Chapter 5

Strong subadditivity inequality for quantum entropies and four-particle entanglement

Entanglement in a composite system refers to certain implicit correlation between the subsystems arising from their interaction. It is the key resource of quantum computation and quantum information processing. Nowadays there is a growing interest in studying entanglement due to its potential application in quantum computing and information processing. In order to be a well defined characteristic, entanglement has to be quantifiable. In the following section we will briefly summarize different works to understand and quantify entanglement.

5.1 Characterization of entanglement

There have been different approaches to understand and to quantify entanglement [117, 118, 119, 120]. For a given state, there are several criteria to understand whether it is entangled or not. A bipartite pure state is said to be separable if it can be written as a direct product of the states of its subsystems. Mathematically this statement can be well understood from Schmidt decomposition as written in Eq. (1.11). If the Schmidt number is greater than one, then the state of the two systems is entangled. Schmidt decomposition is also very useful to study continuous variable bipartite systems [121]. For mixed state the situation is less
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simple. Mixed states are generally expressed in terms of the density matrix $\rho$. We have mentioned earlier that a mixed state $\rho$ is separable if it can be expressed as in Eq. (1.15), otherwise entangled. Peres and Horodecki independently derived a necessary condition for separability of $\rho$, which states that all eigenvalues of the matrix obtained by partial transpose of $\rho$ has to be non-negative eigenvalues [122, 123]. To derive this condition one has to write the density matrix explicitly with all their elements [124]. For example the separable state, given by Eq. (1.15) can be written as

$$\rho_{m,n} = \sum_k p_k (\rho_A^k)_{mn} (\rho_B^k)_{mn}.$$  \hfill (5.1)

Latin indices refer to the first subsystem and the Greek indices refer to the second subsystem. Now if one takes the partial transpose on any one subsystem (say, subsystem A), then one can define a new matrix with elements given by

$$\sigma_{m,n} \equiv \rho_{m,n}.$$  \hfill (5.2)

where the Latin indices of $\rho$ have been transposed. Then one can write

$$\sigma_{m,n} = \sum_k p_k (\rho_A^k)^T \otimes (\rho_B^k).$$  \hfill (5.3)

such that

$$\sigma = \sum_k p_k (\rho_A^k)^T (\rho_B^k).$$  \hfill (5.4)

which is again a genuine density matrix as the matrix $\rho$ itself is a genuine separable density matrix. Thus the matrix $\sigma$ doesn’t have any negative eigenvalues.

The next and more crucial question is how to quantify entanglement. Entanglement in a bipartite pure state can be well understood by the “measure of uncertainty” in a state of quantum system. The von Neumann entropy [125] of a quantum system described by a density matrix $\rho$ is defined by $S(\rho) = -\text{Tr}(\rho \ln \rho)$. However in the context of quantum information theory, one prefers to use the base 2 in the logarithm and thus the von Neumann entropy is redefined as [126, 127, 128]

$$S(\rho) = -\text{Tr}(\rho \log_2 \rho).$$  \hfill (5.5)

which is the quantum partner of the Shannon’s entropy in classical information theory. In case of an entangled state of two subsystems $A$ and $B$, the uncertainty
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In subsystem $B$ before we measure $A$ is $S(\rho_B)$, where $\rho_B$ is the reduced density matrix of the subsystem $B$ defined as

$$\rho_B = \text{Tr}_A \rho_{AB}.$$  \hfill (5.6)

So the gain in information or measure of entanglement is given by $S(\rho_H) - S(\rho_{AB})$. Note that $A$ and $B$ are most entangled when their reduced density matrices are maximally mixed.

In general, the quantities $S(j)$ ($j = A, B$), where $S(j) = S(\rho_j)$, satisfy an inequality called the Araki-Lieb inequality [129] which is one of the most important inequalities in the quantum theory of correlation. The inequality can be stated as

$$|S(A) - S(B)| \leq S(A, B) \leq S(A) + S(B).$$  \hfill (5.7)

where $S(A, B)$ is the joint entropy of the composite system comprising $A$ and $B$. The second part of the above inequality is known as subadditivity inequality [130] which physically implies that we have more information (less uncertainty) for an entangled state instead than if the two states are treated separately. This is due to the correlation between the subsystems. For a pure state, $S(A, B) = 0$ and thus $S(A) = S(B)$. The equality sign in the above relation (5.7) holds good if and only if the composite density matrix $\rho_{AB}$ can be written as a tensor product of its two reduced density matrices $\rho_A$ and $\rho_B$, i.e., if the system is in a disentangled state. One can define the index of correlation $I_c$ given by the expression $S(A) + S(B) - S(A, B)$ [131], which can also be interpreted as information entropy in quantum information point of view. We note that Kim et al. have calculated the entropies of different kinds of pure states including two-mode Fock states and squeezed states [132]. Further, the above relation for entropy has been studied in the context of entangled Gaussian states [133].

