldots, n - k + 1 \)

Now, \( \sum_{i=1}^{n-k+1} E(T_{i1}) \) gives the expected time server remains continuously in the system which is equal to

\[ (n - k + 1) \left[ \frac{\lambda + \beta + \gamma}{\gamma(\mu - \lambda)} + \frac{\beta\lambda(\lambda + \mu + \gamma)}{\gamma(\lambda + \gamma)(\lambda + \mu + \beta)^2(\mu - \lambda)} + \frac{\lambda^2\mu(\lambda + \beta + \gamma)}{\gamma(\lambda + \gamma)(\mu - (\lambda + \mu + \beta)^2(\lambda + \gamma))} \right] \]

\[ - \frac{\mu}{(\mu - \lambda)} \left( 1 - (\lambda/\mu)^{n-k+1} \right) \left[ \frac{\lambda + \beta + \gamma}{\gamma(\mu - \lambda)} + \frac{\beta\lambda(\lambda + \mu + \gamma)}{\gamma(\lambda + \gamma)(\lambda + \mu + \beta)^2(\mu - \lambda)} + \frac{\lambda^2\mu(\lambda + \beta + \gamma)}{\gamma(\lambda + \gamma)(\mu - \lambda)(\lambda + \mu + \beta)^2(\lambda + \gamma)} \right] \]

\[ + \frac{\mu}{(\mu - \lambda)} \left( 1 - (\lambda/\mu)^{n-k+1} \right) \frac{\lambda\mu(\lambda + \beta + \gamma)}{\gamma(\lambda + \mu + \beta)^2(\mu - (\lambda + \gamma))} \]

\[ - \frac{\lambda^2\mu(\lambda + \beta + \gamma)}{\gamma^2(\lambda + \mu + \beta)^2(\mu - (\lambda + \gamma))} \left( 1 - (\lambda/\mu)^{n-k+1} \right) \]}
5.4.6 Expected time server remains in down state

The expected number of visits to \((i, 2)\) before visiting the state \((0,0)\) is \(\frac{q_{i2}}{q_{00}}\). Expected time the system remains in the state \((i, 2)\) is \(\frac{1}{\lambda + \gamma}\), when \(i = 1, 2, \ldots, n - k\) and the expected time the system remains in the state \((n - k + 1, 2)\) is \(\frac{1}{\gamma}\). Hence, expected time the server is in breakdown state during a cycle is

\[
\sum_{i=1}^{n-k} \frac{1}{\lambda + \gamma} \frac{q_{i2}}{q_{00}} + \frac{1}{\gamma} \frac{q_{n-k+1,2}}{q_{00}}
\]

Thus, expected time server remains busy in a cycle

\[
\sum_{i=1}^{n-k} (E(T_{i1}) - \frac{q_{i2}}{q_{00}} \frac{1}{\lambda + \gamma}) + E(T_{n-k+1,2}) - \frac{q_{n-k+1,2}}{q_{00}} \frac{1}{\gamma}
\]

where \(q_{i2}/q_{00}\) can be computed for given parameters of the process.

5.5 \(k\)-out-of-\(n\) system with activation time

Lifetimes of the components are assumed to be exponentially distributed with parameter \(\lambda\). Server is switched on after the elapse of \(T\) time units since the epoch of its inactivation (ie. completion of repair of all failed units in the previous cycle) or until accumulation of \(n - k\) failed units, whichever occurs first. The server does not get activated the moment it is switched on. It takes a random length of time \(U\) which is assumed to be exponentially distributed with rate \(\theta\). \(T\) is exponentially distributed with rate \(\alpha\) and repair time exponentially distributed with rate \(\mu\). Hence the time elapsed until activation starting from all units operational, has generalized Erlang distribution. In chapter 3, we considered the case when activation time is zero. We get the results there by taking \(\lim_{\theta \to \infty}\) in this section.