So far we have discussed about the measurement of entanglement in a bipartite pure state. If the composite system is in a mixed state (defined by the density operator $\rho$), then there is no unique way to quantify the entanglement of the state. Several measurements have been proposed for the entanglement of a bipartite mixed state. Let us very briefly discuss some of them.

**Entanglement of formation**: Suppose there are $k$ numbers of Bell states (maximally entangled bipartite pure states) shared by two distant parties $A$ (say, Alice)
and B (say, Bob). If they can prepare by local operation and classical communication (LOCC) \( n \) numbers of mixed states given by \( \rho_{AB} \), asymptotically, out of these, then the entanglement of formation (sometimes called the entanglement of creation) is defined by

\[
E_F = \lim_{n \to \infty} \frac{k_{\min}}{n} .
\]  

(5.8)

Bennett et al. [127] have defined \( E_F \) as an explicit function of \( \rho_{AB} \) by,

\[
E_F(\rho) = \min \sum_i p_i S(\rho_{iA}^i) .
\]

(5.9)

where the minimum is taken over all the possible realizations of the state \( \sum_j p_j |\psi_j\rangle \langle \psi_j| \). Wootters have found a closed form of this measure [134].

**Entanglement of distillation:** Related to the above measure one can also define the entanglement of distillation [127]. It defines the amount of entanglement of a state \( \rho \) as a proportion to the singlets (Bell states) that can be distilled using the purification procedure. However, no closed form or analytical expression is available for entanglement of distillation.

**Concurrence:** Wootters has introduced the notion "concurrence" [134] in the context of measure of entanglement of formation of bipartite system. For a bipartite pure state \( |\psi\rangle \) one can write

\[
|\tilde{\psi}\rangle = \sigma_y^{\otimes 2} |\psi^*\rangle .
\]

(5.10)

\( |\psi^*\rangle \) is the complex conjugate of \( |\psi\rangle \), and \( \sigma_y \) is a one-qubit spin-flip operator expressed in the \( \{|0\rangle, |1\rangle\} \) basis as \( \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \). The concurrence \( C \) is defined as \( C(\psi) = |\langle \psi | \tilde{\psi}\rangle| \) and the entanglement of formation is given by

\[
E(\psi) = \mathcal{E}(C(\psi)) .
\]

(5.11)

where the function \( \mathcal{E} \) is given by \( \mathcal{E}(C) = h \left( (1 + \sqrt{1 - C^2})/2 \right) \), and \( h(x) = -x \log_2 x - (1-x) \log_2 (1-x) \). For a maximally entangled state, such as the Bell states, concurrence is unity. On the other hand, for a disentangled state, concurrence is zero.

For a mixed bipartite state \( \rho \), concurrence is defined as \( C(\rho) = \max \{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 \} \) and \( \lambda_i \)'s are the eigenvalues, in decreasing order, of the Hermitian matrix \( R = \sqrt{\rho \mathcal{P}} \sqrt{\rho} \), where \( \mathcal{P} = (\sigma_y \otimes \sigma_y)\rho(\sigma_y \otimes \sigma_y) \).
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Entanglement monotone: One can also properly quantify the entanglement in a bipartite state from the separability criteria. It is based on the trace norm of the partial transpose $\rho_A^T$ [see Eq. (5.4)] and essentially measures the degree to which $\rho_A^T$ fails to be positive. Using trace norm of $\rho_A^T$ denoted by $\|\rho_A^T\|_1$, one can define negativity $N(\rho)$ as,

$$N(\rho) = \frac{\|\rho_A^T\|_1 - 1}{2} \tag{5.12}$$

which corresponds to the absolute value of the sum of negative eigenvalues of $\rho_A^T$. [135, 136], and which vanishes for disentangled states. $N(\rho)$ does not increase under LOCC and therefore is an entanglement monotone [137] and as such it can be used to quantify the degree of entanglement of a composite system.

Despite many approaches to define entanglement for a bipartite system, there have been only a few approaches to quantify entanglement in the composite systems of three or more particles [134, 138, 139, 140]. Coffman et al. [141] proposed a measure of entanglement in a tripartite system in terms of concurrences of the pairs of subsystems. This measure is invariant under permutations of the subsystems. An average entanglement in a four-partite state has been defined in terms of von Neumann entropies of the pairs of subsystems [142]. Very recently, Yukalov has addressed the question more generally and quantified multipartite entanglement [143] in terms of the ratio of norms of an entangling operator and of a disentangling operator in the relevant disentangled Hilbert space.