5.5.1 Mathematical Formulation

Let \(X(t)\) represent the number of failed units at time \(t\).

\[
Y(t) = \begin{cases} 
2 & \text{if server is active at time } t \\
1 & \text{if server is only switched on but not active at } t \\
0 & \text{otherwise}
\end{cases}
\]
\{(X(t), Y(t)), t \in R_+\} \text{ is a Markov chain with state space.}

\[ A = \{(i,0)/0 \leq i \leq n-k-1\} \cup \{(i,1)/0 \leq i \leq n-k+1\} \cup \{(i,2)/0 \leq i \leq n-k+1\} \]

The difference-differential equations satisfied by \( P_{ij}(t) \) are

\[
\begin{align*}
P'_{00}(t) &= -(\lambda + \alpha)P_{00}(t) + \mu P_{12}(t) \\
P'_{01}(t) &= -(\lambda + \theta)P_{01}(t) + \alpha P_{00}(t) \\
P'_{02}(t) &= -\lambda P_{02}(t) + \theta P_{01}(t) \\
P'_{10}(t) &= -(\lambda + \alpha)P_{10}(t) + \lambda P_{1-1,0}(t); \quad 1 \leq i \leq n-k-1 \\
P'_{11}(t) &= -(\lambda + \theta)P_{11}(t) + \lambda P_{1-1,1}(t) + \alpha P_{10}(t)(1 - \delta_{n-k}) + \lambda \delta_{n-k}P_{1-1,0}(t); \\
& \quad 1 \leq i \leq n-k \\
P'_{12}(t) &= -(\lambda(1 - \delta_{n-k+1}) + \mu)P_{12}(t) + \lambda P_{1-1,2}(t) + \mu P_{i+1,2}(t)(1 - \delta_{n-k+1}) \\
& \quad + \beta(1 - \delta_{n-k+1})P_{11}(t) + \lambda \delta_{n-k+1}P_{1-1,1}(t); \quad 1 \leq i \leq n-k+1
\end{align*}
\]

### 5.6 Steady state distribution

Let \( q_{ij} = \lim_{t \to \infty} P_{ij}(t) \). Then the steady state probabilities are given by

\[
q_{01} = \frac{\alpha}{\alpha + \theta} q_{00}; \quad q_{02} = \frac{\delta \alpha}{\lambda \alpha + \delta \theta} q_{00}; \quad q_{10} = \frac{\lambda}{(\lambda + \alpha)} q_{00}, 1 \leq i \leq n-k-1
\]

\[
q_{11} = \frac{\alpha}{\theta - \alpha} q_{01} + \frac{1}{\lambda + \theta} \left( \frac{1}{(\theta + \lambda)^{i+1}} q_{00} \right) \quad 1 \leq i \leq n-k-1
\]

\[
q_{n-k,1} = \frac{\lambda}{\lambda + \theta} \left( \frac{1}{(\theta + \alpha) \lambda^{n-k-1}} + \frac{1}{(\theta + \lambda)^{n-k-2}} + \ldots \right)
\]

\[
= \frac{1}{(\theta + \lambda)^{n-k-1}} + \frac{(\lambda + \theta)^{n-k-1}}{(\alpha + \lambda)^{n-k-1}} q_{00}
\]

\[
= \frac{\lambda}{\lambda + \theta} \left( \frac{1}{(\theta + \alpha) \lambda^{n-k-1}} \right) \left( 1 - \frac{1}{(\alpha + \lambda)^{n-k-1}} q_{00} \right)
\]
\[ q_{2i} = \left(\frac{\alpha + \lambda}{\mu}\right) \left[\frac{1 - (\lambda/\mu)^i}{1 - (\lambda/\mu)}\right] = \frac{\alpha \theta \lambda}{\mu (\theta + \lambda)} \left(\frac{1}{\lambda} + \frac{\lambda}{\alpha + \lambda} + \frac{1}{\theta + \lambda}\right) \]

\[- \sum_{j=2}^{i-1} \frac{\alpha \theta \lambda j}{\mu j (\theta + \lambda)} \left(\frac{\mu^{j-1}}{(\theta + \lambda)^j} + \frac{\mu^{j-1}}{(\theta + \lambda)^j (\alpha + \lambda)}\right) + \cdots \]

\[+ \frac{\mu^{i-1}}{(\alpha + \lambda)^i} + \cdots + \frac{1}{\lambda} + \frac{1}{\alpha + \lambda} + \frac{1}{\theta + \lambda} \]