5.1.1 Measuring the entanglement of a four-particle entangled state

In this section, we put forward a possible measure of entanglement in a four-particle system [144] by studying the entropy of the reduced three-particle system. We have already mentioned above that, von Neumann entropy is a good measure for entanglement in a bipartite system. For a tripartite composite state, this entropy satisfies a strong subadditivity inequality [SSI] [145], which has many important implications in the subject of quantum information theory. We will study the properties of a four-particle entangled state through the three-particle entropy and the SSI.
5.2 Strong subadditivity inequality

We have already mentioned that for a bipartite composite system of two particles $A$ and $B$, the joint entropy $S(A,B)$ satisfies the subadditivity inequality (5.7). For a composite system of three particles $A$, $B$, and $C$, this inequality can be extended to a more stronger inequality of the following form [146]:

$$S(A, B, C) + S(B) \leq S(A, B) - S(B, C).$$

(5.13)

This inequality is known as strong subadditivity inequality. The most obvious situation that the equality sign holds in (5.13) is when the composite density matrix $\rho_{ABC}$ can be written as the tensor product of its three reduced density matrices as $\rho_A \otimes \rho_B \otimes \rho_C$, i.e., when the system is in a disentangled state. However, the more stringent condition for this reads as [130]

$$\log_2(\rho_{ABC}) - \log_2(\rho_{AB}) = \log_2(\rho_{BC}) - \log_2(\rho_C).$$

(5.14)

There have been numerous implications of the above inequality (5.13) in quantum information theory [1]. Firstly, it refers to the fact that the conditioning on the subsystem always reduces the entropy, i.e., $S(AB, C) < S(A, B)$, where $S(AB) = S(A, B) - S(B)$ is the entropy of $A$ conditional on knowing the state of $B$. Secondly, the above inequality implies that discarding a quantum system never increases mutual information, i.e., $S(A : B) < S(A : B, C)$, where $S(A : B) = S(A) + S(B) - S(A, B)$ is the mutual information of the subsystems $A$ and $B$. Thirdly, mutual information of two subsystems never increases by quantum operations. This means that if the mutual information of the two subsystems $A$ and $B$ becomes $S'(A : B)$ after trace-preserving operation on $B$, then $S'(A : B) < S(A : B)$. Further, this inequality (5.13) implies that the conditional entropy of the subsystems $A$, $B$, and $C$ is also subadditive, i.e., $S(A, B | C) \leq S(A | C) + S(B | C)$.

To verify SSI, one needs to calculate the entropies like $S(A, B, C)$ which clearly requires a three-particle mixed state which we can produce using a pure four-particle entangled state [146]. In the next subsection, we discuss how one can prepare a pure four-particle entangled state so that we can study SSI for the first time for a system realizable using cavity QED methods.
5.2.1 Preparation of four-particle entangled state

In order to prepare a four particle entangled state we would consider a situation as discussed in Chap. 4, where two three-level atoms (A and B) in Λ configuration [see Fig. 4.1] are interacting with a high-Q bimodal optical cavity. Here \( a \) and \( b \) are the specified annihilation operators for the cavity modes.

The interaction Hamiltonian of the system under rotating wave approximation can be written as

\[
H = \hbar \sum_{k=A,B} [g_{1k}|e_k\rangle\langle g|ae^{-i\Delta t} + g_{2k}|e_k\rangle_A (f^\dagger be^{-i\Delta t} - H.c.) \right],
\]

where \( g_{jk} \) \((j = 1,2)\) provides the atom-cavity coupling and \( \Delta \) is common one-photon detuning of the system. We assume \( g_{jk} \)'s to be real and function of time. Further each atom would be in its ground state \( |g\rangle \) at large negative times and we further assume the field in the Fock state which can be prepared by using excited atoms [106].