\[= \left(\frac{\alpha + \lambda}{\mu - \lambda}\right) - \frac{\alpha \theta \lambda}{\mu (\theta + \lambda)} \left(\frac{1}{\lambda} + \frac{1}{\alpha + \lambda} + \frac{1}{\theta + \lambda}\right) - \frac{\alpha \theta \lambda^2}{(\alpha + \lambda) (\theta - \alpha) (\lambda + \alpha - \mu)} \left(\frac{(\lambda/\mu)^{i-2}}{(\mu - \lambda)} - \frac{\alpha}{(\lambda/\mu)^{i-2}} \right) \]

\[+ \frac{\alpha \theta \lambda^2 (\theta + \lambda)}{(\theta - \alpha) (\alpha + \lambda)^2 (\theta + \lambda - \mu)} \left(\frac{(\lambda/\theta + \lambda))^{i-2}}{\theta} - \frac{(\lambda/\mu)^{i-2}}{(\mu - \lambda)} \right) \]

\[q_{00} \text{ can be obtained using the normalizing condition} \]

\[\sum_{i=0}^{n-k-1} q_{i0} + \sum_{i=1}^{n-k} q_{i1} + \sum_{i=1}^{n-k+1} q_{2i} = 1 \]

The system reliability is given by \(1 - q_{n-k+1,2}\). Fraction of time the system is down is \(q_{n-k+1,2}\)

### 5.7 Some performance measures

#### 5.7.1 Distribution of duration of server availability

Consider the Markov chain on \(\{(0,0), (1,2), (2,2), \ldots, (n-k+1,2)\}\). We have to compute the distribution of time until the system reaches \((0,0)\) starting from one of the transient states \((i,2)\). Consider the infinitesimal generator \(\begin{bmatrix} S & S^0 \\ 0 & 0 \end{bmatrix}\), where \(S\) is the matrix obtained by deleting the row and column corresponding to the state \((0,0)\). Then distribution of time to reach \((0,0)\) starting from \((i,2)\) is phase type given by \(1 - \gamma_1 \exp(Sx)\gamma_1\), where \(\gamma_1\) is the initial probability vector.
5.7.2 Expected duration of time the system remains non-functional in a cycle

\[ \frac{\alpha}{\mu} \] gives the expected number of visits to \((n - k + 1, 2)\) before first return to \((0,0)\). Further \(\frac{1}{\mu}\) is the expected amount of time system remains in that state during each visit to it. Hence expected duration of time system is down is \(\frac{1}{\mu} \sum_{k=0}^{n-k-1,2} \) which is equal to

\[
\begin{align*}
1 \left( \frac{(\alpha + \lambda)}{(\mu - \lambda)} - \frac{\alpha}{\mu} \right) &+ \frac{1}{\mu} \left( \frac{(\alpha + \lambda)}{\lambda + \mu} \right) \left( \frac{1}{\lambda + \mu} + \frac{1}{\theta + \lambda} \right) - \frac{\lambda^3}{(\alpha + \lambda)^2(\theta - \alpha)(\mu - \lambda)} \\
&+ \frac{\alpha}{\mu} \left( \frac{\alpha \theta \lambda^2}{\lambda + \mu} \right) \left( \frac{(\lambda/\mu)^{n-k-1}}{(\mu - \lambda)} - \frac{\alpha}{(\lambda/(\lambda + \alpha))^{n-k-1}} \right) \\
&+ \frac{\alpha}{\mu} \left( \frac{\alpha \theta \lambda^2}{\lambda + \mu} \right) \left( \frac{(\lambda/(\theta + \lambda))^{n-k-1}}{\theta} - \frac{(\lambda/\mu)^{n-k-1}}{(\mu - \lambda)} \right)
\end{align*}
\]

5.7.3 Expected time server remains active during a cycle

Define \(T_{i2}\) as the time to reach \((i - 1, 2)\) starting from \((i, 2)\). The following transitions are possible

\((i, 2) \rightarrow (i + 1, 2) \rightarrow (i, 2) \rightarrow (i - 1, 2)\)

\((i, 2) \rightarrow (i - 1, 2)\)

\[
E(T_{i2}) = \frac{1}{\lambda + \mu} \left( \frac{1}{\lambda + \mu} + E(T_{i+1,2}) + E(T_{i2}) \right)
\]

Hence \(E(T_{i2}) = \frac{1}{\mu} + \frac{1}{\mu} E(T_{i+1,2})\) recursively we get

\[
E(T_{i2}) = \frac{1 - (\lambda/\mu)^{n-k+2-i}}{\mu - \lambda}
\]

for \(i = 1, 2, \ldots, n - k + 1\) starting with \(E(T_{n-k+1,2}) = \frac{1}{\mu}\). The expected time to reach \((0,0)\) starting from \((i, 2)\) is \(\sum_{i=0}^{n-k+1} E(T_{i2})\), where \(T_{02}\) denote the time to reach \((0, 0)\) starting from \((0, 2)\).

5.7.4 Expected amount of time server is inactive in a cycle

From the state \((0,0)\), the system goes to \((1, 0)\) on failure of one unit or it goes to \((0,1)\) on elapse of \(T\) time units.
From the state \((0,1)\), system goes to \((0,2)\) on elapse of activation time or it goes to \((1,1)\) on failure of one unit. The process goes on in this fashion. A possible transition is indicated in the diagram.

\[
\begin{array}{cccccccc}
(0,2) & \rightarrow & (1,2) & \rightarrow & (2,2) & \rightarrow & \cdots & \rightarrow & (n-k-1,2) & \rightarrow & (n-k,2) & \rightarrow & (n-k+1,2) \\
(0,1) & \rightarrow & (1,1) & \rightarrow & (2,1) & \rightarrow & \cdots & \rightarrow & (n-k-1,1) & \rightarrow & (n-k,1) \\
(0,0) & \rightarrow & (1,0) & \rightarrow & (2,0) & \rightarrow & \cdots & \rightarrow & (n-k-1,0) & \rightarrow & (n-k,0) \\
\end{array}
\]

The expected time server is not in the system

\[
= \left(\frac{1}{\alpha} + \frac{1}{\theta}\right) P(T + U < S_1) + \left(\frac{1}{\alpha} + \frac{1}{\theta} + \frac{1}{\lambda}\right) P(S_1 < T + U < S_2) + \cdots + \left(\frac{1}{\alpha} + \frac{n-k-1}{\lambda}\right) P(S_{n-k-1} < T + U < S_{n-k}) + \frac{n-k}{\lambda} P(T + U > S_{n-k})
\]

where

\[
P(S_i < T + U < S_{i+1}) = \int_{u=0}^{\infty} \int_{v=0}^{u} \frac{e^{-\lambda u}(\lambda u)^{i-1}}{(i-1)!} \frac{\lambda \alpha (e^{-\sigma v} - e^{-\theta v})}{(\theta - \alpha)} e^{-\mu(v + u)} dv du
\]

\[
= \frac{\alpha \lambda \theta}{(\theta - \alpha)} \left(\frac{1}{(\lambda + \alpha)^{i+1}} - \frac{1}{(\lambda + \theta)^{i+1}}\right)
\]

for \(i = 0, 1, \ldots, n-k-1\) and \(P(T + U > S_{n-k}) = \frac{\alpha \lambda^{n-k}}{(\theta - \alpha)} \left(\frac{1}{(\lambda + \alpha)^{n-k+1}} - \frac{1}{(\lambda + \theta)^{n-k+1}}\right)\). Thus we get expected time server is not in the system

\[
= 2\left(\frac{1}{\alpha} + \frac{1}{\theta}\right) - \left(\frac{\lambda}{\lambda + \alpha}\right)^{n-k} \left(\frac{2\theta + \alpha}{\alpha(\theta - \alpha)}\right) - \left(\frac{\lambda}{\lambda + \theta}\right)^{n-k} \left(\frac{2\alpha + \theta}{\theta(\theta - \alpha)}\right)
\]

### 5.7.5 Expected cycle length

\(E(\tau) = E(\text{busy period}) + E(\text{time server remains inactive in the system})\)