The corresponding zero eigenvalue eigenstate of the Hamiltonian \( H \) is

\[
|\psi_0\rangle = \frac{1}{P} \left[ \alpha |g_A, g_B, n, \mu\rangle + 3 \delta f_A, f_B, n - 2, \mu + 2 \right. \\
- \left. \gamma |g_A, f_B, n - 1, \mu + 1\rangle - 3 \delta f_A, g_B, n - 1, \mu + 1 \right],
\]

where

\[
\alpha = g_{2A}g_{2B} \sqrt{(\mu - 1)(\mu - 2)} \cdot \beta = g_{1A}g_{1B} \sqrt{n(n + 1)}, \\
\gamma = g_{1A}g_{2B} \sqrt{n(\mu + 2)} \cdot \delta = g_{1A}g_{2B} \sqrt{n(\mu + 2)},
\]

\( P = \sqrt{\alpha^2 + \beta^2 + \gamma^2 + \delta^2} \)

and \( n \) and \( \mu \) are the initial photon numbers in the modes \( a \) and \( b \), respectively. Clearly, this state is an entangled state of four particles, namely, the atoms A and B, and the two modes \( a \) and \( b \).
5.2.2 Study of strong subadditivity inequality

Let us assume that \( \Delta = 0 \) and initially both the atoms are in \(|y\rangle\) state. The cavity fields are time dependent and applied in a counter-intuitive sequence as

\[
\begin{align*}
g_{1A} &= g_{1B} = g_{10} \exp \left[-(t - T)^2/\tau^2\right], \\
g_{2A} &= g_{2B} = g_{20} \exp \left(-t^2/\tau^2\right).
\end{align*}
\]

Under the action of these time-dependent cavity fields, the system remains in the state \(|\psi_0\rangle\) for all times. At the end of the evolution, the population of both the atoms simultaneously transfer to the state \(|f\rangle\). However, if the atoms are not in one-photon resonance, i.e., if \(\Delta \tau \neq 0\), then this transfer process is not complete. This happens because the system does not remain confined in the null adiabatic state \(|\psi_0\rangle\) for \(\Delta \tau \neq 0\) \cite{50}. We will discuss all these issues in more detail in the next chapter.

We now investigate the validity of SSI for any trio of quantum systems in the present process. We can express this inequality for any three subsystems, namely, atom A, atom B, and cavity mode a with \(n\) photons out of the four subsystems under consideration as

\[
E = S(A, B) + S(A, n) - S(A, B, n) - S(A) \leq 0.
\]

Here, \(S\) defines the joint von Neumann entropy of the relevant subsystems [see Eq. (5.5)]. This can be calculated from the entire state of the (atoms + cavity modes) system by tracing over the other subsystems, e.g.,

\[
S(A, B) = -\text{Tr}_{AB}(\rho_{AB} \log_2 \rho_{AB})
\]

where, \(\rho_{AB}\) is the reduced density matrix of the atoms A and B and is given by

\[
\rho_{AB} = \text{Tr}_{n,a}(|\psi(t)\rangle\langle \psi(t)|).
\]

where \(|\psi(t)\rangle\) is spanned over all the basis states of the four-particle system. We show the time variation of \(E\) in Fig. 5.1(a). Clearly, \(E(t)\) never becomes negative during the evolution and thus the SSI (5.19) holds for all times.

From Fig. 5.1(a), one clearly sees that for \(\Delta \tau = 0\), in long time limit, \(E\) becomes zero. This means that the subsystems (A, B, and the mode a with photon number
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Figure 5.1: (a) Time evolution of the parameter $\Delta \tau$ for $\Delta \tau = 60$ (dashed curve). This clearly shows that the strong subadditivity inequality remains valid in the present physical situation. The parameters in section a are $n = 2$, $\mu = 0$, $g_{2}\tau = 15$ ($j = 1, 2$), $T = 4\tau / 3$. $\tau$ and $T$ are the width and separation between the cavity pulses respectively [Eq. (5.18)]. The coefficients $a/P$, $b/P$, $c/P$, and $d/P$ with time. The parameters $\Delta \tau = 60$. (b) as in (a).

$n$ become disentangled. This happens because of complete and successive transfer of population to the level $|f\rangle$ of both the atoms at long time limit. The entire process can be written as

$$g_{A} g_{B} \cdot \mu \cdot \mu \rightarrow f_{A} f_{B} \cdot \mu \cdot \mu$$

We have shown the time-variation of the coefficients $a$, $b$, $c$, and $d$ [see Eq. (5.17)] in Fig. 5.1(b). This figure reveals the above evolution according to the state $|\psi_0\rangle$ under the action of the pulses (5.18). On the other hand for $\Delta \tau = 0$, since complete population transfer does not occur, the system remains entangled in a coherent superposition of all the possible basis states at long time limit. This is clear from the dashed curve of Fig. 5.1(a), as the equality $\Delta \tau = 0$ no longer holds at this time limit.