From 5.7.3 and 5.7.4, we get

\[
E(\tau) = \frac{(n-k)(\mu - \lambda) + 2\mu - 3\lambda}{(\mu - \lambda)^2} + \frac{\lambda}{(\mu - \lambda)^2} \left(\frac{\lambda}{\mu}\right)^{n-k+2} + 2\left(\frac{1}{\alpha} + \frac{1}{\theta}\right)
\]

\[
- \left(\frac{\lambda}{\lambda + \alpha}\right)^{n-k} \left(\frac{2\theta + \alpha}{\alpha(\theta - \alpha)}\right) - \left(\frac{\lambda}{\lambda + \theta}\right)^{n-k} \left(\frac{2\alpha + \theta}{\theta(\theta - \alpha)}\right)
\]
5.7.6 Cost Analysis

Let \( C_1 \) be the fixed cost of hiring the server, \( w \) be the wage of the server per unit time and \( C_2 \) be the cost per unit time due to the system remaining non-functional.

Then, total expected cost per unit time,

\[
TEC = \frac{C_1}{E(\tau)} + w \left[ \frac{(n - k + 2)}{(\mu - \lambda)} \right] - \frac{\lambda}{(\mu - \lambda)^2} \left(1 - \left(\frac{\lambda}{\mu}\right)^{n-k+2}\right)/E(\tau)
\]

\[
+ C_2 \frac{q_{n-k+1.2}}{\mu q_{\infty}}
\]

where \( \frac{q_{n-k+1.2}}{\mu q_{\infty}} \) is the expected time the system remains non-functional is a cycle and is given in 5.7.2. \( TEC \) is convex in \( \alpha \). Hence, global minimum value \( \alpha^* \) that minimizes \( TEC \) exits. Numerically, \( TEC \) is evaluated for given set of parameters and various values of \( \alpha \) and is given below.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( C_1 = 100, w = 50, \lambda = 5, \mu = 10 )</th>
<th>( C_1 = 30, w = 20 \lambda = 15 \mu = 20 )</th>
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<tbody>
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<td>18.679</td>
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<td>67.903</td>
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<td>68.281</td>
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<tr>
<td>2.9</td>
<td>68.624</td>
<td>19.709</td>
</tr>
<tr>
<td>3.0</td>
<td>68.926</td>
<td>19.782</td>
</tr>
</tbody>
</table>

5.8 \( k \)-out-of-\( n \) system with inactivation time

Model 3

5.8.1 Mathematical modelling and analysis

Lifetimes of components are assumed to be exponentially distributed with rate \( \lambda \). Service time follows on exponential distribution with rate \( \mu \). \( T \) is exponentially distributed with rate \( \alpha \).
Let $X(t)$ denote the number of failed units at time $t$.

\[
Y(t) = \begin{cases} 
0 & \text{if server is inactive at } t \\
1 & \text{if server is active at } t \\
2 & \text{if server is switched off, but has not become inactive at } t
\end{cases}
\]

In models discussed earlier from the state $(1, 1)$, the state $(0,0)$ is reached on completion of repair of the failed units provided no units fail in between. Here the system goes to $(0, 2)$ from $(1, 1)$ before reaching $(0,0)$ provided no failure takes place during this period.

\{(X(t), Y(t)), \ t \in \mathbb{R}^+\} \text{ from a Markov chain on the state space.}

\[
B = \{(i, 0)|0 \leq i \leq n-k-1\} \cup \{(i, 1)|0 \leq i \leq n-k-1\} \cup \{(0,2)\}
\]

The difference-differential equations satisfied by $P_{ij}(t)$ are given by

\[
P'_{00}(t) = -(\lambda + \alpha)P_{00}(t) + \eta P_{02}(t)
\]

\[
P'_{02}(t) = -(\lambda + \eta)P_{02}(t) + \mu P_{11}(t)
\]

\[
P'_{n-k,1}(t) = -(\lambda + \mu)P_{n-k,1}(t) + \lambda P_{n-k-1,0}(t) + \lambda P_{n-k-1,1}(t)
\]

\[
+ \mu P_{n-k+1,1}(t)
\]

\[
P'_{m0}(t) = -(\lambda + \alpha)P_{m0}(t) + \lambda P_{m-1,0}(t); \quad 0 < m \leq n-k+1
\]

\[
P'_{m1}(t) = -(\lambda + \mu)P_{m1}(t) + \mu P_{m+1,1}(t) + \lambda P_{m-1,1}(t) + \alpha P_{m0}(t) + \lambda \delta_{m1}P_{02}(t)
\]

\[
0 < m \leq n-k-1
\]

\[
P'_{01}(t) = -\lambda P_{01}(t) + \alpha P_{00}(t)
\]

\[
P'_{n-k+1,1}(t) = \lambda P_{n-k,1}(t) - \mu P_{n-k+1,1}
\]