Thus we can recognize the expression $E$ [see Eq. (5.19)] as a measure of four-particle entanglement in the present process. Precisely, $E > 0$, where the equality sign holds good for the disentangled states. An increase in value of $E$ refers to increase in entanglement. Thus, during the evolution, the system gets more entangled for $\Delta \tau = 0$ than for $\Delta \tau = 60$. However, at the end of the evolution, the
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entanglement persists only for nonzero $\Delta \tau$. We must emphasize here that the present definition of entanglement measurement satisfies all the relevant criteria, namely, (a) it is semipositive, i.e., $E > 0$. (b) $E = 0$ for a disentangled state, (c) the function $E(t)$ is continuous in time domain, (d) $E$ is invariant and non-increasing under local unitary operations as the von Neumann entropies defining the parameter $E$ [Eq. (5.19)] hold the similar properties. Note that temporal evolution of the entanglement for non-equilibrium processes has also been studied [147]. A very recent paper has suggested a quantitative relation between entanglement and linear entropy for two-qubit-cavity system governed by unitary evolutions [148].

The authors have proposed a measure of bipartite entanglement for a particular physical system.

We should mention here that the parameter $E$ can also be calculated with other different combinations of subsystems. For example, another choice of $E$ could be $E = S(A, B) + S(A, \mu) - S(A, B, \mu) - S(A)$. So we can average this quantity over all the choices to obtain an equivalent entanglement measurement $E_{\text{equ}}$. We have also calculated the entanglement following the prescription of Yukalov [143].
He has explained entanglement as a process by which disentangled states are transformed into entangled ones by an entangling operator $A$. There always exist certain operators $A^\otimes$ which correspond to $A$ which do not entangle. Yukalov has defined the measure of entanglement, produced by the operator $A$ in terms of the ratios of the norms of $A$ and $A^\otimes$ as

$$\mathcal{E}(A) = \log \frac{\|A\|}{\|A^\otimes\|}.$$  \hspace{1cm} (5.23)

where the norms are to be calculated in the disentangled Hilbert space.

In context of the four-particle entangled state $|\psi_0\rangle$ [see Eq. (5.17)], we identify the entangling operator $A$ as $\rho_0 = |\psi_0\rangle \langle \psi_0|$. Thus the disentangling operator corresponding to $\rho_0$ is given by $\rho_0^\otimes = \rho_{ABH} \rho_H$, where $\rho_k (k = A, B, \ldots)$ is the reduced density matrix of the subsystem $k$. Here we have used the fact that $Tr[\rho_{ABH}] = 1$. Further as $\rho_0^\otimes$ is a disentangling operator, its trace in the disentangling subspace is also unity. We have calculated the Hilbert-Schmidt norms of both $A$ and $A^\otimes$ in the following way:

$$\|\rho_0\| = \sqrt{\text{Tr}(\rho_0^2)}; \quad \|\rho_0^\otimes\| = \sqrt{\text{Tr}(\rho_0^\otimes)^2}.$$  \hspace{1cm} (5.24)

We have used the relation (5.23) to calculate the entanglement $E_{\text{yukalov}}$ in the state $|\psi_0\rangle$ numerically. Furthermore for a four-particle entangled state Higuchi et al. [142] suggested a measure of average entanglement given by $E_{\text{avg}} = E_{\text{yukalov}} + E_{\text{Br}} + E_{\text{min}})/3$. We have plotted all the three entanglement measures namely, $E_{\text{max}}$, $E_{\text{yukalov}}$, and $E_{\text{avg}}$ in Fig. 5.2. It is clear from the figure that all the three measures exhibit similar temporal behavior though they differ from each other by a constant amount.

Using our four-particle entanglement, we can also study the inequality (5.7) involving the entropies of two particles (Araki-Lieb inequality), say, atoms A and B. They remain strongly correlated during the evolution, as $I_c$ remains much larger than zero. At the end of the evolution, $I_c$ becomes zero for $\Delta \tau \to 0$ which means that the subsystems become uncorrelated. In other words, the entanglement between them vanishes.

We note in passing that for a four-particle GHZ state defined by

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$$  \hspace{1cm} (5.25)
of four qubits A, B, C, and D, one is led to a three-particle mixed state $\rho_{ABC}$ defined by

$$\rho_{ABC} = \frac{1}{2}(|000\rangle\langle 000| + |111\rangle\langle 111|).$$

(5.26)

In this case, $S(A, B, C) = S(A) = S(A, B) = S(B, C) = \log_2 2 = 1$. Therefore, the parameter $E$ becomes zero, as from Eq. (5.19). So we have a counter-example, in which the equality sign in (5.13) holds for an entangled state, too. However, we note that the above state (5.26) satisfies the condition (5.14) and thus the said equality.