### 5.8.2 Steady state probabilities

Let $q_{ij} = \lim_{t \to \infty} P_{ij}(t), \ i, j \in B$ On solving the equations, we get

\[
q_{01} = \frac{\alpha}{\lambda} q_{00} \quad q_{00} = \left(\frac{\lambda}{\lambda + \alpha}\right)^{q_{00}}; \quad i = 1, 2, \ldots, n-k-1
\]

\[
q_{02} = \left(\frac{\lambda + \alpha}{\eta}\right) q_{00} \quad q_{11} = \left(\frac{\lambda + \eta}{\mu}\right)\left(\frac{\lambda + \alpha}{\eta}\right) q_{00}
\]
\[ q_{21} = \left[ \frac{\lambda^2}{\mu(\lambda + \alpha)} + \left( \frac{\lambda}{\mu} \right) \left( \frac{\lambda + \eta}{\mu} \right) \frac{\lambda}{\eta} \right] q_{00} \]

\[ q_{31} = \left[ \frac{\lambda^3}{\mu(\lambda + \alpha)^2} + \frac{\lambda^3}{\mu^2(\lambda + \alpha)} + \left( \frac{\lambda}{\mu} \right)^2 \left( \frac{\lambda + \eta}{\mu} \right) \frac{\lambda}{\eta} \right] q_{00} \]

\[ q_{i1} = \left[ \frac{\lambda^i (1 - (\frac{\lambda + \alpha}{\mu})^{i-3})}{(\lambda + \alpha)^{i-1}(\mu - (\lambda + \alpha))} + \frac{\lambda^i (\lambda + \alpha + \mu)}{\mu^{n-k}(\lambda + \alpha)^2} + \left( \frac{\lambda}{\mu} \right)^{i-1} \left( \frac{\lambda + \eta}{\mu} \right) \frac{\lambda}{\eta} \right] q_{00} \]

\[ 4 \leq i \leq n - k \]

\[ q_{n-k+1,1} = \left[ \frac{\lambda^{n-k+1}(1 - (\frac{\lambda + \alpha}{\mu})^{n-k-3})}{\mu(\lambda + \alpha)^{n-k-1}(\mu - (\lambda + \alpha))} + \frac{\lambda^{n-k+1}(\lambda + \alpha + \mu)}{\mu^{n-k}(\lambda + \alpha)^2} + \frac{(\lambda^i)^{n-k}(\lambda + \alpha + \mu)}{\mu^{n-k}(\lambda + \alpha)^2} \right] q_{00} \]

System reliability is given by \( 1 - q_{n-k+1,1} \). Fraction of time system is down is \( q_{n-k+1,1} \).

\( q_{00} \) can be obtained using normalising condition

\[ \sum_{i=0}^{n-k-1} q_{i0} + \sum_{i=0}^{n-k+1} q_{i1} + q_{02} = 1 \]

\[ q_{00} = \left[ \left( \frac{\lambda + \alpha}{\alpha} \right) (1 - \frac{\lambda}{\lambda + \alpha})^{n-k} + \frac{\lambda^2}{\mu(\lambda + \alpha)} + \frac{\lambda + \alpha}{\eta} \right] \frac{\lambda + \alpha}{\mu} \]

\[ + \frac{1}{\mu(\mu - \lambda)} \left( 1 - \frac{\lambda}{\mu} \right)^{n-k+1} \frac{(\lambda + \eta)(\lambda + \alpha)\mu}{\eta} + \frac{\lambda^3(\lambda + \alpha + \mu)}{(\lambda + \alpha)^2} \]

\[ + \frac{\lambda^i}{\alpha(\lambda + \alpha)^2(1 - \frac{\lambda}{\mu})^{n-k-3}} - \frac{\lambda^i}{\alpha(\lambda + \alpha)^2} \frac{\lambda}{(\lambda + \alpha)^3} \]

\[ - \frac{\lambda^{n-k+2}}{\mu(\lambda + \alpha)^{n-k-1}(\mu - (\lambda + \alpha))} + \frac{(\lambda^i)^{n-k+2}}{\alpha(\lambda + \alpha)^2} \frac{\lambda^{n-k+3}(\mu - (\lambda + \alpha))}{\mu^{n-k-3}(\mu - (\lambda + \alpha))} \]

\[ \{ 0, 1, 2, 3, \ldots, (n - k - 1, 1), (n - k, 1), (n - k + 1, 1) \} \]

\[ 5.8.3 \text{ Distribution of time duration server remains continuously in the system} \]

This is the distribution of time from activation till it becomes inactive. Consider the Markov chain on the state space \{ (0, 0), (0, 2), (1, 1), \ldots, (n - k - 1, 1), (n - k, 1), (n - k + 1, 1) \}. The distribution of time taken to reach (0,0) starting from any of the transient states (i, 1) is given by \( G_1(x) = 1 - \alpha \exp(Dr) \) where \( D \) is the matrix obtained by deleting the row and column corresponding to state (0,0) and \( \alpha \) is the initial probability vector.
5.8.4 Expected time server remains busy in a cycle

First, we find expected time to reach cycle \((0,2)\) starting from some \((i, 1)\). Define \(T_{i1}\), for \(i = 1, 2, \ldots, n - k + 1\) as the time required to reach \((0, 2)\) from \((i, 1)\). Following are the possible transitions

\[
(i, 1) ightarrow (i - 1, 1) \quad (i, 1) ightarrow (i + 1, 1) \rightarrow (i, 1) \rightarrow (i - 1, 1)
\]

Then,

\[
E(T_{i1}) = \frac{1}{\lambda + \mu} \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} \left( \frac{1}{\lambda + \mu} + E(T_{i+11}) \right) + E(T_{i1})
\]

ie. \(E(T_{i1}) = \frac{1}{\mu} + \frac{\lambda}{\mu} E(T_{i+11})\)

for \(i = 1, 2, \ldots, n - k + 1\). From the above relation, recursively we get

\[
E(T_{n-k+11}) = \frac{1}{\mu} \sum_{i=1}^{n-k+1} E(T_{i1})
\]

starting from \(E(T_{n-k+11}) = \frac{1}{\mu}\). \(\sum_{i=1}^{n-k+1} E(T_{i1})\) gives the expected time to reach \((0, 2)\) starting from \((n - k + 1, 1)\). Next, we find the expected time to reach \((0,0)\) starting from \((0,2)\). The following transitions are possible

\[
(0, 2) \rightarrow (0, 0) \\
(0, 2) \rightarrow (1, 1) \rightarrow (0, 2) \rightarrow (0, 0)
\]

Let \(T_{02}\) denote the time to reach \((0,0)\) starting from \((0,2)\). Then,

\[
E(T_{02}) = \frac{1}{\lambda + \eta} \frac{\eta}{\lambda + \eta} + \frac{\lambda}{\lambda + \eta} \left( \frac{1}{\lambda + \eta} + E(T_{11}) + E(T_{02}) \right)
\]

ie. \(E(T_{02})(1 - \frac{\lambda}{\lambda + \eta}) = \frac{\lambda + \eta}{(\lambda + \eta)^2} + \frac{\lambda}{\lambda + \eta} E(T_{11})\)

ie. \(E(T_{02}) = \frac{1}{\eta} + \frac{\lambda}{\eta} \frac{(1 - (\lambda/\mu)^{n-k+1})}{\mu - \lambda}\)

Expected time server remains continuously in the system is the expected time to reach \((0, 2)\) from \((0,0)\) and to reach \((0,0)\) from \((0, 2)\). Thus, expected time server remains continuously
in the system is given by
\[ \sum_{i=1}^{n-k+1} E(T_{i1}) + E(T_{i2}) = \frac{(n - k + 2)}{(\mu - \lambda)} - \frac{\lambda}{(\mu - \lambda)^2} \left(1 - \left(\frac{\lambda}{\mu}\right)^{n-k+2}\right) + \frac{1}{\eta} + \frac{\lambda (1 - (\lambda/\mu)^{n-k+1})}{\mu - \lambda} \]

5.8.5 Expected time system remains non-functional

\[ \frac{q_{n-k+1}}{\mu} \] gives the expected number of visits to \((n - k + 1, 1)\) before first return to \((0, 0)\). 

\[ \frac{1}{\mu} \] is the expected sojourn time in the state \((n - k + 1, 1)\). Thus, expected time system remains non-functional is given by

\[ \frac{1}{\mu} q_{n-k+1} = \frac{\lambda^{n-k+1}(1 - (\frac{\lambda+\alpha}{\mu})^{n-k-3})}{\mu^2(\lambda + \alpha)^{n-k-1}(\mu - (\lambda + \alpha))} + \frac{\lambda^{n-k+1}(\lambda + \alpha + \mu)}{\mu^{n-k+1}(\lambda + \alpha)^2} + \left(\frac{\lambda}{\mu}\right)^{n-k+1} \frac{1}{\mu} \left(\frac{\lambda + \eta}{\mu}\right) \left(\frac{\lambda + \alpha}{\eta}\right) \]

5.8.6 Expected time server remains inactive during a cycle

From state \((0, 0)\) system goes to \((1, 0)\) or \((0, 1)\) on failure of one unit or on elapse of \(T\) time units respectively. From state \((1, 0)\), system goes to \((1, 1)\) or \((2, 0)\). The process goes on like this. The possible transition are given below.

Expected time server remains inactive

\[ \frac{1}{\alpha} P(T < S_1) + \left(\frac{1}{\alpha} + \frac{1}{\lambda}\right) P(S_1 < T < S_2) + \ldots + \left(\frac{1}{\alpha} + \frac{n-k-1}{\lambda}\right) P(S_{n-k-1} < T < S_{n-k}) + \frac{n-k}{\lambda} P(T > S_{n-k}) \] where \(S_i\) denote the time till \(i\) failures take place. Write \(S_0 = 0\)
and $S_0 < S_1 < S_2 \cdots < S_{n-k}$.

$$P(S_i < T < S_{i+1}) = \frac{\alpha \lambda^i}{(\lambda + \alpha)^{i+1}} \quad \text{for} \ 0 \leq i \leq n - k - 1$$

This, expected time server remains inactive during a cycle $= \frac{2}{\alpha} (1 - \left(\frac{\lambda}{\lambda + \alpha}\right)^{n-k})$

### 5.8.7 Cost Analysis

Let $C$ denote the cost per unit time due to the machine remaining non-functional and $w$ be the wage of the server per unit time.

Then, total expected profit per unit time,

$$TEP = w - \frac{2}{\alpha} (1 - \left(\frac{\lambda}{\lambda + \alpha}\right)^{n-k}) C \frac{q_{n-k+1,1}}{q_{00}} = \frac{2}{\alpha} (1 - \left(\frac{\lambda}{\lambda + \alpha}\right)^{n-k})$$

$$- \frac{C}{\mu} \left[ \frac{\lambda^{n-k+1} (1 - (\frac{\lambda}{\lambda + \alpha})^{n-k-3})}{\mu^{n-k}(\lambda + \alpha)^2} + \frac{\lambda^{n-k+1} (\lambda + \alpha + \mu)}{\mu^{n-k}(\lambda + \alpha)} \right]$$

It is seen that $TEP$ is concave in $\alpha$. The objective is to find an optimal $\alpha$ which maximizes the profit. Numerically $TEP$ is evaluated for given parameters and for various values of $\alpha$ and is given below.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$C = 30, \lambda = 10, \mu = 12$</th>
<th>$C = 50, w = 100 \lambda = 5 \mu = 12$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k = 5, w = 50, n = 30 \eta = 5$</td>
<td>$n = 50, k = 5, \eta = 2$</td>
<td></td>
</tr>
<tr>
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